For Most Frequencies, Strong Trapping Has a Weak Effect in Frequency-Domain Scattering

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Abstract

It is well-known that when the geometry and/or coefficients allow stable trapped rays, the outgoing solution operator of the Helmholtz equation grows *exponentially* through a sequence of real frequencies tending to infinity.

In this paper we show that, even in the presence of the strongest possible trapping, if a set of frequencies of arbitrarily small measure is excluded, the Helmholtz solution operator grows at most *polynomially* as the frequency tends to infinity.

One significant application of this result is in the convergence analysis of several numerical methods for solving the Helmholtz equation at high frequency that are based on a polynomial-growth assumption on the solution operator (e.g. hp-finite elements, hp-boundary elements, and certain multiscale methods). The result of this paper shows that this assumption holds, even in the presence of the strongest possible trapping, for most frequencies. © 2021 The Authors. *Communications on Pure and Applied Mathematics* published by Wiley Periodicals LLC.

1 Introduction

1.1 Motivation: Bounds on the Solution Operator under Trapping

Trapping and nontrapping are central concepts in scattering theory. This paper is concerned with the behaviour of the outgoing solution operator in frequency domain scattering problems (aka the resolvent) in the presence of strong trapping. Our results hold for a wide variety of boundary value problems where the differential operator is the Helmholtz operator $\Delta + k^2$ outside some compact set; indeed, we work in the framework of *black box scattering* introduced by Sjöstrand-Zworski in [107] and recalled briefly in §2. For simplicity, in this introduction we focus on the exterior Dirichlet problem (EDP) for the Helmholtz equation; i.e. the problem of, given a bounded, open set $\mathcal{O}_- \subset \mathbb{R}^n$, $n \geq 2$, such that the open complement

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$\mathcal{O}_+ := \mathbb{R}^n \setminus \overline{\mathcal{O}_-}$ is connected and $\partial\mathcal{O}_+$ is Lipschitz, $f \in L^2(\mathcal{O}_+)$ with compact support, and frequency $k > 0$, finding $u \in H^1_{\rm loc}(\mathcal{O}_+)$ such that

$$\Delta u + k^2 u = -f \quad \text{in } \mathcal{O}_+, \quad \gamma u = 0 \quad \text{on } \partial\mathcal{O}_+, \tag{1.1}$$

(where γ denotes the trace operator on $\partial\mathcal{O}_+$) and

$$\frac{\partial u}{\partial r}(x) - \mathrm{i}ku(x) = o\left(\frac{1}{r^{(d-1)/2}}\right), \tag{1.2}$$

as $r \to \infty$, uniformly in $\widehat{x} := x/r$ (with this last condition known as the *Sommerfeld radiation condition*, and solutions satisfying this condition known as *outgoing*). A classic result of Rellich (see, e.g., [43, theorems 3.33 and 4.17]) implies that the solution of the EDP is unique for all k. Formulating the EDP as a variational problem in a large ball as in §1.3 below, one can then apply the Fredholm alternative (see, e.g., [90, theorem 2.6.6]) to obtain that the solution exists for all k and, given $R > 0$ such that $\operatorname{supp} f \subset B_R := \{x : |x| < R\}$ and $k_0 > 0$,

$$\|\nabla u\|_{L^2(\mathcal{O}_+\cap B_R)} + k\|u\|_{L^2(\mathcal{O}_+\cap B_R)} \leq \Upsilon(k, \mathcal{O}_-, R, k_0)\|f\|_{L^2(\mathcal{O}_+)} \tag{1.3}$$

for all $k \geq k_0$, where $\Upsilon(k, \mathcal{O}_-, R, k_0)$ is some (a priori unknown) function of $k, \mathcal{O}_-, R$, and k_0.

It is convenient to write bounds such as (1.3) in terms of the outgoing cutoff resolvent $\chi R(k)\chi : L^2(\mathcal{O}_+) \to H^1(\mathcal{O}_+)$ for $k \in \mathbb{R} \setminus \{0\}$, where $\chi \in C^\infty_{\rm comp}(\overline{\mathcal{O}_+})$ and

$$R(k) := -(\Delta + k^2)^{-1}$$

on $\partial\mathcal{O}_+$ is defined by analytic continuation from $R(k) : L^2(\mathcal{O}_+) \to L^2(\mathcal{O}_+)$ for $\Im k > 0$ (this definition implies that the radiation condition (1.2) is satisfied for $k \in \mathbb{R} \setminus \{0\}$); see, e.g., [43, §3.6, theorem 4.4, and example 2 on p. 229]. The bound (1.3) then becomes

$$\begin{aligned} \|\chi R(k)\chi\|_{L^2(\mathcal{O}_+)\to L^2(\mathcal{O}_+)} &\leq \frac{\Upsilon(k, \mathcal{O}_-, \chi, k_0)}{k}, \\ \|\chi R(k)\chi\|_{L^2(\mathcal{O}_+)\to H^1(\mathcal{O}_+)} &\leq \frac{\Upsilon(k, \mathcal{O}_-, \chi, k_0)}{\min(k_0, 1)}, \end{aligned} \tag{1.4}$$

for all $k \geq k_0$.

Having obtained an $L^2 \to L^2$ bound on $\chi R(k)\chi$, an $L^2 \to H^1$ bound can be obtained from Green's identity (i.e., multiplying the PDE in (1.1) by $\overline{u}$ and integrating by parts; see, e.g., [109, lemma 2.2]), and so we focus on $L^2 \to L^2$ bounds from now on. The Schwartz kernel of the outgoing resolvent, often referred to as the outgoing Green function, is necessarily singular at the diagonal, so it is L^2 mapping estimates that seem most natural in this context.

When $\mathcal{O}_+$ has C^∞ boundary and is *nontrapping*, i.e., all billiard trajectories starting in an exterior neighbourhood of $\mathcal{O}_-$ escape from that neighbourhood after some uniform time, one can show that Υ in (1.4) is independent of k, i.e., given

$k_0 > 0$,

$$\|\chi R(k)\chi\|_{L^2(\mathcal{O}_+)\to L^2(\mathcal{O}_+)} \lesssim \frac{1}{k} \quad \text{for all } k \ge k_0, \tag{1.5}$$

where the notation $a \lesssim b$ means that there exists a $C > 0$ that is independent of k (but dependent on k_0, $\mathcal{O}_-$, and χ) such that $a \le Cb$. This classic nontrapping resolvent estimate was first obtained by the combination of the results on propagation of singularities for the wave equation on manifolds with boundary by Andersson-Melrose [4], Melrose [84], Taylor [117], and Melrose-Sjöstrand [86,87] with either the parametrix method of Vainberg [118] (see [98]) or the methods of Lax-Phillips [73] (see [85]). (See [51] for precise estimates on the omitted constant in the inequality (1.5).)

On the other hand, when $\mathcal{O}_-$ is *trapping*, a loss is unavoidable in the cutoff resolvent; indeed, at least in the analogous case of semiclassical scattering by a potential, if trapping exists then one has a semiclassical lower bound by [13, théorème 2] (see also [43, theorem 7.1]), which in our notation implies that there exists a sequence of frequencies $0 < k_1 < k_2 < \cdots$, with $k_j \to \infty$, such that

$$\|\chi R(k_j)\chi\|_{L^2\to L^2} \gtrsim \frac{\log(2+k_j)}{k_j}, \quad j = 1, 2, \dots, \tag{1.6}$$

and one expects the strength of the loss to depend on the strength of the trapping. In the standard example of *hyperbolic trapping*, when $\mathcal{O}_-$ equals the union of two disjoint convex obstacles with strictly positive curvature (see Figure 1.1(a), the lower bound (1.6) is achieved, since

$$\|\chi R(k)\chi\|_{L^2(\mathcal{O}_+)\to L^2(\mathcal{O}_+)} \lesssim \frac{\log(2+k)}{k} \quad \text{for all } k \ge k_0,$$

by [18, prop. 4.4] (which is based on now classic work of Ikawa [66]). In the standard example of *parabolic trapping*, when $\mathcal{O}_-$ equals the union of two disjoint, aligned squares, in two dimensions, or cubes, in three dimensions (see Figure 1.1(b)), the cutoff resolvent suffers a polynomial loss over the nontrapping estimate, with the bound

$$\|\chi R(k)\chi\|_{L^2(\mathcal{O}_+)\to L^2(\mathcal{O}_+)} \lesssim k \quad \text{for all } k \ge k_0$$

proved in [28, theorem 1.10]; variable-power polynomial losses have also been exhibited in [35, theorem 2] in cases of degenerate-hyperbolic trapping in the setting of scattering by metrics.

For general $\mathcal{O}_-$ with C^∞ boundary, the cutoff resolvent can grow at most exponentially in k by the bound of Burq [16, theorem 2]

$$\|\chi R(k)\chi\|_{L^2(\mathcal{O}_+)\to L^2(\mathcal{O}_+)} \lesssim e^{\alpha k} \quad \text{for all } k \ge k_0$$

for some $\alpha = \alpha(\mathcal{O}_-, k_0) > 0$. This exponential growth of the cutoff resolvent is achieved in the presence of the strongest possible trapping—so-called elliptic trapping. Indeed, if $\mathcal{O}_-$ has an ellipse-shaped cavity (see Figure 1.1(c)), then there

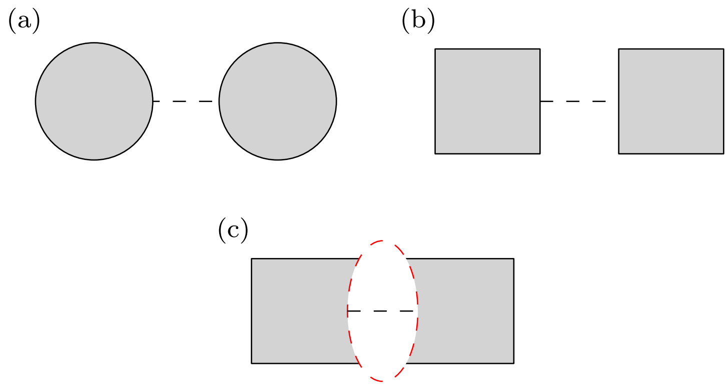

FIGURE 1.1. Examples of (a) hyperbolic trapping, (b) parabolic trapping, and (c) elliptic trapping, with a trapped ray for each indicated by a black dashed line.

exists a sequence of frequencies $0 < k_1 < k_2 < \cdots$ with $k_j \to \infty$, and $\alpha > 0$ such that

$$\left\| \chi R(k_j) \chi \right\|_{L^2(\mathcal{O}_+) \to L^2(\mathcal{O}_+)} \gtrsim \mathrm{e}^{\alpha k_j}, \quad j = 1, 2, \ldots, \tag{1.7}$$

see, e.g., [11, §2.5]. More generally, if there exists an elliptic trapped ray (i.e., an elliptic closed broken geodesic), and $\partial\mathcal{O}_+$ is analytic in neighbourhoods of the vertices of the broken geodesic, then the resolvent can grow at least as fast as $\exp(\alpha k_j^q)$, through a sequence k_j as above and for some range of $q \in (0, 1)$, by the quasi mode construction of Cardoso-Popov [21] (note that Popov proved *superalgebraic* growth for certain elliptic trapped rays when $\partial\Omega_-$ is smooth in [96]).

The question this paper answers is *how does the cutoff resolvent behave under elliptic trapping when k is not equal to one of the "bad" frequencies k_j?*

Our answer to this question uses the fact that the growth (1.7) of the cutoff resolvent through the real sequence k_j under trapping is due to the presence of (complex) resonances lying in the lower-half complex k-plane, close to the real axis. The "bad" real frequencies k_j then correspond to the real parts of these (complex) resonances. The strength of the trapping and how close the resonances are to the real axis are intimately related. Indeed, in elliptic trapping, the resonances are superalgebraically close to the real axis, causing at least superalgebraic growth of the cutoff resolvent, whereas in hyperbolic trapping the resonances stay a fixed distance away from the real axis, hence the weak logarithmic loss over the nontrapping resolvent estimate; see the recent overview discussion in [125, §2.4] and the references therein.

1.2 Statement of Main Results (in the Setting of Impenetrable-Dirichlet-Obstacle Scattering)

In the setting of scattering by an impenetrable Dirichlet obstacle our main result is the following. This result is valid (and hence stated) for all Lipschitz obstacles, but is of primary interest when the obstacle contains an elliptic trapped ray.

THEOREM 1.1 (Polynomial resolvent estimate for most frequencies). *Let $\mathcal{O}_- \subset \mathbb{R}^n$, $n \geq 2$, be a bounded open set such that the open complement $\mathcal{O}_+ := \mathbb{R}^n \setminus \overline{\mathcal{O}_-}$ is connected and $\partial\mathcal{O}_+$ is Lipschitz. Let $R(k)$ be defined as in* §1.1. *Then, given $k_0 > 0$, $\delta > 0$, and $\varepsilon > 0$, there exists $C = C(k_0, \delta, \varepsilon, n) > 0$ and a set $J \subset [k_0, \infty)$ with $|J| \leq \delta$ such that*

$$\|\chi R(k)\chi\|_{L^2(\mathcal{O}_+)\to L^2(\mathcal{O}_+)} \leq Ck^{5n/2+\varepsilon} \quad \textit{for all } k \in [k_0, \infty) \setminus J. \tag{1.8}$$

In other words, even in the presence of elliptic trapping, outside an arbitrarily small set of frequencies, the resolvent is always polynomially bounded, with an exponent depending only on the dimension. We make the following remarks:

(1) The analogue of Theorem 1.1 in the black-box-scattering framework is given as Theorem 3.3 below—a resolvent estimate identical to (1.8) in its k-dependence is therefore valid in a wide range of settings, including scattering by an impenetrable Neumann obstacle, by a penetrable obstacle, by a potential, by elliptic and compactly supported perturbations of Laplacian, and on finite volume surfaces (see §2 and the references therein).

