UNC MINISCHOOL PROBLEMS

(1) Show that the kernel of the resolvent $(\Delta - \lambda^2)^{-1}$ is given in $\mathbb{R}^3$ by

$$R_0(\lambda) = \frac{e^{i\lambda|x-y|}}{4\pi|x-y|}.$$

You can do this by Fourier inversion, directly by verifying an equation on distributions, or by Fourier transform the spherical means kernel for the wave equation.

Check directly that this operator is L^2-bounded for $\operatorname{Im}\lambda > 0$.

(2) Check that on $\mathbb{R}^n$, $[\Delta, rD_r] = (2/i)\Delta$. Then compute $[\Delta, D_r]$ while you're at it. (Be very careful with the latter part in low dimension, where some wacky stuff happens owing to the fact that D_r is *not* a differential operator with smooth coefficients.)

(3) (Local smoothing for Schrödinger) Let $U(t) = e^{-it\Delta}$ denote the Schrödinger propagator (on Euclidean space or a perturbation thereof).

We claim that T is bounded from L^2 to $L^2(\mathbb{R}; H^{1/2})$ ("local smoothing"). We will also prove (indeed, prove first) an inhomogeneous version of the estimate: if

$$(D_t + \Delta)v = \chi f, \quad v = 0 \text{ for } t < 0,$$

then

$$\|\chi v\|_{L^2(\mathbb{R};H^{1/2})} \lesssim \|\chi f\|_{L^2(\mathbb{R};H^{-1/2})}. \tag{1}$$

(a) First prove the inhomogeneous estimate (1) by Fourier–Laplace transforming v from t to $\mu + i\epsilon$ (with the ϵ making this convergent) and using the "nontrapping resolvent estimate" between Sobolev spaces

$$\|\chi R(\lambda)\chi\|_{H^s \to H^{s+r}} \lesssim \langle\lambda\rangle^{-1+r}$$

on the resulting inhomogeneous Helmholtz equation, and the Plancherel theorem.

(b) Consider the map $T : f \mapsto \chi U(t)f$ (the solution map for the Schrödinger IVP). We want to show that T is bounded from L^2 to $L^2(\mathbb{R}; H^{1/2})$.

Begin by showing that T satisfies this bound iff TT^* is bounded on $L^2(\mathbb{R}; H^{1/2})$.

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(c) Compute T^* and then show that

(2) $$TT^*f(t,\bullet) = \int_0^t \chi U(t-s)\chi f(s,\bullet)\,ds.$$

(d) Split the integral in (2) into the pieces where $s \gtrless t$ and show that each of these pieces solves an *inhomogeneous* Schrödinger equation. Conclude from our inhomogeneous estimate that TT^* is bounded, and hence T is.

Overall hint: in case of trouble, see Theorem 7.2 of the unpublished book of Dyatlov–Zworski or N. Burq, *Smoothing effect for Schrödinger boundary value problems.*

(4) On $X = S^1 \times S^1$ consider a sequence of eigenfunctions of the Laplacian given by $e^{imx+iny}$ with $h = (m^2+n^2)^{-1/2}$. as $(m,n) \to \infty$. What are the possible defect measures associated to the sequence if $(m,n) = k(m_0,n_0)$ for fixed m_0, n_0 and $k \in \mathbb{N}$?

(Very hard:) Do you think you can produce *some* sequence of eigenfunctions with defect measure whose projection to the base is not AC with respect to Lebesgue? Remember that there is multiplicity, so you can in general use combinations of the complex exponentials with $m^2+n^2 = \lambda^2 = h^{-2}$ fixed.

(5) Find the defect measure(s) and wavefront set of a Gaussian wavepacket on $\mathbb{R}^n$:
$$e^{(-(x-a)^2/2+ix\cdot\alpha)/h}$$

(6) Find the defect measure(s) and wavefront set of the family of highest weight spherical harmonics Y^m_m on S^2 (with h^2 equal to the reciprocal of the eigenvalue). You may need to read up on Legendre function asymptotics. Remark on the differences with the eigenfunctions you investigated in Problem 4.

(7) Remember that the functional calculus for semiclassical pseudodifferential operators tells us that if $P = h^2\Delta + V$ is a semiclassical Schrödinger operator on a compact manifold X, and if $\chi \in \mathcal{C}_c^\infty$,
$$\chi(P) \in \Psi^{-\infty}$$
and has principal symbol $\chi \circ p$ (with $p = |\xi|^2 + V$ the principal symbol of P). Consequently, we can obtain asymptotics for $\operatorname{Tr}\chi(P)$ by writing it as a quantization.

Finish the proof that this gives the Weyl law for the eigenvalues E_j of P :
$$\#\{E_j \in [a,b]\} = (2\pi h)^{-n}\big(\operatorname{Vol}(p^{-1}([a,b]) + o(1)\big)$$
by taking χ's that approximate the function $1_{[a,b]}$ from above and below.

(8) In this problem, we compute resonances (and lack thereof) in some simple 1d problems. It's already not so easy.

(a) Consider the semiclassical Schrödinger equation on $\mathbb{R}^1$ with potential $V = H(x)$, where $H(x)$ denotes the Heaviside function:

$$(-h^2\partial_x^2 + H(x) - E)u = 0.$$

Show that for $E > 1$ there exist "outgoing" solutions corresponding to an incident plane wave of the form:

$$u(x) = \begin{cases} e^{i\sqrt{E}x/h} + Re^{-i\sqrt{E}x/h} & x < 0, \\ Te^{i\sqrt{E-1}x/h} & x \geq 0, \end{cases}$$

for some values of R, T (compute them).

(b) What kind of solutions do we get if $E \in (0, 1)$?

(c) Show that there are *no* nonzero solutions of the form

$$u(x) = \begin{cases} Ae^{-i\sqrt{E}x/h} & x \ll 0, \\ Be^{+i\sqrt{E-1}x/h} & x \gg 0, \end{cases}$$

("outgoing solutions"), even if we let E take on complex values.

(d) Now consider the case where the potential is the indicator function of the interval $[0, 1]$ (and again $E > 1$). Here you will have to define your solution piecewise on $(-\infty, 0)$, $[0, 1]$, and $(1, \infty)$. Show that there *do* exist outgoing solutions for certain complex values of the spectral parameter E ("resonances"). Approximately where do these lie as $h \downarrow 0$ and E near the real axis? (This part may be challenging—it involves an unpleasant transcendental equation for E.)