CHAPTER 2

Applications of Integration

2.1. More about Areas

2.1.1. Area Between Two Curves. The area between the curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$ ($f$, $g$ continuous and $f(x) \geq g(x)$ for $x$ in $[a, b]$) is

$$A = \int_{a}^{b} f(x) \, dx - \int_{a}^{b} g(x) \, dx = \int_{a}^{b} [f(x) - g(x)] \, dx.$$  

Calling $y_T = f(x)$, $y_B = g(x)$, we have:

$$A = \int_{a}^{b} (y_T - y_B) \, dx$$

Example: Find the area between $y = e^x$ and $y = x$ bounded on the sides by $x = 0$ and $x = 1$.

Answer: First note that $e^x \geq x$ for $0 \leq x \leq 1$. So:

$$A = \int_{0}^{1} (e^x - x) \, dx = \left[ e^x - \frac{x^2}{2} \right]_{0}^{1} = \left( e^1 - \frac{1^2}{2} \right) - \left( e^0 - \frac{0^2}{2} \right) = e - \frac{1}{2} - 1 = e - \frac{3}{2}.$$  

The area between two curves $y = f(x)$ and $y = g(x)$ that intersect at two points can be computed in the following way. First find the intersection points $a$ and $b$ by solving the equation $f(x) = g(x)$. Then find the difference:

$$\int_{a}^{b} f(x) \, dx - \int_{a}^{b} g(x) \, dx = \int_{a}^{b} [f(x) - g(x)] \, dx.$$  

If the result is negative that means that we have subtracted wrong. Just take the result in absolute value.
2.1. MORE ABOUT AREAS

*Example:* Find the area between \( y = x^2 \) and \( y = 2 - x \). *Solution:* First, find the intersection points by solving \( x^2 - (2 - x) = x^2 + x - 2 = 0 \). We get \( x = -2 \) and \( x = 1 \). Next compute:

\[
\int_{-2}^{1} (x^2 - (2 - x)) \, dx = \int_{-2}^{1} (x^2 + x - 2) \, dx = -\frac{9}{2}.
\]

Hence the area is \( \frac{9}{2} \).

Sometimes it is easier or more convenient to write \( x \) as a function of \( y \) and integrate respect to \( y \). If \( x_L(y) \leq x_R(y) \) for \( p \leq y \leq q \), then the area between the graphs of \( x = x_L(y) \) and \( x = x_R(y) \) and the horizontal lines \( y = p \) and \( y = q \) is:

\[
A = \int_{p}^{q} (x_R - x_L) \, dy
\]

*Example:* Find the area between the line \( y = x - 1 \) and the parabola \( y^2 = 2x + 6 \).

*Answer:* The intersection points between those curves are \((-1, -2)\) and \((5, 4)\), but in the figure we can see that the region extends to the left of \( x = -1 \). In this case it is easier to write

\[
x_L = \frac{1}{2} y^2 - 3, \quad x_R = y + 1,
\]

and integrate from \( y = -2 \) to \( y = 4 \):

\[
A = \int_{-2}^{4} (x_R - x_L) \, dx = \int_{-2}^{4} \left\{ (y + 1) - \left( \frac{1}{2} y^2 - 3 \right) \right\} \, dx
\]

\[
= \int_{-2}^{4} \left( -\frac{1}{2} y^2 + y + 4 \right) \, dx
\]

\[
= \left[ -\frac{y^3}{6} + \frac{y^2}{2} + 4y \right]_{-2}^{4}
\]

\[
= 18
\]