(2) The proof of Theorem 1.1 uses (i) a polynomial bound on the density of resonances ((2.5) below), (ii) a bound on the resolvent away from resonances (Theorem 3.2 below), and (iii) the semiclassical maximum principle (Theorem 3.1 below). The bounds in (i) were originally pioneered by Melrose, and then further developed by Sjöstrand, Sjöstrand-Zworski, Vodev, and Zworski (see the references below (2.5), and also the literature overviews in [43, §3.13 and §4.7]). The results in (ii) and (iii) are due to Tang-Zworski [116]. We highlight that, in fact, [116, prop. 4.6] notes that the cutoff resolvent is bounded polynomially in regions of the complex plane that include intervals of the real axis away from resonances; the difference here is that we seek to control the measure of these intervals.

(3) When we have finer information about the distribution of resonances, we can lower the exponent in (1.8) and also obtain a bound on the measure of the set $\{k : \|\chi R(k)\chi\|_{L^2\to L^2} > \lambda^s\} \cap [\lambda, \lambda+1)$; see Theorem 3.5. In particular, for scattering by an obstacle with $C^{1,\sigma}$ boundary (for some $0 < \sigma < 1$), known results on Weyl laws for resonances [95] allow us to improve the exponent in (1.8) to $5n/2 - 1 + \varepsilon$ for all $\varepsilon > 0$; see Corollary 3.7. Another scenario where we have an improvement in the exponent is that of scattering by a smooth, strictly convex, penetrable obstacle; see Corollary 3.8.

(4) We do not know the sharp value of the exponent in the bound (1.8). Under a hypothesis that there exist quasi-mode solutions to the equation (often easy to construct in strong trapping situations) whose frequencies are well distributed, we

obtain a lower bound for all frequencies of $\|\chi R(k)\chi\|_{L^2\to L^2} \gtrsim k^{n-2}$: see Lemma 3.9 below.

(5) Similar results to Theorem 1.1 about relatively "good" behaviour of the Helmholtz solution operator under elliptic trapping as long as k is outside some finite set were proved by Capdeboscq and co-workers for scattering by a penetrable ball in [19, theorem 6.5] for two dimensions and [20, theorem 2.5] for three dimensions. These results use the explicit expression for the solution in terms of an expansion in Fourier series (two dimensions) or spherical harmonics (three dimensions), with coefficients given by Hankel and Bessel functions, to bound the scattered field outside the obstacle in terms of the incident field, with a loss of derivatives (corresponding to a loss of powers of k). At least when the contrast in wave speeds inside and outside the obstacle is sufficiently large, [19, lemma 6.2] and [20, lemma 3.6] show that the scattered field everywhere outside the obstacle is polynomially bounded in k for k outside a set of small, finite measure; see Remark 3.10 below for more discussion on the results of [19, 20].

(6) As noted in §1.1, when the obstacle $\mathcal{O}_-$ contains an ellipse-shaped cavity, the resolvent grows exponentially through a sequence k_j (1.7); in this situation Theorem 1.1 implicitly contains information about the widths of the peaks in the norm of the resolvent at k_j. We are not aware of any results in the literature about the widths of these peaks in the setting of obstacle scattering, but precise information about the widths and heights of peaks in the transmission coefficient for model resonance problems in one space dimension can be found in [1, 105].

(7) Complementary results (in a different direction to Theorem 1.1) about "good" behaviour of the resolvent in trapping scenarios can be found in in [24, theorem 1.1], [17, theorem 4], and [39, theorems 1.1, 1.2]. Indeed, [24, theorem 1.1] proves that, even in the presence of trapping, the nontrapping resolvent estimate (1.5) holds when the support of χ is sufficiently far away from the obstacle ([17, theorem 4] proves this result up to factors of $\log k$). The results [39, theorems 1.1, 1.2] prove the analogue of this result in the setting of scattering by a potential and/or by a metric when the cutoff functions are replaced by semiclassical pseudodifferential operators restricting attention to areas of phase space isolated from the trapped set.

(8) A result similar in spirit to Theorem 1.1 in the case of bounded domains and eigenfunctions is [59, theorem 1]; this result obtains an improvement on previous bounds about concentration of eigenfunctions for frequencies outside a specific set (corresponding to eigenvalues of a subdomain).

Using the results of [9] (a sharpening of previous arguments in [71, 109] and written down in [28, lemma 4.2] for a resolvent estimate with arbitrary k-dependence), the resolvent estimate (1.8) immediately implies bounds on the Dirichlet-to-Neumann (DtN) map described in the following corollary.

To state these bounds we first recall the definition of the weighted H^1 norm:

$$\|v\|^2_{H^1_k(D)} := \|\nabla v\|^2_{L^2(D)} + k^2\|v\|^2_{L^2(D)} \quad \text{for } D \text{ an open set.}$$

We use this definition below, both with $D = \mathcal{O}_+$ and with $D = \partial\mathcal{O}_+$; in the latter case the gradient is understood as the surface gradient on $\partial\mathcal{O}_+$; see, e.g., [78, pp. 98–99]. The weighted Sobolev spaces $H^s_k(\partial\mathcal{O}_+)$ for $s \in (0,1)$ are then defined by, e.g., [78, chap. 3], with the norms defined by interpolation; see, e.g., [28, §2.3] and [26]. Finally, let ∂_ν denote the normal-derivative operator defined by, e.g., [78, lemma 4.3] (recall that this operator is such that, when $v \in H^2_{\text{loc}}(\mathcal{O}_+)$, $\partial_\nu v = \nu \cdot \gamma\nabla v$).

COROLLARY 1.2 (Bounds on the DtN map for most frequencies). *Let $\mathcal{O}_+$ be as in Theorem* 1.1. *Let $u \in H^1_{\text{loc}}(\mathcal{O}_+)$ be a solution to the Helmholtz equation $\Delta u + k^2 u = 0$ in $\mathcal{O}_+$ that satisfies the Sommerfeld radiation condition* (1.2) *and the boundary condition $\gamma u = g$. Then, given $\chi \in C^\infty_{\text{comp}}(\overline{\mathcal{O}_+})$, $k_0 > 0$, $\delta > 0$, and $\varepsilon > 0$, there exists $C' = C'(k_0, \delta, \varepsilon, n, \mathcal{O}_-, \chi) > 0$ and a set $J \subset [k_0, \infty)$ with $|J| \le \delta$ such that*

$$\|\chi u\|_{H^1_k(\mathcal{O}_+)} + \|\partial_\nu u\|_{L^2(\partial\mathcal{O}_+)} \le C' k^{5n/2+\varepsilon}\|g\|_{H^1_k(\partial\mathcal{O}_+)} \quad \textit{for all } k \in [k_0,\infty) \setminus J,$$

if $g \in H^1(\partial\mathcal{O}_+)$. Furthermore, uniformly for $0 \le s \le 1$, and provided $g \in H^s(\partial\mathcal{O}_+)$,

$$\|\partial_\nu u\|_{H^{s-1}_k(\partial\mathcal{O}_+)} \lesssim k^{5n/2+\varepsilon}\|g\|_{H^s_k(\partial\mathcal{O}_+)}$$

and

$$\|\partial_\nu u\|_{H^{s-1}(\partial\mathcal{O}_+)} \lesssim k^{5n/2+1+\varepsilon}\|g\|_{H^s(\partial\mathcal{O}_+)}$$

for all $k \in [k_0, \infty) \setminus J$.

1.3 Applications to Numerical Analysis of Helmholtz Scattering Problems

The use of bounds on the resolvent in numerical analysis

The Helmholtz equation is arguably the simplest possible model of wave propagation, and therefore there has been considerable research into designing accurate and efficient methods for solving it numerically, especially when the frequency is large and the solution is highly oscillatory. A bound on the solution operator for a boundary value problem underpins the numerical analysis of any numerical method for solving that particular problem; consequently, the nontrapping resolvent estimate (1.5) for the Helmholtz equation has been widely used by the numerical analysis community in the frequency-explicit analysis of numerical methods for Helmholtz problems.

The following is a nonexhaustive list of papers on the frequency-explicit convergence analysis of numerical methods for solving the Helmholtz equation where a central role is played by *either* the nontrapping resolvent estimate (1.5) *or* its analogue (with the same k-dependence) for the commonly used approximation of the exterior problem where the exterior domain $\mathcal{O}_+$ is truncated and an impedance boundary condition is imposed:

- conforming FEMs (including continuous interior-penalty methods) [77, prop. 2.1], [79, prop. 8.1.4], [103, theorem 2.2], [104, theorem 3.1], [57, lemma 2.1], [82, lemma 3.5], [83, assumptions 4.8 and 4.18], [45, §2.1], [121, theorem 3.1], [124, §3.1], [46, §3.2.1], [41, remark 3.2], [42, remark 3.1], [30, assumption 1], [31, def. 2], [56, theorem 3.2], [51, lemma 6.7], [15, eq. 4], [70, eq. 1.20],
- least squares methods [34, assumption A1], [10, remark 1.2], [64, assumption 1 and equation after eq. 5.37],
- DG methods based on piece-wise polynomials [47, theorem 2.2], [48, theorem 2.1], [40, assumption 3], [49, §3], [63, assumption A (eq. 4.5)], [81, eq. 4.4], [36, remark 3.2], [33, eq. 2.4], [89, eq. 4.3], [123, remark 3.1], [101, theorem 2.2],
- plane-wave/Trefftz-DG methods [3, theorem 1], [60, eq. 3.5], [61, theorem 2.2], [2, lemma 4.1], [62, prop. 2.1],
- multiscale finite-element methods [52, eq. 2.3], [14, §1.2], [94, assumption 5.3], [8, theorem 1], [93, assumption 3.8], [32, assumption 1],
- integral-equation methods [76, eq. 3.24], [80, eq. 4.4], [25, chap. 5], [54, theorem 3.2], [122, remark 7.5], [44, theorem 2], [50, theorem 3.2], [53, assumption 3.2],

In addition, the following papers focus on proving bounds on the solution of Helmholtz boundary value problems (with these bounds often called "stability estimates") motivated by applications in numerical analysis: [6, 7, 9, 11, 27–29, 37, 51, 55, 56, 58, 75, 88, 100, 109]. Of these papers, all but [6, 11, 28, 75] are in nontrapping situations, [6, 28, 75] are in parabolic trapping scenarios, and [11] proves the exponential growth (1.7) under elliptic trapping.

How do numerical methods behave in the presence of trapping?

We highlight three features of the behaviour of numerical methods in the presence of trapping:

First, one finds general "bad behaviour" compared to nontrapping scenarios, independent of the frequency, because of the increased number of multiple reflections. For an example of this phenomenon, see [65, fig. 8, right panel], where "bad behaviour" here means a lower compression rate of BEM matrices for trapping obstacles compared to nontrapping obstacles (and with the compression rate dependent on the strength of trapping, and worst for elliptic trapping).

Second, one finds extremely bad behaviour at real frequencies corresponding to the real parts of the (complex) resonances lying under the real axis. For example, [38] shows the condition number of integral-equation formulations spiking at such frequencies under parabolic trapping [38, fig. 18] and elliptic trapping [38, fig. 19, right panel].

Third, this extremely bad behaviour at certain real frequencies is very sensitive to the frequency. For example, calculations in [76, fig. 4.7] of the norm of inverse of the integral operator $A_{k,\eta}$ defined in (1.14) below find that $\|A_{k,\eta}^{-1}\|_{L^2\to L^2} \sim 10^{11}$ at k corresponding to a resonance, but changing the fifth significant figure of k reduces the norm to $\sim 10^4$. Furthermore, this sensitivity means that verifying the exponential blowup in (1.7) is challenging. Indeed, the exponential growth of the resolvent implies exponential growth of $\|A_{k,\eta}^{-1}\|_{L^2\to L^2}$ (see [11, theorem 2.8], [25, eq. 5.39]). In the setting where the elliptic trapping is due to an ellipse-shaped cavity in the obstacle, the "bad" frequencies correspond to certain eigenvalues of the ellipse; even knowing these eigenvalues (corresponding to the zeros of a Mathieu function; see [11, appendix]) to high precision, [11, §4.8] could only verify numerically the exponential growth of $\|A_{k,\eta}^{-1}\|_{L^2\to L^2}$ up to $k \approx 100$ (where the obstacle had characteristic length scale ~ 1). To our knowledge, Theorem 1.1 is the first result rigorously describing this sensitivity of the resolvent to a frequency under elliptic trapping.

Three immediate applications of Theorem 1.1

The resolvent estimate in Theorem 1.1 can be immediately applied in all the analyses listed at the beginning of §1.3 to prove results about these methods under elliptic trapping for most frequencies.

The most exciting applications are for numerical methods whose analyses require the resolvent to be *polynomially bounded in k, with the method depending only mildly on the degree of this polynomial*. Three such methods are the following:

(1) The hp-finite-element method (hp-FEM), where, under the assumption that the resolvent is polynomially bounded in k, the results of [45, 82, 83] establish that the finite-element method when $h_{\text{FEM}} \sim k^{-1}$ and $p \sim \log k$ does not suffer from the pollution effect;[1] i.e., under this choice of h_{FEM} and p, for which the total number of degrees of freedom $\sim k^n$, the method is quasi-optimal with constant independent of k (see, e.g., (1.12) below). Similar results were then obtained for DG methods in [81, 101], and for least-squares methods in [10, 34].

(2) The hp-boundary-element method (hp-BEM), where, under a polynomial-boundedness assumption on the solution operator, the results of [76, 80] establish that the boundary-element method when $h_{\text{FEM}} \sim k^{-1}$ and $p \sim \log k$ does not suffer from the pollution effect.

(3) The multiscale finite-element method of [14, 52, 94], which, under the assumption that the resolvent is polynomially bounded in k, computes solutions that are uniformly accurate in k but with a total number of degrees of freedom $\sim k^n$, provided that a certain oversampling parameter grows logarithmically with k.

[1] We use h_{FEM} (as opposed to h) to denote the maximal element diameter in a finite-element method to distinguish it from the semiclassical parameter $h = 1/k$ used in §2 and §3.

The next two subsections give the details of the results outlined in points 1 and 2 above for obstacles with strong trapping (for brevity we do not give the details of the results in point 3).

Quasi-optimality of hp-FEM for trapping domains for most frequencies

Given $R > \max_{x\in\partial\mathcal{O}_+}|x|$, let $\mathcal{O}_R := \mathcal{O}_+ \cap B_R$, and let the Hilbert space $V_R := \{w|_{\mathcal{O}_R} : w \in H^1_{\mathrm{loc}}(\Omega_+) \text{ and } \gamma w = 0\}$. A standard reformulation of the EDP, and the starting point for discretisation by FEMs, is the variational problem

$$\text{find } u_R \in V_R \text{ such that } a(u_R, v) = F(v) \text{ for all } v \in V_R, \tag{1.9}$$

where

$$a(u,v) := \int_{\mathcal{O}_R} (\nabla u \cdot \overline{\nabla v} - k^2 u\overline{v})\mathrm{d}x - \int_{\partial B_R} \overline{\gamma v} T_R \gamma u \,\mathrm{d}s, \quad F(v) := \int_{\mathcal{O}_R} f\,\overline{v}\,\mathrm{d}x,$$

where T_R is the DtN map for the exterior problem with obstacle B_R; see, e.g., [27, eq. 3.5 and 3.6], [82, eq. 3.7 and 3.10] for the definition of T_R in terms of Hankel functions and polar coordinates (when $d = 2$) or spherical polar coordinates (when $d = 3$). This setup implies that the solution u_R to the variational problem (1.9) is $u|_{\mathcal{O}_R}$, where u is the solution of the EDP described in §1.1. Let C_{cont} be the continuity constant of the sesquilinear form $a(\cdot,\cdot)$ in the norm $\|\cdot\|_{H^1_k(\mathcal{O}_R)}$, i.e., $a(u,v) \leq C_{\mathrm{cont}}\|u\|_{H^1_k(\mathcal{O}_R)}\|v\|_{H^1_k(\mathcal{O}_R)}$ for all $u, v \in V_R$ and for all $k \geq k_0$; by the Cauchy-Schwarz inequality and the bound on T_R in [82, lemma 3.3], C_{cont} is independent of k (but dependent on k_0).

Let $\mathcal{T}_{h_{\mathrm{FEM}}}$ be a quasi-uniform triangulation of $\mathcal{O}_R$ in the sense of [83, assumption 5.1], with

$$h_{\mathrm{FEM}} := \max_{K\in\mathcal{T}_{h_{\mathrm{FEM}}}} \mathrm{diam}(K)$$

the maximum element diameter. Let $\mathcal{S}_0^{p,1}(\mathcal{T}_{h_{\mathrm{FEM}}}) := \mathcal{S}^{p,1}(\mathcal{T}_{h_{\mathrm{FEM}}}) \cap V_R$, where $\mathcal{S}^{p,1}(\mathcal{T}_{h_{\mathrm{FEM}}})$ is the space of continuous, piecewise polynomials of degree $\leq p$ on the triangulation $\mathcal{T}_{h_{\mathrm{FEM}}}$ [83, eq. 5.1]. The hp-FEM then seeks u_{hp}—an approximation of u_R in the subspace $\mathcal{S}_0^{p,1}(\mathcal{T}_{h_{\mathrm{FEM}}})$—as the solution of

$$\text{find } u_{hp} \text{ such that } a(u_{hp}, v_{hp}) = F(v_{hp}) \text{ for all } v_{hp} \in \mathcal{S}_0^{p,1}(\mathcal{T}_{h_{\mathrm{FEM}}}). \tag{1.10}$$

Theorem 1.1 implies that the polynomial-boundness assumption ([83, assumption 4.18]) in the analysis of the hp-FEM in [83] is satisfied for most frequencies, and [83, theorem 5.18] then implies the following.

COROLLARY 1.3 (k-independent quasi-optimality of hp-FEM for most frequencies). *Let $\mathcal{O}_+$ be as in Theorem* 1.1*, and assume further that $\partial\mathcal{O}_+$ is analytic. Given $k_0 > 0$, $\delta > 0$, and $\varepsilon > 0$, there exists $C_j = C_j(k_0, \delta, \varepsilon, n, \mathcal{O}_-) > 0$, $j = 1, 2$, and a set $J \subset [k_0, \infty)$ with $|J| \leq \delta$ such that, if*

$$\frac{kh}{p} \leq C_1 \quad \textit{and} \quad p \geq 1 + C_2 \log(2 + k), \tag{1.11}$$

then, for all $k \in [k_0, \infty) \setminus J$*, the Galerkin solution* u_{hp} *defined by* (1.10) *exists, is unique, and satisfies the quasi-optimal error estimate*

$$\big\|u_R - u_{hp}\big\|_{H_k^1(\mathcal{O}_R)} \leq 2\big(1 + C_{\rm cont}\big) \min_{v_{hp}\in \mathcal{S}_0^{p,1}(\mathcal{T}_{h_{\rm FEM}})} \big\|u_R - v_{hp}\big\|_{H_k^1(\mathcal{O}_R)}. \tag{1.12}$$

In this corollary we assumed that $\partial\mathcal{O}_+$ is analytic; this is so we could directly apply [83, theorem 4.18], but we highlight that analogous quasi-optimality results under polynomial-boundedness of the resolvent are obtained for nonconvex polygonal domains in [45].

The significance of the quasi-optimality results for the hp-FEM in [45, 82, 83] is that they show that the hp-FEM does not suffer from the pollution effect, in that the constant $2(1 + C_{\rm cont})$ on the right-hand side of (1.12) is independent of k, and h and p satisfying (1.11) can be chosen so that the total number of degrees of freedom (i.e., the dimension of the subspace $\mathcal{S}_0^{p,1}(\mathcal{T}_{h_{\rm FEM}})$) grows like k^n (see [83, remark 5.9] for more details). The resolvent estimate of Theorem 1.1 now shows that this property is enjoyed even for strongly trapping obstacles, at least for most frequencies.

Quasi-optimality of hp-BEM for trapping domains for most frequencies

Integral equations for the exterior Dirichlet problem In this subsection, we let $u \in H^1_{\rm loc}(\mathcal{O}_+)$ be a solution to the Helmholtz equation $\Delta u + k^2 u = 0$ in $\mathcal{O}_+$ that satisfies the Sommerfeld radiation condition (1.2) and the boundary condition $\gamma u = g$ for $g \in H^1(\partial\mathcal{O}_+)$ (note that if the data g arises from plane-wave or point-source scattering, this regularity of g is guaranteed; see [25, def. 2.11]).

We now briefly state the standard second-kind integral-equation formulations of this problem. Let $\Phi_k(x, y)$ be the fundamental solution of the Helmholtz equation given by

$$\Phi_k(x,y) = \frac{\mathrm{i}}{4} H_0^{(1)}(k|x-y|), \;\; d = 2, \quad \Phi_k(x,y) = \frac{\mathrm{e}^{\mathrm{i}k|x-y|}}{4\pi|x-y|}, \;\; d = 3$$

and let S_k, D_k, and D_k' be the single-layer, double-layer, and adjoint-double-layer operators defined by

$$\begin{aligned} S_k\phi(x) &:= \int_{\partial\mathcal{O}_+} \Phi_k(x,y)\phi(y)\,\mathrm{d}s(y), \\ D_k\phi(x) &:= \int_{\partial\mathcal{O}_+} \frac{\partial\Phi_k(x,y)}{\partial n(y)}\phi(y)\,\mathrm{d}s(y), \\ D_k'\phi(x) &:= \int_{\partial\mathcal{O}_+} \frac{\partial\Phi_k(x,y)}{\partial n(x)}\phi(y)\,\mathrm{d}s(y), \end{aligned}$$

for $\phi \in L^2(\partial\mathcal{O}_+)$ and $x \in \partial\mathcal{O}_+$. The standard second-kind combined-field "direct" formulation (that arises from Green's integral representation) and "indirect" formulation (that arises from an ansatz of layer potentials not related to Green's integral representation) are, respectively,

$$A'_{k,\eta}\partial_\nu u = f_{k,\eta} \quad \text{and} \quad A_{k,\eta}\phi = g, \tag{1.13}$$

where

$$A'_{k,\eta} := \frac{1}{2}I + D'_k - \mathrm{i}\eta S_k, \quad A_{k,\eta} := \frac{1}{2}I + D_k - \mathrm{i}\eta S_k, \tag{1.14}$$

where $\eta \in \mathbb{R} \setminus \{0\}$ is an arbitrary coupling parameter. In (1.13) the unknown $f_{k,\eta}$ is given in terms of the Dirichlet data g by, e.g., [25, eqs. 2.69 and 2.114], and in the indirect formulation the solution u can be recovered from the potential ϕ; see, e.g., [25, eq. 2.70].

The operators $A^{-1}_{k,\eta}$ and $(A'_{k,\eta})^{-1}$ can be expressed in terms of (i) the exterior Dirichlet-to-Neumann map and (ii) the interior impedance-to-Dirichlet map, see [25, theorem 2.33], and therefore bounds on $A^{-1}_{k,\eta}$ and $(A'_{k,\eta})^{-1}$ can be obtained from bounds on these maps [9, 27, 28, 109]. Inputting into [28, lemma 6.2] the bound on (i) from Corollary 1.2 and the bounds on (ii) from [9, cor. 1.9], [109, cor. 4.7], we obtain the following corollary. For simplicity, we only state bounds on the $L^2(\partial\mathcal{O}_+) \to L^2(\partial\mathcal{O}_+)$ norms of $A^{-1}_{k,\eta}$ and $(A'_{k,\eta})^{-1}$, but we highlight that bounds in the spaces $H^s(\partial\mathcal{O}_+)$ and $H^s_k(\partial\mathcal{O}_+)$ can also be obtained; see [28, lemma 6.2].

COROLLARY 1.4 (Bounds on $(A'_{k,\eta})^{-1}$ and $A^{-1}_{k,\eta}$ for most frequencies). *Let $\mathcal{O}_+$ be as in Theorem* 1.1.

(i) *Given $k_0 > 0$, $\delta > 0$, and $\varepsilon > 0$, there exists*

$$C'' = C''(k_0, \delta, \varepsilon, n, \mathcal{O}_-) > 0$$

and a set $J \subset [k_0, \infty)$ with $|J| \leq \delta$ such that, if $\eta = ck$, for some $c \in \mathbb{R} \setminus \{0\}$, then

$$\begin{aligned}\|(A'_{k,\eta})^{-1}\|_{L^2(\partial\mathcal{O}_+)\to L^2(\partial\mathcal{O}_+)} &= \|A^{-1}_{k,\eta}\|_{L^2(\partial\mathcal{O}_+)\to L^2(\partial\mathcal{O}_+)} \\ &\leq C'' k^{5n/2+1/2+\varepsilon}\end{aligned} \tag{1.15}$$

for all $k \in [k_0, \infty) \setminus J$.

(ii) *If the boundaries of the* (*finite number of*) *disjoint components of $\mathcal{O}_-$ are each piecewise smooth, then the exponent in* (1.15) *reduces to $5n/2 + 1/4 + \varepsilon$.*

(iii) *If* either *the components are star-shaped with respect to a ball* or *the boundaries are C^∞, then the exponent reduces to $5n/2 + \varepsilon$.*

The hp-BEM. To simplify the exposition, we focus on the Galerkin method applied to the direct equation $A'_{k,\eta}\partial_\nu u = f_{k,\eta}$, but everything below holds also for the indirect equation $A_{k,\eta}\phi = g$. Assume that $\partial\mathcal{O}_+$ is analytic, and that $\mathcal{T}_{h_{\mathrm{FEM}}}$ is a quasi-uniform triangulation with mesh size h of Γ in the sense of [76, def. 3.15]. Let $\mathcal{S}^p(\mathcal{T}_{h_{\mathrm{FEM}}})$ denote the space of continuous, piecewise polynomials of degree

$\leq p$ on the triangulation $\mathcal{T}_{h_{\mathrm{FEM}}}$. The hp-BEM then seeks $(\partial_\nu u)_{hp}$—an approximation of $\partial_\nu u$ in the subspace $\mathcal{S}^p(\mathcal{T}_{h_{\mathrm{FEM}}})$—as the solution of

$$\big(A'_{k,\eta}(\partial_\nu u)_{hp}, v_{hp}\big)_\Gamma = (f_{k,\eta}, v_{hp})_\Gamma \quad \text{for all } v_{hp} \in \mathcal{S}^p(\mathcal{T}_{h_{\mathrm{FEM}}}), \tag{1.16}$$

where $(\cdot,\cdot)_\Gamma$ denotes the inner product on $L^2(\Gamma)$.

Corollary 1.4 implies that the polynomial-boundedness assumption [76, eq. 3.24] in the analysis of the hp-BEM in [76] is satisfied for most frequencies, and [76, cor. 3.18] then implies the following:

COROLLARY 1.5 (k-independent quasi-optimality of the hp-BEM for most frequencies). *Let $\mathcal{O}_+$ be as in Theorem* 1.1 *and assume further that $\partial\mathcal{O}_+$ is analytic. Assume that $\eta = ck$, for some $c \in \mathbb{R} \setminus \{0\}$. Given $k_0 > 0$, $\delta > 0$, and $\varepsilon > 0$, there exists $C_j = C_j(k_0, \delta, \varepsilon, n, \mathcal{O}_-, c) > 0$, $j = 1, 2$, $C_3 = C_3(\mathcal{O}_-) > 0$, and a set $J \subset [k_0, \infty)$ with $|J| \leq \delta$ such that, if $k \geq k_0$ and* (1.11) *holds, then, for all $k \in [k_0, \infty) \setminus J$, the Galerkin solution $(\partial_\nu u)_{hp}$ defined by* (1.16) *exists, is unique, and satisfies the quasi-optimal error estimate*

$$\big\|(\partial_\nu u)_{hp} - v_{hp}\big\|_{L^2(\Gamma)} \leq C_3 \inf_{v_{hp}\in\mathcal{S}^p(\mathcal{T}_{h_{\mathrm{FEM}}})} \|\partial_\nu u - v_{hp}\|_{L^2(\Gamma)}.$$

The significance of the quasi-optimality results for the hp-BEM in [76] is that they show that the hp-BEM does not suffer from the pollution effect, in that the constant C_3 in (1.12) is independent of k, and h and p satisfying (1.11) can be chosen so that the total number of degrees of freedom grows like k^{n-1} (see [76, remark 3.19] for more details). Just as in the hp-FEM case, the resolvent estimate of Theorem 1.1 (via Corollary 1.4) now shows that this property is enjoyed even for strongly trapping obstacles, at least for most frequencies.

2 Recap of the Black-Box Scattering Framework

2.1 Abstract Framework

We now briefly recap the abstract framework of *black-box scattering* introduced in [107]; for more details, see the comprehensive presentation in [43, chap. 4].[2]

Let $\mathcal{H}$ be an Hilbert space with an orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_{R_0} \oplus L^2(\mathbb{R}^n \setminus B(0, R_0)),$$

and let P be a self-adjoint operator $\mathcal{H} \to \mathcal{H}$ with domain $\mathcal{D} \subset \mathcal{H}$ (so, in particular, $\mathcal{D}$ is dense in $\mathcal{H}$). We require that the operator P be $-\Delta$ outside $\mathcal{H}_{R_0}$ in the sense that

$$\mathbf{1}_{\mathbb{R}^n\setminus B(0,R_0)} P = -\Delta|_{\mathbb{R}^n\setminus B(0,R_0)}, \quad \mathbf{1}_{\mathbb{R}^n\setminus B(0,R_0)}\mathcal{D} \subset H^2(\mathbb{R}^n \setminus B(0, R_0)), \tag{2.1}$$

We further assume that

$$v \in H^2(\mathbb{R}^n),\ v|_{B(0,R_0+\varepsilon)} = 0 \quad \text{implies that} \quad v \in \mathcal{D},$$

[2] In this section, we recap the black-box framework for non-semiclassically-scaled operators, as in [116, §2]. We highlight that [43, chap. 4] deals with semiclassically-scaled operators, but transferring the results from [43, chap. 4] into the former setting is straightforward.

and that

$$\mathbf{1}_{B(0,R_0)}(P+\mathrm{i})^{-1} \text{ is compact from } \mathcal{H} \to \mathcal{H}. \tag{2.2}$$

Under these assumptions, the resolvent

$$R(k) = (P-k^2)^{-1} : \mathcal{H} \to \mathcal{D} \tag{2.3}$$

is meromorphic for $\operatorname{Im} k > 0$ and extends to a meromorphic family of operators of $\mathcal{H}_{\text{comp}} \to \mathcal{D}_{\text{loc}}$ in the whole complex plane when n is even and in the logarithmic plane when n is odd [43, theorem 4.4]. The poles of $(P-k^2)^{-1}$ are called the *resonances* of P, and we denote them by $\operatorname{Res} P$.

To study the resonances of P, we define a *reference operator* $P^\sharp$ associated to P but acting in a compact manifold: we glue our black box into a torus in place of $\mathbb{R}^n$. For a precise definition, see [43, §4.3], but we note here that $P^\sharp$ is defined in

$$\mathcal{H}^\sharp = \mathcal{H}_{R_0} \oplus L^2((\mathbb{R}/R_1\mathbb{Z})^n \setminus B(0,R_0)),\ R_1 > R_0,$$

and can be thought of as P in $\mathcal{H}_{R_0}$ and $-\Delta$ in $(\mathbb{R}/R_1\mathbb{Z})^n \setminus B(0,R_0)$. We assume that the eigenvalues of $P^\sharp$ satisfy the *polynomial growth of eigenvalues condition*

$$N(P^\#, [-C,\lambda]) = O(\lambda^{n^\#/2}), \tag{2.4}$$

where $n^\# \geq n$ and $N(P^\#, I)$ is the number of eigenvalues of $P^\sharp$ in the interval I, counted with their multiplicity.[3] When $n^\# = n$, the asymptotics (2.4) correspond to a Weyl-type upper bound, and thus (2.4) can be thought of as a weak Weyl law. One can then show that the resonances of P grow in the same way, that is,

$$N(P,r,\theta) \lesssim r^{n^\#} \tag{2.5}$$

where $N(P,r,\theta)$ is the number of resonances of P (counted with their multiplicity) in the sector $\{|z| \leq r, \arg z < \theta\}$, and the omitted constant in (2.5) depends on θ; see [107, 119, 120] and [43, theorem 4.13] for this result for resonances in the disc of radius r, and [106, text after eq. 2.10] and [116, eq. 2.1] for resonances in a sector.

In the proof of Theorem 1.1 (and its black-box analogue Theorem 3.3 below), it is convenient to work with the semiclassical operator $h^2 P$, where $h > 0$ is a small parameter. We define the *semiclassical resolvent*, $R(z,h)$, by

$$R(z,h) := (h^2 P - z)^{-1}, \tag{2.6}$$

and we let $\mathcal{R}$ be the set of the poles of the meromorphic continuation of $R(z,h)$, i.e., the semiclassical resonances. Observe that

$$z \in \mathcal{R}(h) \text{ implies } h^{-1} z^{1/2} \in \operatorname{Res} P \quad \text{and} \quad k \in \operatorname{Res} P \text{ implies } h^2 k^2 \in \mathcal{R}(h).$$

[3] Note that here we use the convention in [116] of counting eigenvalues in $[-C,\lambda]$ instead of using the convention of [43, eq. 4.3.10] of counting eigenvalues in $[-C,\lambda^2]$.

2.2 Scattering Problems Fitting in the Black-Box Framework

Scattering problems fitting in the black-box framework include scattering by impenetrable and penetrable obstacles, scattering by a compactly supported potential (i.e., $P = -\Delta + V$), scattering by elliptic, compactly supported perturbations of the Laplacian, and scattering on finite volume surfaces; see [43, §4.1].

Here we focus on scattering by impenetrable and penetrable obstacles. In the literature, these are usually placed in the black-box framework when the boundary of the obstacle is C^∞; here we show that obstacles with Lipschitz boundaries can also be put into this framework.

LEMMA 2.1 (Scattering by an impenetrable Dirichlet or Neumann Lipschitz obstacle in a black-box framework). *Let $\mathcal{O}_- \subset \mathbb{R}^n$, $n \geq 2$, be a bounded open set with Lipschitz boundary such that the open complement $\mathcal{O}_+ := \mathbb{R}^n \setminus \overline{\mathcal{O}_-}$ is connected and such that $\mathcal{O}_- \subset B(0, R_0)$. Let $A \in C^{0,1}(\mathcal{O}_+, \mathbb{R}^{d\times d})$ be such that* $\operatorname{supp}(I - A) \subset B(0, R_0)$*, A is symmetric, and there exists $A_{\min} > 0$ such that*

$$(A(x)\xi) \cdot \overline{\xi} \geq A_{\min}|\xi|^2 \quad \text{for almost every } x \in \mathcal{O}_+ \text{ and for all } \xi \in \mathbb{C}^d. \tag{2.7}$$

Let ν be the unit normal vector field on $\partial\mathcal{O}_-$ pointing from $\mathcal{O}_-$ into $\mathcal{O}_+$, and let $\partial_{\nu,A}$ denote the corresponding conormal derivative defined by, e.g., [78, lemma 4.3] (*recall that this is such that, when $v \in H^2(\mathcal{O}_+)$, $\partial_{\nu,A} v = \nu \cdot \gamma(A\nabla v)$*). *Then the operator $Pv := -\nabla \cdot (A\nabla v)$ with either one of the domains*

$$\mathcal{D}_D := \left\{v \in H^1(\mathcal{O}_+),\ \nabla \cdot (A\nabla v) \in L^2(\mathcal{O}_+),\ \gamma v = 0\right\}$$

or

$$\mathcal{D}_N := \left\{v \in H^1(\mathcal{O}_+),\ \nabla \cdot (A\nabla v) \in L^2(\mathcal{O}_+),\ \partial_{\nu,A} v = 0\right\}$$

fits into the black-box framework with

$$\mathcal{H} = L^2(\mathcal{O}_+) \quad \text{and} \quad \mathcal{H}_{R_0} = L^2(B(0, R_0) \cap \mathcal{O}_+).$$

Furthermore, the corresponding reference operator $P^\#$ (defined precisely in [43, §4.3]) *satisfies* (2.4) *with $n^\# = n$.*

PROOF. Since C^∞ functions with compact support are both dense in $L^2(\mathcal{O}_+)$ and contained in $\mathcal{D}_D$ and $\mathcal{D}_N$ when A is Lipschitz, $\mathcal{D}_D$ and $\mathcal{D}_N$ are both dense in $\mathcal{H}$. The definitions of $\mathcal{D}_{D/N}$ imply that P is self-adjoint; the definitions of $\mathcal{D}_{D/N}$ and $\mathcal{H}$ imply that $P : \mathcal{D}_{D/N} \to \mathcal{H}$ and that the resolvent $R : \mathcal{H} \to \mathcal{D}_{D/N}$. The operator P is then self-adjoint by Green's second identity (valid in Lipschitz domains by, e.g., [78, theorem 4.4(iii)]). The first condition in (2.1) is satisfied since $\operatorname{supp}(I - A) \subset B(0, R_0)$, and the second condition in (2.1) is satisfied due to interior regularity of the Laplacian (see, e.g., [78, theorem 4.16]). The condition (2.2) follows from the compact embedding of $H^1(B(0, R_0) \cap \mathcal{O}_+)$ in $L^2(B(0, R_0) \cap \mathcal{O}_+)$; see, e.g., [78, theorem 3.27]. The polynomial growth of eigenvalues condition (2.4) follows from results about heat kernel asymptotics from [92]; see Lemma B.1.

Note that in [43, chap. 4] (our default reference for the black-box framework), the (semiclassically scaled) norm defined by $\|u\|_{\mathcal{D}}^2 := \|u\|_{\mathcal{H}}^2 + h^4\|Pu\|_{\mathcal{H}}^2$ is placed on $\mathcal{D}$; in our setting this would correspond to the norm squared being $\|u\|_{L^2}^2 + h^4\|\nabla\cdot(A\nabla u)\|_{L^2}^2$. However, the results in [43] also hold with the norm squared being $\|u\|_{L^2}^2 + h^2\|\nabla u\|_{L^2}^2 + h^4\|\nabla\cdot(A\nabla u)\|_{L^2}^2$. Indeed, the only place the form of the norm on $\mathcal{D}$ is used in [43] is in the bounds of [43, lemma 4.3], which are used in the proof of meromorphic continuation of the resolvent [43, theorem 4.4]. However, the bounds in [43, lemma 4.3] hold also (at least in this obstacle setting) with the norm squared being $\|u\|_{L^2}^2+h^2\|\nabla u\|_{L^2}^2+h^4\|\nabla\cdot(A\nabla u)\|_{L^2}^2$, since control of the ∇u term follows from control of u and Δu via, e.g., Green's identity. $\square$

Remark 2.2 (Exterior Dirichlet or Neumann scattering problem). With P and A as in Lemma 2.1, given $f \in L^2(\mathcal{O}_+)$ with compact support and $k > 0$, $u := R(k)f$ satisfies either one of the boundary-value problems: $u \in H^1_{\mathrm{loc}}(\mathcal{O}_+)$,

$$\nabla\cdot(A\nabla u) + k^2 u = -f \quad \text{in } \mathcal{O}_+, \quad \text{either } \gamma u = 0 \text{ or } \partial_\nu u = 0 \text{ on } \partial\mathcal{O}_+,$$

and the radiation condition (1.2) at infinity.

LEMMA 2.3 (Scattering by an penetrable Lipschitz obstacle in a black-box framework). *Let $\mathcal{O}_-$ be as in Lemma* 2.1. *Let $A \in C^{0,1}(\mathbb{R}^d, \mathbb{R}^{d\times d})$ be such that* $\operatorname{supp}(I - A) \subset B(0, R_0)$, *$A$ is symmetric, and there exists $A_{\min} > 0$ such that* (2.7) *holds (with $\mathcal{O}_+$ replaced by $\mathbb{R}^d$). Let ν be the unit normal vector field on $\partial\mathcal{O}_-$ pointing from $\mathcal{O}_-$ into $\mathcal{O}_+$, and let $\partial_{\nu,A}$ be the corresponding conormal derivative from either $\mathcal{O}_-$ or $\mathcal{O}_+$. For D an open set, let $H^1(D, \nabla\cdot(A\nabla\cdot)) := \{v : v \in H^1(D), \nabla\cdot(A\nabla v) \in L^2(D)\}$. Let $c, \alpha > 0$ and set*

$$\mathcal{H}_{R_0} = L^2(\mathcal{O}, c^{-2}\alpha^{-1}\mathrm{d}x) \oplus L^2(B(0, R_0) \setminus \overline{\mathcal{O}}), \tag{2.8}$$

so that

$$\mathcal{H} = L^2\big(\mathcal{O}; c^{-2}\alpha^{-1}\mathrm{d}x\big) \oplus L^2\big(B(0, R_0) \setminus \overline{\mathcal{O}}\big) \oplus L^2\big(\mathbb{R}^n \setminus B(0, R_0)\big).$$

Let

$$\begin{aligned}\mathcal{D} := \big\{v = (v_1, v_2, v_3) \text{ where } &v_1 \in H^1(\mathcal{O}_-, \nabla\cdot(A\nabla\cdot)),\\ &v_2 \in H^1(B(0, R_0) \setminus \overline{\mathcal{O}_-}, \nabla\cdot(A\nabla\cdot)),\\ &v_3 \in H^1(\mathbb{R}^n \setminus \overline{B(0, R_0)}, \Delta),\\ &\gamma v_1 = \gamma v_2 \text{ and } \partial_{\nu,A} v_1 = \alpha\, \partial_{\nu,A} v_2 \text{ on } \partial\mathcal{O}_-,\\ &\gamma v_2 = \gamma v_3 \text{ and } \partial_{\nu,A} v_2 = \partial_{\nu,A} v_3 \text{ on } \partial B(0, R_0)\big\}\end{aligned} \tag{2.9}$$

(*observe that the conditions on v_2 and v_3 on $\partial B(0, R_0)$ in the definition of $\mathcal{D}$ are such that $(v_2, v_3) \in H^1(\mathbb{R}^n \setminus \overline{\mathcal{O}_-}, \nabla\cdot(A\nabla\cdot))$). Then the operator*

$$Pv := -\big(c^2\nabla\cdot(A\nabla v_1), \nabla\cdot(A\nabla v_2), \Delta v_3\big),$$

defined for $v = (v_1, v_2, v_3)$, fits in the the black-box framework, and the the corresponding reference operator $P^\#$ (defined precisely in [43, §4.3]) *satisfies* (2.4) *with $n^\# = n$.*

PROOF. The domain $\mathcal{D}$ contains C^∞ functions that are zero in a neighbourhood of $\partial\mathcal{O}_-$, and these are dense in $L^2(\mathbb{R}^n)$. The scalings in the measure imposed on $\mathcal{O}_-$ in (2.8) imply that P is self-adjoint by Green's identity. The conditions (2.1) and (2.2) are satisfied by the same arguments in Lemma 2.1. The proof that the corresponding reference operator $P^\#$ satisfies (2.4) with $n^\# = n$ is given in Lemma B.2. The remarks in the proof of Lemma 2.1 about the norm applied on $\mathcal{D}$ in [43, chap. 4] also apply here. □

Remark 2.4 (Scattering by a penetrable obstacle (aka the transmission problem)). With $\mathcal{O}_-$, A, and P as in Lemma 2.3, given $f \in L^2(\mathbb{R}^n)$ with compact support and $c, \alpha, k > 0$, and let $u := R(k)f$. Then, with the notation $u_{\mathrm{in}} = u|_{\mathcal{O}_-}$ ($= u_1$ in the notation of Lemma 2.1) and $u_{\mathrm{out}} = u|_{\mathcal{O}_+}$ ($= (u_2, u_3)$), u satisfies the boundary-value problem: $u \in H^1_{\mathrm{loc}}(\mathbb{R}^n \setminus \partial\mathcal{O}_-)$,

$$\begin{aligned} &\nabla\cdot(A\nabla u_{\mathrm{in}}) + \frac{k^2}{c^2}u_{\mathrm{in}} = -f \quad \text{in } \mathcal{O}_-, \\ &\nabla\cdot(A\nabla u_{\mathrm{out}}) + k^2 u_{\mathrm{out}} = -f \quad \text{in } \mathcal{O}_+, \end{aligned} \tag{2.10}$$

$$\gamma u_{\mathrm{in}} = \gamma u_{\mathrm{out}} \quad \text{and} \quad \partial_{\nu,A} u_{\mathrm{in}} = \alpha\,\partial_{\nu,A} u_{\mathrm{out}} \quad \text{on } \partial\mathcal{O}_-, \tag{2.11}$$

$$u_{\mathrm{out}} \text{ satisfies the radiation condition (1.2).} \tag{2.12}$$

By rescaling, any other transmission problem with constant real coefficients can be written as (2.10)–(2.12); see [88, def. 2.3 and following paragraph].

Remark 2.5 (Trapping by penetrable obstacles). When $c < 1$ and $\partial\mathcal{O}_-$ is C^∞ with strictly positive curvature, then the boundary-value problem (2.10)-(2.12) is trapping; see [19, 20, 97, 112] and [88, §6].

3 Polynomial Resolvent Estimates Away from "Bad Frequencies" (Including the Proof of Theorem 1.1)

For completeness, we state the two main ingredients of our proofs, namely (i) the semiclassical maximum principle of [115, lemma 2] and [116, lemma 4.2] (see also [43, lemma 7.7]), and (ii) exponential resolvent bounds away from resonances from [115, lemma 1], [116, prop. 4.3] (see also [43, theorem 7.5]).

THEOREM 3.1 (Semiclassical maximum principle [115, 116]). *Let $\mathcal{H}$ be an Hilbert space and $z \mapsto Q(z,h) \in \mathcal{L}(\mathcal{H})$ an holomorphic family of operators in a neighbourhood of*

$$\Omega(h) := \big(w - 2a(h), w + 2a(h)\big) + i\big(-\delta(h)h^{-L}, \delta(h)\big),$$

where

$$0 < \delta(h) < 1 \quad \textit{and} \quad a(h)^2 \geq Ch^{-3L}\delta(h)^2 \tag{3.1}$$

for some $L > 0$ *and* $C > 0$. *Suppose that*

$$\|Q(z,h)\|_{\mathcal{H}\to\mathcal{H}} \le \exp(Ch^{-L}), \quad z \in \Omega, \tag{3.2}$$

$$\|Q(z,h)\|_{\mathcal{H}\to\mathcal{H}} \le \frac{1}{\operatorname{Im} z}, \quad \operatorname{Im} z > 0, \;\; z \in \Omega. \tag{3.3}$$

Then

$$\|Q(z,h)\|_{\mathcal{H}\to\mathcal{H}} \le \delta(h)^{-1}\exp(C+1) \quad \textit{for all } z \in [w-a(h), w+a(h)]. \tag{3.4}$$

REFERENCES FOR PROOF. Let $f, g \in \mathcal{H}$ with $\|f\|_{\mathcal{H}} = \|g\|_{\mathcal{H}} = 1$, and let

$$F(z,h) := \langle Q(z+w,h)g, f\rangle_{\mathcal{H}}.$$

The result (3.4) follows from the "three-line theorem in a rectangle" (a consequence of the maximum principle) stated as [43, lemma D.1] applied to the holomorphic family $(F(\cdot,h))_{0<h\ll 1}$ with

$$\begin{aligned} R &= 2a(h), \quad \delta_+ = \delta(h), \quad \delta_- = \delta(h)h^{-L}, \\ M &= M_- = \exp(Ch^{-L}), \quad M_+ = \delta(h)^{-1}. \end{aligned}$$

□

THEOREM 3.2 (Bounds on the resolvent away from resonances [115, 116]). *Let* P *satisfy the assumptions in* §2 *and let* $n^\#$ *be the exponent in the condition* (2.4)*, and let* $\Omega \Subset \{\operatorname{Re} z > 0\}$ *be a precompact neighbourhood of* $E \in \mathbb{R}^+$. *Let* $h \mapsto g(h)$ *be a positive function. Then there exist* $h_0 > 0$ *and* $C_1 > 0$ (*both depending on* Ω) *such that, for* $0 < h < h_0$, *the resolvent* (2.6) *satisfies*

$$\begin{aligned} &\|\chi R(z,h)\chi\|_{\mathcal{H}\to\mathcal{H}} \le C_1 \exp\left(C_1 h^{-n^\#}\log\left(\frac{1}{g(h)}\right)\right) \\ &\qquad \textit{for all } z \in \Omega \setminus \bigcup_{z_j\in\mathcal{R}_P} D(z_j, g(h)) \end{aligned} \tag{3.5}$$

(*where* $D(z_j, g(h))$ *is the open disc of radius* $g(h)$ *centred at* $z_j \in \mathbb{C}$).

The significance of Theorem 3.2 is that it provides one of the two bounds needed to apply the semiclassical maximum principle to the resolvent $R(z,h)$, namely (3.2). The second bound, (3.3), is given by the following bound on the resolvent, valid when P satisfies the assumptions in §2, and $\operatorname{Im} z > 0$,

$$\|R(z,h)\|_{\mathcal{H}\to\mathcal{H}} \le \frac{1}{\operatorname{Im} z}. \tag{3.6}$$

(A simple proof of this is to take the inner product of the equation $(h^2P - z)u = f$ with u and use the self-adjointness of P.)

THEOREM 3.3 (Black-box analogue of Theorem 1.1). *Let* P *satisfy the assumptions in* §2 *and let* $n^\#$ *be the exponent in the condition* (2.4). *Then, given* $k_0 > 0$, $\delta > 0$, *and* $\varepsilon > 0$, *there exists a* $C = C(k_0, \delta, \varepsilon, n^\#) > 0$ *and a set* J *with* $|J| \le \delta$ *such that the resolvent* (2.3) *satisfies*

$$\|\chi R(k)\chi\|_{\mathcal{H}\to\mathcal{H}} \le Ck^{5n^\#/2+\varepsilon} \quad \textit{for all } k \in [k_0,\infty)\setminus J. \tag{3.7}$$

PROOF. Let $\Omega \Subset \{\mathrm{Re}\, z > 0\}$ be a neighbourhood of $E \in \mathbb{R}^+$ such that $\Omega \cap \mathbb{R} = (E/2, 2E)$ and $(E/2, 2E) + \mathrm{i}[-1, 1] \subset \Omega$. Moreover, let $m > 0$, to be fixed later. Let $I_1, \dots, I_{N(h)}$ be a partition of $(E/2, 2E)$ into intervals, i.e.,

$$(E/2, 2E) = \bigcup_{j=1\dots N(h)} I_j, \tag{3.8}$$

with $|I_j| = 10C_{\mathrm{w}}h^m$ for $j = 1, \dots, N(h) - 1$ and $|I_N| \le 10C_{\mathrm{w}}h^m$, where C_{w} will be chosen later (the subscript w in C_{w} emphasises that this constant dictates the width of the intervals in the partition of $(E/2, 2E)$). Let

$$J'(h) := \bigcup_{(I_j + \mathrm{i}[-1,1]) \cap \mathcal{R} \neq \varnothing} I_j. \tag{3.9}$$

The set $J'(h)$ can be written as a disjoint union

$$J'(h) = [a_i, b_i] \cap \Omega \tag{3.10}$$

(where the intersection with Ω is taken to ensure that $J'(h) \subset (E/2, 2E)$, as implied by its definition (3.9)). Let

$$J''(h) := \bigcup [a_i - 3C_{\mathrm{w}}h^m, b_i + 3C_{\mathrm{w}}h^m]. \tag{3.11}$$

This setup implies that every point of $(E/2, 2E) \setminus J''(h)$ has a neighbourhood of the form

$$[w - 2C_{\mathrm{w}}h^m, w + 2C_{\mathrm{w}}h^m] + \mathrm{i}[-1/2, 1/2]$$

that is disjoint from

$$\bigcup_{z \in \mathcal{R}} D(z, C_{\mathrm{w}}h^m).$$

Theorem 3.2 therefore implies that in these neighbourhoods the semiclassical resolvent $R(w, h)$ satisfies

$$\begin{aligned}\|\chi R(w,h)\chi\|_{\mathcal{H}\to\mathcal{H}} &\le C_1 \exp\biggl(C_1 h^{-n^\#} \log\biggl(\frac{1}{C_{\mathrm{w}}h^m}\biggr)\biggr),\\ &= C_1 \exp\biggl(C_1 h^{-n^\#}\biggl[\log\biggl(\frac{1}{C_{\mathrm{w}}}\biggr) + m\log\biggl(\frac{1}{h}\biggr)\biggr]\biggr),\end{aligned}$$

for all $0 < h < h_0$, where h_0 and C_1 are given in (3.5) and depend on Ω, and hence on E. Therefore, given $\eta > 0$, by choosing $h_1 = h_1(h_0, \eta, C_{\mathrm{w}}, m)$ sufficiently small,

$$\|\chi R(w,h)\chi\|_{\mathcal{H}\to\mathcal{H}} \le C_1 \exp\bigl(C_1 m h^{-(n^\# + \eta)}\bigr) \quad \text{for all } 0 < h < h_1.$$

Since the resolvent also satisfies the bound (3.6), we can apply Theorem 3.1 (the semiclassical maximum principle) with $Q = \chi R\chi/C_1$, $C = C_1 m$, $a(h) = C_{\mathrm{w}}h^m$, $L = n^\# + \eta$ with $\eta > 0$ arbitrarily small, and the largest possible $\delta(h)$ permitted by (3.1), namely

$$\delta(h) = ch^{m+3L/2} = ch^{m+3n^\#/2+3\eta/2}$$

where $c > 0$ is sufficiently small (depending on m and C_{w}); the result is that there exists a $C_2 > 0$ (depending on C_1, m, and C_{w}) such that

$$\|\chi R(w,h)\chi\|_{\mathcal{H}\to\mathcal{H}} \le C_2 h^{-(m+3n^\#/2+3\eta/2)} \quad \text{for all } w \in (E/2,2E)\setminus J''(h) \tag{3.12}$$

and for all $0 < h < h_1$. Observe that, at the price of making C_2 bigger, we can set $h_1 = 1$. More precisely, (3.12) and the fact that $\|\chi R(w,h)\chi\|_{\mathcal{H}\to\mathcal{H}}$ is bounded for all $h > 0$ imply that there exists $C_3 > 0$ (depending on C_1, m, C_{w}, and h_1, and thus on $C_1, m, C_{\mathrm{w}}, h_0$, and η) such that

$$\|\chi R(w,h)\chi\|_{\mathcal{H}\to\mathcal{H}} \le C_3 h^{-(m+3n^\#/2+3\eta/2)} \quad \text{for all } w \in (E/2,2E)\setminus J''(h) \tag{3.13}$$

and for all $0 < h \le 1$.

We now need to estimate the size of $J''(h)$. For $z \in (E/2,2E) + \mathrm{i}[-1,1]$, $h^{-1}z^{1/2}$ is contained in a ball of radius proportional to h^{-1} in an angular sector with angle independent of h. Therefore, by the bound (2.5) on the number of resonances of P, there exists $C^\# > 0$ such that

$$\operatorname{card}(\Omega \cap \mathcal{R}) \le C^\# h^{-n^\#}, \tag{3.14}$$

and so we also have that $\operatorname{card}\{j,\ (I_j + \mathrm{i}[-1,1]) \cap \mathcal{R} \neq \varnothing\} \le C^\# h^{-n^\#}$. The measure of $J''(h)$ is bounded by the number of intervals in the definition (3.11) multiplied by the width of the intervals, and thus

$$|J''(h)| \le C^\# h^{-n^\#} \times 6C_{\mathrm{w}} h^m = 6C^\# C_{\mathrm{w}} h^{m-n^\#}. \tag{3.15}$$

The plan for the rest of the proof is to obtain the bound (3.7) on the nonsemiclassical resolvent $\chi(P-k^2)^{-1}\chi$ for $k \in [k_0,\infty)$ (i.e., $k^2 \in [k_0^2,\infty)$) by taking $E = k_0^2$, writing

$$[k_0^2,\infty) = \bigcup_{\ell=0}^{\infty} [2^\ell E, 2^{\ell+1}E),$$

applying the resolvent estimate (3.13) in each interval, choosing m so that the union of the excluded sets has finite measure, and finally choosing C_{w} so that this measure is bounded by δ. Indeed, if $k^2 \in [2^\ell E, 2^{\ell+1}E)$, then $2^{-\ell}k^2 \in [E,2E) \subset (E/2,2E)$. We now apply the estimate (3.12) with $h = 2^{-\ell/2}$ and $w = h^2k^2$; observe that the smallest ℓ, namely $\ell = 0$, corresponds to $h = 1$, i.e., the largest h for which the estimate (3.13) is valid. The result is that

$$\|\chi(P-k^2)^{-1}\chi\|_{\mathcal{H}\to\mathcal{H}} = h^2\|\chi(h^2P-h^2k^2)^{-1}\chi\|_{\mathcal{H}\to\mathcal{H}} \le C_3 h^2 h^{-(m+3n^\#/2+3\eta/2)} \le \tag{3.16}$$

$$\leq C_3\left(\frac{k}{\sqrt{E}}\right)^{(-2+m+3n^\#/2+3\eta/2)}$$
$$\leq Ck^{(-2+m+3n^\#/2+3\eta/2)},$$

for all $h^2k^2 \in (E/2, 2E)\setminus J''(h)$, and in particular for all $k^2 \in [2^\ell E, 2^{\ell+1}E)\setminus \widetilde{J}_\ell$, where

$$\widetilde{J}_\ell := \left\{z \in [2^\ell E, 2^{\ell+1}E) : 2^{-\ell}z \in J''(2^{-\ell/2})\right\}.$$

The bound (3.16) will become the bound (3.7) in the result (after m is specified). Observe that the constant C in (3.16) depends on $C_3, E, m, n^\#$, and η; tracking through the dependencies of C_3 (described above), and using the fact that $E = k_0^2$, we find that C depends on $k_0, m, n^\#, \eta, C_{\mathrm{w}}, C_1$, and h_0.

We then set

$$\widetilde{J} := \bigcup_{\ell=0}^{\infty} \widetilde{J}_\ell, \tag{3.17}$$

so that the bound (3.16) holds for $k^2 \in [k_0^2, \infty) \setminus \widetilde{J}$. We now choose m so that $\widetilde{J}$ has finite measure; indeed, by (3.15),

$$\left|\widetilde{J}_\ell\right| \leq 2^\ell 6C^\# C_{\mathrm{w}} 2^{-\ell(m-n^\#)/2} = 6C^\# C_{\mathrm{w}} 2^{-\ell(m-n^\#-2)/2}. \tag{3.18}$$

Taking

$$m = n^\# + 2 + \widetilde{\varepsilon}, \tag{3.19}$$

and using (3.17) and (3.18) yields

$$|\widetilde{J}| \leq 6C^\# C_{\mathrm{w}} \sum_{\ell=0}^{\infty} 2^{-\ell\widetilde{\varepsilon}/2} = 6C^\# C_{\mathrm{w}} \frac{1}{1-2^{-\widetilde{\varepsilon}/2}}, \tag{3.20}$$

and so $|\widetilde{J}| < \infty$ for every $\widetilde{\varepsilon} > 0$. We now use the freedom we have in choosing C_{w} to make $|\widetilde{J}|$ arbitrarily small: given $\delta' > 0$ and $\widetilde{\varepsilon} > 0$, let

$$C_{\mathrm{w}} := \frac{\delta'(1-2^{-\widetilde{\varepsilon}/2})}{6C^\#},$$

so that $|\widetilde{J}| \leq \delta'$ by (3.20). We now define J so that

$$k \in [k_0, \infty) \setminus J \quad \text{if and only if} \quad k^2 \in [k_0^2, \infty) \setminus \widetilde{J}. \tag{3.21}$$

Since $|J| \leq |\widetilde{J}|/(2k_0)$ given $\delta > 0$, let $\delta' := 2\delta k_0$, so that $|J| \leq \delta$. We have therefore proved that the bound (3.16) holds with m given by (3.19) for all $k \in [k_0, \infty) \setminus J$. The bound (3.7) then follows from (3.16) with $\varepsilon := 3\eta/2 + \widetilde{\varepsilon}$. The constant C in (3.7) depends on $k_0, n^\#, \delta, \varepsilon, C^\#, C_1$, and h_0, where C_1 and h_0 are defined in Theorem 3.2 and depend on k_0, and $C^\#$ is defined in (3.14) and arises from the bound (2.5) on the number of resonances. $\square$

Remark 3.4 (Multiplicities). In (3.14) we are concerned with the *distinct locations* of resonances in Ω, while the bound (3.7) is unaffected by their multiplicity. If we assume that the multiplicity of all but finitely many resonances is proportional to k^{ρ}, the number of distinct locations is reduced, and the bound (3.14) is replaced by $\mathrm{card}(\Omega \cap \mathcal{R}) \lesssim h^{-n^{\#}+\rho}$; one can then take $m = n^{\#} + 2 + \widetilde{\varepsilon} - \rho$, and the bound (3.7) is improved by a factor of $k^{-\rho}$. Although such multiplicity assumptions are highly nongeneric—see, e.g., [43, theorem 4.39]—they hold, however, in certain situations with a high degree of symmetry; see Corollary 3.8 below for an example.

THEOREM 3.5 (Improvement of Theorem 3.3 under stronger assumption on location of resonances). *Assume that, given $c_j > 0$, $j = 1, 2$, the number of resonances of P in the box*

$$[r, r + c_1 r^{-1}] + \mathrm{i}[-c_2, c_2] \tag{3.22}$$

is $\lesssim r^p$ for some $p > 0$ and for all $r > 0$.

(i) *Given $\varepsilon > 0$, $\delta > 0$, and*

$$s \geq -\frac{1}{2} + \frac{n^{\#}}{2} + \frac{4\varepsilon}{5}, \tag{3.23}$$

there exists $\lambda_0 = \lambda_0(\varepsilon, \delta, s, n^{\#}) > 1$ such that

$$\left|\{w, \|\chi(P - w)^{-1}\chi\|_{\mathcal{H}\to\mathcal{H}} > \lambda^s\} \cap [\lambda, \lambda + 1]\right| \leq \delta\lambda^{-s-1+3n^{\#}/4+p/2+\varepsilon} \tag{3.24}$$

for all $\lambda > \lambda_0$.

(ii) *Given $k_0 > 0$, $\delta > 0$, and $\varepsilon > 0$, there exists a constant $C(k_0, \delta, \varepsilon, n^{\#}) > 0$ and a set J with $|J| \leq \delta$ such that the resolvent* (2.3) *satisfies*

$$\|\chi R(k)\chi\|_{\mathcal{H}\to\mathcal{H}} \leq Ck^{3n^{\#}/2+p+\varepsilon} \quad \text{for all } k \in [k_0, \infty) \setminus J.$$

Before proving Theorem 3.5, we note that two situations where the hypotheses of Theorem 3.5 on the number of resonances can be verified are: (a) where one has a sharp Weyl remainder in the asymptotics of the eigenvalue counting function for the reference operator $P^{\#}$—see Corollaries 3.6 and 3.7 below, and (b) where one has Weyl-type asymptotics for the counting function of the resonances of P—see Corollary 3.8 below for the specific case of a penetrable obstacle. We highlight that the hypotheses can be verified in (a) thanks to the results of [12, 95, 108] (see Corollary 3.6 below for more detail). We also note that, in both cases (a) and (b), the number of resonances in $[r, r + c_1 r^{-1}] + \mathrm{i}[-c_2, c_2]$ is estimated by the number in $[r, r + c_1] + \mathrm{i}[-c_2, c_2]$; although we see below that the former set arises naturally in the proof of Theorem 3.5, rigorous results about resonance distribution on the r^{-1} scale seem well out of reach of current methods.

PROOF OF THEOREM 3.5.

Proof of Part (i). We argue as in Theorem 3.3 except that now we work in an interval of size h^2 instead of $3E/2$ and choose the intervals comprising J' to have smaller imaginary parts. Indeed, let $\Omega \Subset \{\mathrm{Re}\, z > 0\}$ be such that $\Omega \cap \mathbb{R} =$

$(1, 1+h^2)$ and $(1, 1+h^2) + \mathrm{i}[-h, h] \subset \Omega$. Let $I_1, \ldots, I_{N(h)}$ be a partition of $(1, 1+h^2)$ into intervals, i.e.,

$$(1, 1+h^2) = \bigcup_{j=1\ldots N(h)} I_j$$

(compare to (3.8)) with $|I_j| = 10C_{\mathrm{w}}h^m$ for $j = 1, \ldots, N(h)-1$ and $|I_N| \leq 10C_{\mathrm{w}}h^m$, where $m > 0$ and $C_{\mathrm{w}} > 0$ will be chosen later. Let

$$J'(h) := \bigcup_{(I_j + \mathrm{i}[-h,h]) \cap \mathcal{R} \neq \varnothing} I_j.$$

With $J'(h)$ written as (3.10), let $J''(h)$ be defined by (3.11). As in the proof of Theorem 3.3, every point of $(\Omega \cap \mathbb{R}) \setminus J''(h)$ has a neighbourhood of the form

$$[w - 2C_{\mathrm{w}}h^m, w + 2C_{\mathrm{w}}h^m] + \mathrm{i}[-h/2, h/2]$$

that is disjoint from

$$\bigcup_{z \in \mathcal{R}} D(z, C_{\mathrm{w}}h^m),$$

and thus where Theorem 3.2 implies that the semiclassical resolvent $R(w, h)$ satisfies

$$\|\chi R(w,h)\chi\|_{\mathcal{H}\to\mathcal{H}} \leq C_1 \exp\left(C_1 h^{-n^\#}\left[\log\left(\frac{1}{C_{\mathrm{w}}}\right) + m\log\left(\frac{1}{h}\right)\right]\right)$$

for all $0 < h < h_0$, where h_0 and C_1 are given in (3.5) and depend on Ω. Arguing as in the proof of Theorem 3.3, we find that, given $\eta > 0$, by choosing $h_1 = h_1(h_0, \eta, C_{\mathrm{w}}, m)$ sufficiently small,

$$\left\|\chi R(w,h)\chi\right\|_{\mathcal{H}\to\mathcal{H}} \leq C_1 \exp(C_1 m h^{-(n^\# + \eta)}) \quad \text{for all } 0 < h \leq h_1.$$

We now use the semiclassical maximum principle, Theorem 3.1, with $C = C_1 m$, $Q = \chi R \chi / C_1$, $a(h) = C_{\mathrm{w}}h^m$, $L = n^\# + \eta$ with $\eta > 0$ arbitrarily small, and the largest possible $\delta(h)$ permitted by (3.1), namely

$$\delta(h) = ch^{m+3L/2} = ch^{m+3n^\#/2+3\eta/2}, \tag{3.25}$$

where $c \leq C_{\mathrm{w}}(C_1 m)^{-1/2}$. Note that, to apply the semiclassical maximum principle, we need $(-\delta(h)h^{-L}, \delta(h)) \subset (-h/2, h/2)$. Therefore, we assume, and check later, that with our choice of c and m,

$$\delta(h) \leq \frac{1}{2}h^{1+L}. \tag{3.26}$$

The result is that there exists $C_2 > 0$ such that

$$\begin{gathered}\|\chi R(w,h)\chi\|_{\mathcal{H}\to\mathcal{H}} \leq C_2 h^{-(m+3n^\#/2+3\eta/2)} \\ \text{for all } w \in (1, 1+h^2) \setminus J''(h)\end{gathered} \tag{3.27}$$

and for all $0 < h \leq h_1$. Just as in the proof of Theorem 3.3, at the price of making C_2 bigger, we can assume that $h_1 = 1$. Observe that, by choosing c sufficiently small in the definition of $\delta(h)$ (3.25), the condition (3.26) is satisfied when

$$h^{m+3n^\#/2+3\eta/2-1} \lesssim h^L \tag{3.28}$$

for h sufficiently small.

As in Theorem 3.3, we bound $|J''(h)|$ by the number of intervals multiplied by their widths. As before, the widths are bounded by $6C_{\rm w}h^m$, but now the number of intervals—corresponding to the number of semiclassical resonances in $[1, 1+h^2]+\mathrm{i}[-h, h]$—is bounded by $C_1^\# h^{-p}$, where $C_1^\#$ depends only on P. Indeed, by Lemma A.1, the image of the box $[1, 1+h^2]+\mathrm{i}[-h, h]$ under the scaling $z \to h^{-1}z^{1/2} = k$ is included in a box of the form $[h^{-1}, h^{-1}(1+c_1h^2)]+\mathrm{i}[-c_2, c_2]$ for some $c_j > 0$, $j = 1, 2$, independent of h, and by the assumption in the theorem, the number of resonances of P in this latter box is bounded, up to a multiplicative constant, which we denote by $C_1^\#$, by h^{-p}. Therefore,

$$|J''(h)| \leq C_1^\# h^{-p} \times 6C_{\rm w}h^m = 6C_{\rm w}C_1^\# h^{m-p}. \tag{3.29}$$

Having obtained the bound (3.27), we now seek an upper bound on the measure of the set where $\|\chi R(w, h)\chi\|_{\mathcal{H}\to\mathcal{H}} > h^{-t} =: B(h)$. The choice of t here will dictate our choice of m (and hence the measure of the set via (3.29)). Observe that

$$C_2 h^{-(m+3n^\#/2+3\eta/2)} \leq h^{-t}$$

if and only if

$$m \leq t + \frac{\log(1/C_2)}{\log(1/h)} - \frac{3n^\#}{2} - \frac{3\eta}{2}. \tag{3.30}$$

Since C_2 is independent of h, there exists an $h_2 > 0$ such that the inequality (3.30) holds when

$$m = t - \eta - 3n^\#/2 - 3\eta/2 \tag{3.31}$$

and $0 < h \leq h_2$. Note that h_2 depends on C_2 and on the choice of m, and hence on $n^\#$, η, and $C_{\rm w}$.

Observe that with the choice of m (3.31), we see that the inequality (3.28) holds, in particular, when

$$t \geq 1 + L + \eta. \tag{3.32}$$

We now input the information about m into our bound on the measure of the set $J''(h)$. Indeed, from (3.27) and our choice of m (3.31), for $0 < h \leq h_2$ and $w \in (1, 1 + h^2)$,

$$\|\chi R(w, h)\chi\|_{\mathcal{H}\to\mathcal{H}} > B(h) \quad \text{implies that} \quad w \in J''(h).$$

Therefore, choosing C_{w} small enough so that $6C_{\mathrm{w}}C_1^{\#} \leq \delta$, we get, by (3.29), for $0 < h \leq h_2$,

$$\begin{aligned} &\left|\{\|\chi R(w,h)\chi\|_{\mathcal{H}\to\mathcal{H}} > B(h)\} \cap [1, 1+h^2]\right| \\ &\quad \leq |J''(h)| \leq 6C_{\mathrm{w}}C_1^{\#}h^{m-p} \leq \delta h^{t-3n^{\#}/2-p-5\eta/2}. \end{aligned} \tag{3.33}$$

Since

$$\begin{cases} w \in [\lambda, \lambda+1], \\ \|\chi(P-w)^{-1}\chi\|_{\mathcal{H}\to\mathcal{H}} > A(\lambda) \end{cases}$$

$$\text{if and only if} \quad \begin{cases} h^2 w \in [1, 1+h^2] & \text{with } h = \lambda^{-1/2} \\ \|\chi R(h^2 w, h)\chi\|_{\mathcal{H}\to\mathcal{H}} > B(h) & \text{with } B(h) = h^{-2}A(h^{-2}), \end{cases}$$

applying this with $A(\lambda) = \lambda^s$ and hence $B(h) = h^{-2s-2}$, i.e., $t = 2s+2$, and using the bound (3.33), we have that, for $\lambda \geq h_2^{-2}$

$$\left|\{\|\chi(P-w)^{-1}\chi\|_{\mathcal{H}\to\mathcal{H}} > A(\lambda)\} \cap [\lambda, \lambda+1]\right| \leq \delta\lambda^{-s-1+3n^{\#}/4+p/2+5\eta/4}.$$

This last bound implies the result (3.24) with $\varepsilon = 5\eta/4$ and $\lambda_0 = h_2^{-2}$. Recalling that $h = \lambda^{-1/2}$, one can check that the condition (3.32) is satisfied by the hypothesis (3.23).

Proof of Part (ii). First of all, observe that it is sufficient to prove that there exists $J \subset [k_1, \infty)$ with $|J| \leq \delta$ such that

$$\|\chi R(k)\chi\|_{\mathcal{H}\to\mathcal{H}} \leq Ck^{3n^{\#}/2+p+\varepsilon} \quad \text{for all } k \in [k_1, \infty) \setminus J, \tag{3.34}$$

where $k_1 > k_0$. Indeed, if (3.34) holds, the result follows by increasing the constant C so that the estimate still holds in $[k_0, \infty) \setminus J$. We therefore now prove (3.34).

Let $\delta_0 > 0$ be a constant to be fixed later, and

$$s := 3n^{\#}/4 + p/2 + 2\varepsilon;$$

observe that this choice satisfies the requirement (3.23). Let $\lambda_0 = \lambda_0(\delta_0, s, 2\varepsilon)$ be given by Theorem 3.5. We set

$$\widetilde{J} := \left\{\|\chi(P-w)^{-1}\chi\|_{\mathcal{H}\to\mathcal{H}} > w^s\right\} \cap [\lambda_0, +\infty), \tag{3.35}$$

so that

$$\left\|\chi(P-\lambda)^{-1}\chi\right\|_{\mathcal{H}\to\mathcal{H}} \leq \lambda^{3n^{\#}/4+p/2+2\varepsilon} \quad \text{for all } \lambda \in [\lambda_0, +\infty) \setminus \widetilde{J}. \tag{3.36}$$

We now bound the measure of $\widetilde{J}$ using Theorem 3.5. Indeed, by Theorem 3.5, for all $\lambda \geq \lambda_0$,

$$\left|\{w, \|\chi(P-w)^{-1}\chi\|_{\mathcal{H}\to\mathcal{H}} > \lambda^s)\} \cap [\lambda, \lambda+1]\right| \leq \delta_0\lambda^{-1-\varepsilon}. \tag{3.37}$$

From the definition of $\widetilde{J}$ (3.35),

$$|\widetilde{J}| = \sum_{k\geq 0,\ \lambda=\lambda_0+k} \big|\{w, \|\chi(P-w)^{-1}\chi\|_{\mathcal{H}\to\mathcal{H}} > w^s\} \cap [\lambda, \lambda+1]\big|$$
$$\leq \sum_{k\geq 0,\ \lambda=\lambda_0+k} \big|\{w, \|\chi(P-w)^{-1}\chi\|_{\mathcal{H}\to\mathcal{H}} > \lambda^s\} \cap [\lambda, \lambda+1]\big|,$$

where this last inequality holds because the function $w \mapsto w^s$ is increasing. Therefore, by (3.37),

$$|\widetilde{J}| \leq \delta_0 \sum_{k\geq 0,\ \lambda=\lambda_0+k} \lambda^{-1-\varepsilon} \leq \delta_0 \int_{\lambda_0}^{\infty} \lambda^{-1-\varepsilon} d\lambda = \delta_0 \frac{1}{\varepsilon}(\lambda_0)^{-\varepsilon} \leq \frac{\delta_0}{\varepsilon};$$

thus, choosing $\delta_0 := 2\delta\varepsilon$, the estimate (3.34) follows from (3.36) with $\lambda = k^2$, $k_1^2 = \lambda_0$, and J defined by (3.21). Observe that, since $k_1 > 1$, arguing in a similar way to the proof of Theorem 3.3 (in the text after (3.21)), we have that $|J| \leq |\widetilde{J}|/2 \leq \delta$. □

COROLLARY 3.6 (Improved resolvent estimate under sharp Weyl remainder for reference operator). *Let the assumption* (2.4) *on the growth of the spectral counting function for the black-box reference operator be replaced by the stronger assumption*

$$N(P^{\#}, (-C, \lambda]) = C\lambda^{n/2} + O(\lambda^{(n-1)/2}) \quad \text{as } \lambda \to \infty. \tag{3.38}$$

Then, given $k_0 > 0$, $\delta > 0$, and $\varepsilon > 0$, there exists a constant $C(k_0, \delta, \varepsilon, n^{\#}) > 0$ and a set J with $|J| \leq \delta$ such that the resolvent (2.3) *satisfies*

$$\|\chi R(k)\chi\|_{\mathcal{H}\to\mathcal{H}} \leq Ck^{5n/2-1+\varepsilon} \quad \text{for all } k \in [k_0, \infty) \setminus J.$$

PROOF. This follows from Theorem 3.5 using the result of Petkov-Zworski [95, prop. 2] that, under the Weyl law assumption on the reference operator (3.38), the number of resonances in $[r, r+c_1] + \mathrm{i}[-c_2, c_2]$ (and hence also in the smaller box (3.22)) is bounded by $C_1 r^{n-1}$ for some $C_1 > 0$; i.e., the assumptions of Theorem 3.5 are satisfied with $p = n-1$. (See also [12, theorem 1] and [108, theorem 2] for later refinements on [95].) □

A particularly important situation where the assumptions of Corollary 3.6 apply is scattering by Dirichlet or Neumann obstacles with $C^{1,\sigma}$ boundaries.

COROLLARY 3.7 (Improved resolvent estimate for scattering by $C^{1,\sigma}$ Dirichlet or Neumann obstacles). *Let $\mathcal{O}_-$, $\mathcal{O}_+$, and A be as in Lemma* 2.1*, and assume further that both A and $\partial\mathcal{O}_+$ are $C^{1,\sigma}$ for some $0 < \sigma < 1$ (observe that this also includes the case when $\mathcal{O}_- = \varnothing$). Then, given $k_0 > 0$, $\delta > 0$, and $\varepsilon > 0$, there exists $C = C(k_0, \delta, \varepsilon, n) > 0$ and a set $J \subset [k_0, \infty)$ with $|J| \leq \delta$ such that*

$$\|\chi R(k)\chi\|_{L^2(\mathcal{O}_+)\to L^2(\mathcal{O}_+)} \leq Ck^{5n/2-1+\varepsilon} \quad \text{for all } k \in [k_0, \infty) \setminus J.$$

PROOF. The result that the asymptotics (3.38) hold when, additionally, A and $\partial\mathcal{O}_-$ are smooth goes back to Seeley [102] and Phạm The Lại [74]. The recent results of Ivrii [67–69] extend this result to much more general classes of coefficients and domains, including those that are $C^{1,\sigma}$ for some $0 < \sigma < 1$ [67]. □

COROLLARY 3.8 (Improved resolvent estimate for scattering by a three-dimensional penetrable ball). *Let $R(k)$ be the resolvent in the case of scattering by a penetrable obstacle* (*described in Lemma* 2.3 *and Remark* 2.4) *when, furthermore, the obstacle $\mathcal{O}_-$ is a three-dimensional ball and $c < 1$ so that the problem is trapping* (*see Remark* 2.5). *Assume that the parameter α in the transmission condition* (2.11) *satisfies $\alpha \leq \alpha_0$, where $\alpha_0 > 0$ is as in* [23, theorem 1.1]. *Then, given $k_0 \geq 1$, $\delta > 0$, and $\varepsilon > 0$, there exists a constant $C(k_0, \delta, \varepsilon) > 0$ and a set J with $|J| \leq \delta$ such that the resolvent* (2.3) *satisfies*

$$\|\chi R(k)\chi\|_{\mathcal{H}\to\mathcal{H}} \leq Ck^{6+1/6+\varepsilon} \quad \textit{for all } k \in [k_0, \infty) \setminus J. \tag{3.39}$$

The exponent $6 + 1/6$ in (3.39) should be compared to the exponent $7 + 1/2$ from Theorem 3.3 (recall that $n = 3$ here).

PROOF OF COROLLARY 3.8. Let the number of resonances in $[0, r]+\mathrm{i}[-c_2, c_2]$ be denoted by $N(r)$. By [23, theorem 1.3], there exists $C_1 > 0$ such that, given $\epsilon > 0$,

$$N(r) = C_1 r^n + O_\epsilon(r^{n-1/3+\epsilon}) \quad \text{as } r \to \infty,$$

where $n = 3$. Then

$$\begin{aligned} &N\big(r + r^{-1}\big) - N(r) \\ &\quad = C_1\big((r + r^{-1})^n - r^n\big) + O\big((r + r^{-1})^{n-1/3+\epsilon}\big) + O(r^{n-1/3+\epsilon}) \\ &\quad = O(r^{n-1/3+\epsilon}), \end{aligned}$$

and the assumptions of Theorem 3.5 hold with $p = n - 1/3 + \epsilon = 3 - 1/3 + \epsilon$ (note that this application makes no use of the fact that the interval $[r, r + r^{-1}]$ is shrinking as $r \to \infty$ rather than having fixed width, i.e., $N(r + 1) - N(r)$ enjoys the same estimate).

The bound (3.39) then follows from Remark 3.4 if we can show that all but finitely many resonances have multiplicity proportional to k. Indeed, assuming this multiplicity property, in the proof of Theorem 3.5, instead of the number of semiclassical resonances in $[1, 1 + h^2] + \mathrm{i}[-h, h]$ being bounded by $C_1^{\#}h^{-p}$, it is bounded by $C_1^{\#}h^{-p+1}$; this factor of $h = k^{-1}$ then propagates through the proof of Theorem 3.5.

To prove this multiplicity property, we first recall that, when $c < 1$ and the problem is trapping, the resonances fall into two groups by [112, §9, p. 137]:

(1) one near the resonances of the exterior Dirichlet problem for the ball—since this latter problem is nontrapping, these resonances lie away from the real axis—and

(2) one near the real axis, with asymptotics given by

$$\frac{1}{c}k_{\nu,i} = \nu + \alpha_i\Big(\frac{\nu}{2}\Big)^{1/3} + O(1) \quad \text{as } \nu \to \infty, \tag{3.40}$$

where α_i denotes the m^{th} zero of the Airy function $\mathrm{Ai}(-z)$ and $\nu := \ell + 1/2$, where ℓ is the angular frequency; see, e.g., [72, eq. 1.1] and [5].

Each resonance has multiplicity $2\ell + 1$ because, by separation of variables, the solution can be expressed in the form

$$\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} a_\ell(kr)Y_{\ell,m}(\theta,\phi),$$

where $a_\ell(\cdot)$ is either a spherical Hankel or spherical Bessel function (defined by [91, §10.47]) and $Y_{\ell,m}(\cdot,\cdot)$ are spherical harmonics (defined by [91, eq. 14.30.1]); see, e.g., [20, eqs. 3.1–3.3]. By (3.40) and the fact that $\nu := \ell + 1/2$, the multiplicity of each resonance is proportional to k, and the proof is complete. □

The final result of this section (Lemma 3.9) is a lower bound on the resolvent for all frequencies in an "equidistribution of resonances" scenario. In fact, it is more convenient to work with *quasi-modes* (sequences of approximate solutions to the Helmholtz equation with real spectral parameter) rather than resonances, since the existence of quasi-modes is usually easier to establish in cases of stable trapping, and in many cases is known to be equivalent to the existence of sequences of resonances approaching the real axis; see [111, 113–115] and [43, §7.3].

LEMMA 3.9 (Lower bound on resolvent under "equidistribution of resonances" scenario). *Assume that there exist a compact subset K and $s \geq 0$ such that, for all $k > 0$ and for all $\lambda \in [k, k+1]$, there exists $C_1 > 0$, $\mu \in B(\lambda, C_1 k^{-s})$, and a μ-quasi-mode for P, denoted by u, supported in K and of order $s-1$, i.e.,*

$$\|(P-\mu^2)(u)\|_{\mathcal{H}} = O(\mu^{-(s-1)}) \quad \textit{with } \|u\|_{\mathcal{H}} = 1.$$

Then there exists a $C_2 > 0$ and a $\chi \in C_c^\infty$ such that the lower bound

$$\|\chi R(k)\chi\|_{\mathcal{H}\to\mathcal{H}} \geq C_2 k^{s-1} \tag{3.41}$$

holds for all $k > 0$.

If the two-term Weyl-type asymptotics,

$$N(r) = C_1 r^n + C_2 r^{n-1} + o(r^{n-1}) \quad \text{as } r \to \infty$$

hold, then, arguing as in the proof of Corollary 3.8, the number of resonances in $[k, k+1]$ is comparable to k^{n-1}. The case $s = n-1$ in Lemma 3.9 therefore assumes that quasi-modes corresponding to these resonances are spread out evenly throughout this interval. The existence of many quasi-modes is relatively easy to arrange (e.g., for a Helmholtz resonator); unfortunately, the *equidistribution* of these quasi-modes' spectral parameters, while highly plausible, seems very difficult to verify.

PROOF OF LEMMA 3.9. Let $\lambda \in [k, k+1]$, u, and μ be as above. Then

$$(P - k^2)u = (P - \mu^2)u + (k^2 - \mu^2)u =: f = O_{\mathcal{H}}(k^{-(s-1)}),$$

with f having support in K as well. Thus, with χ compactly supported and equal to 1 on K, $u = \chi u$ and $f = \chi f$, so in particular, $(P - k^2)(u) = \chi f$. Since u is certainly outgoing (because it has compact support), $u = R(k)\chi f$, i.e.,

$$u = \chi R(k) \chi f,$$

and this proves the lower bound. □

Remark 3.10 (Comparison with the results of [19, 20]). As noted in §1.2, in the case of scattering by a two- or three-dimensional penetrable ball [19, lemma 6.2] and [20, lemma 3.6] show that, for k outside a set of small measure, the scattered field everywhere outside the obstacle is bounded in terms of the incident field with a loss of $2+\alpha$ derivatives, with $\alpha > 0$ arbitrary. The nontrapping resolvent estimate (1.5) (which holds for the transmission problem when $c > 1$ by [22, theorem 1.1]; see also [88, theorem 3.1]) can be used to show that $\|u^S\|_{L^2} \lesssim \|u^I\|_{L^2}$; see, e.g., [70, lemma 6.5]. With each derivative corresponding to a power of k, the results of [19, 20] therefore indicate a loss of $k^{2+\alpha}$ over the nontrapping estimate (compare to the loss of k when $s = n - 1$ and $n = 2$ in (3.41)). The lowest loss over the nontrapping resolvent estimate we can prove is a loss of $5 + 2/3 + \varepsilon$ ($= 1 + 5 \times 2/2 - 1/3 + \varepsilon + \epsilon$) from Theorem 3.5 with $n = 2$ and $p = n - 1/3 + \epsilon$ by the results in [23] used in the proof of Corollary 3.8. However (as highlighted above) our results hold in much more general settings, not least scattering by a smooth obstacle with strictly positive curvature that is not a ball, whereas the results of [19, 20] use the explicit expression for the solution when the obstacle is a ball and so are restricted to this setting.

Appendix A Images of Boxes under Semiclassical Scaling

LEMMA A.1 (Images of boxes in $\mathbb{C}$ under semiclassical scaling). *Given* $0 < h_0 < 1$, *let* $0 < h \leq h_0$. *The image of the box* $[1, 1 + h^2] + \mathrm{i}[-h, h]$ *under the mapping* $z \to h^{-1} z^{1/2}$ *is included in a box of form* $[h^{-1}, h^{-1}(1 + c_1 h^2)] + \mathrm{i}[-c_2, c_2]$ *for some* $c_1, c_2 > 0$ *dependent on* h_0 *but independent of* h.

PROOF. Let

$$a := 1 + \mathrm{i}h, \quad b := 1 + h^2 + \mathrm{i}h$$

(so that the box $[1, 1 + h^2] + \mathrm{i}[-h, h]$ has vertices $a, b, \bar{a}$, and $\bar{b}$), and let A and B be the images of a and b under the mapping $z \to h^{-1} z^{1/2}$. One can check that $\operatorname{Im} B < \operatorname{Im} A$ for all $0 < h < 1$. Let

$$l_-(h) := h^{-1}, \quad l_+(h) := \operatorname{Re} B, \quad \text{and} \quad p(h) := \operatorname{Im} A;$$

then the image of $[1, 1 + h^2] + \mathrm{i}[-h, h]$ is included in

$$[l_-(h), l_+(h)] + \mathrm{i}[p(h), -p(h)].$$

Now, with $\theta(h) := \arg a$ and $\psi(h) := \arg b$, we have

$$\theta(h) = \arctan h = h + O(h^3) \quad \text{as } h \to 0$$

and

$$\psi(h) = \arctan\left(\frac{h}{1+h^2}\right) = h + O(h^3) \quad \text{as } h \to 0;$$

hence

$$\begin{aligned} l_+(h) &= h^{-1}|b|^{1/2}\cos\big(\psi(h)/2\big) \\ &= h^{-1}\big((1+h^2)^2+h^2\big)^{1/4}\big(1-h^2/8+O(h^3)\big) \\ &= h^{-1}\big(1+5h^2/8+O(h^3)\big) \end{aligned}$$

and

$$\begin{aligned} p(h) = h^{-1}|a|^{1/2}\sin\big(\phi(h)/2\big) &= h^{-1}(1+h^2)^{1/4}(h+O(h^3)) \\ &= 1+O(h^2), \end{aligned}$$

and the result follows with $c_1 > 5/8$ and $c_2 > 1$. □

Appendix B Weyl-Type Upper Bound for Reference Operator for Penetrable- and Impenetrable-Obstacle Problems

The aim of this appendix is to show that the reference operator $P^\#$ associated with *either* the impenetrable obstacle problem of Lemma 2.1 *or* the transmission problem of Lemma 2.3 satisfies the Weyl-type upper bound (2.4).

LEMMA B.1. *The reference operator $P^\#$ associated to either the Dirichlet or the Neumann obstacle problems of Lemma* 2.1 (*in particular with A Lipschitz*) *satisfies the Weyl-type upper bound*

$$N(P^\#, [-C, \lambda]) \lesssim \lambda^{d/2}. \tag{B.1}$$

PROOF. We use the results of [92] on heat kernel asymptotics in Lipschitz Riemannian manifolds. Indeed, taking the measure density to be 1, [92] covers both Dirichlet and Neumann realisations of the divergence form operator $\nabla\cdot(A\nabla\cdot)$ with A Lipschitz. By [92, theorem 1.1], the heat kernel (for either Dirichlet or Neumann boundary conditions), $p(t,x,y)$, satisfies

$$t\log p(t,x,y) \longrightarrow -\frac{1}{4}d(x,y)^2 \quad \text{as } t \to 0,$$

and therefore, in particular,

$$p(t,x,y) \lesssim \exp\left(-\frac{1}{4t}d(x,y)^2\right). \tag{B.2}$$

Recall, however, that

$$p(t,x,y) = \sum_{n\in\mathbb{N}} \phi_n(x)\phi_n(y)\exp(-t\lambda_n),$$

where ϕ_n is the eigenfunction of L^2-norm 1 associated with λ_n. Therefore, taking the square of (B.2) and integrating with respect to x and y we obtain, by orthogonality of the eigenfunctions,

$$\sum_{n\in\mathbb{N}} \exp(-2t\lambda_n) \lesssim \iint \exp\left(-\frac{1}{2t}d(x,y)^2\right)\mathrm{d}x\,\mathrm{d}y = Ct^{-d/2};$$

the result (B.1) follows by a weak version of the Karamata Tauberian theorem appearing in, e.g., [110, prop. B.0.12]. □

LEMMA B.2. *The reference operator $P^\#$ associated to the transmission problem of Lemma* 2.3 (*in particular with A Lipschitz*) *satisfies the Weyl-type upper bound* (B.1).

PROOF. By the min-max principle for self-adjoint operators with compact resolvent (see, e.g., [99, p. 76, theorem 13.1]) we have

$$\begin{aligned}\lambda_n &= \inf_{X\in\Phi_n(\mathcal{D})}\ \sup_{u\in X,\|u\|_{L^2_{\alpha,c}}=1} \langle P^\# u,u\rangle_{\alpha,c} \\ &= \inf_{X\in\Phi_n(\mathcal{D})}\ \sup_{u\in X,\|u\|_{L^2_{\alpha,c}}=1} \big(\langle A\nabla u,\nabla u\rangle_{L^2(\mathbb{T}_d\setminus\mathcal{O})} + \alpha^{-1}\langle A\nabla u,\nabla u\rangle_{L^2(\mathcal{O})}\big)\end{aligned}$$

where $\langle\ ,\ \rangle_{\alpha,c}$ is the scalar product defined implicitly in Lemma 2.3 by (2.8), $\|\cdot\|_{L^2_{\alpha,c}}$ is the induced norm, $(\lambda_n)_{n\geq 1}$ denotes the ordered eigenvalues of $P^\#$, $\mathcal{D}$ is the domain of $P^\#$ defined by (2.9), and $\Phi_n(\mathcal{D})$ the set of all n-dimensional subspaces of $\mathcal{D}$. By rescaling the norms, we then have that

$$\lambda_n = \inf_{X\in\Phi_n(\mathcal{D})}\ \sup_{u\in X,\|u\|_{L^2}=1} \big(\langle A\nabla u,\nabla u\rangle_{L^2(\mathbb{T}_d\setminus\mathcal{O})} + c^2\langle A\nabla u,\nabla u\rangle_{L^2(\mathcal{O})}\big). \tag{B.3}$$

Observe that

$$\mathcal{D} \subset \big\{(u_1,u_2)\in H^1(\mathbb{T}_n\setminus\mathcal{O})\oplus H^1(\mathcal{O}) \text{ s.t. } u_1=u_2 \text{ on } \partial\mathcal{O}\big\} = H^1(\mathbb{T}_d),$$

and thus, by (B.3),

$$\begin{aligned}\lambda_n \geq \inf_{X\in\Phi_n(H^1(\mathbb{T}_d))}\ \sup_{u\in X,\|u\|_{L^2}=1} &\big(\langle A\nabla u,\nabla u\rangle_{L^2(\mathbb{T}_d\setminus\mathcal{O})} \\ &+ c^2\langle A\nabla u,\nabla u\rangle_{L^2(\mathcal{O})}\big).\end{aligned} \tag{B.4}$$

Now, note that if $c \geq 1$ we have

$$\langle A\nabla u,\nabla u\rangle_{L^2(\mathbb{T}_d\setminus\mathcal{O})} + c^2\langle A\nabla u,\nabla u\rangle_{L^2(\mathcal{O})} \geq \langle A\nabla u,\nabla u\rangle_{L^2(\mathbb{T}_d)},$$

and thus, by (B.4) and the min-max principle on the torus,

$$\lambda_n \geq \lambda_n(A,\mathbb{T}_d),$$

and the result follows by the Weyl-type upper bound on Lipschitz compact manifolds. In the same way, if $c \leq 1$, then $\lambda_n \geq c^2\lambda_n(A,\mathbb{T}_d)$ and the result follows as well. □

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