POSITIVE COMMUTATORS AT THE BOTTOM OF THE SPECTRUM

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Abstract. Bony and Häfner have recently obtained positive commutator estimates on the Laplacian in the low-energy limit on asymptotically Euclidean spaces; these estimates can be used to prove local energy decay estimates if the metric is non-trapping. We simplify the proof of the estimates of Bony-Häfner and generalize them to the setting of scattering manifolds (i.e. manifolds with large conic ends), by applying a sharp Poincaré inequality. Our main result is the positive commutator estimate

$$\chi_I(H^2\Delta_g)\frac{i}{2}[H^2\Delta_g, A]\chi_I(H^2\Delta_g) \geq C\chi_I(H^2\Delta_g)^2,$$

where $H \uparrow \infty$ is a *large* parameter, I is a compact interval in $(0,\infty)$, and χ_I its indicator function, and where A is a differential operator supported outside a compact set and equal to $(1/2)(rD_r + (rD_r)^*)$ near infinity. The Laplacian can also be modified by the addition of a positive potential of sufficiently rapid decay—the same estimate then holds for the resulting Schrödinger operator.

1. Introduction

The purpose of this paper is to clarify an intricate argument recently introduced by Bony and Häfner [1] and use these ideas to generalize certain of the results of [1]. The central thrust of [1] is first of all to obtain certain kinds of commutator estimates for the the Laplacian and its square root on asymptotically Euclidean space. The authors then employ those estimates to yield energy decay results for the wave equation, and, ultimately, global existence results for quadratically semilinear wave equations on these spaces. In a subsequent note [2], applications of the linear results to the low frequency limiting absorption principle were shown. The novel tool central to all of these applications is the commutator estimate

$$\chi_I(H^2\Delta_g)\frac{i}{2}[H^2\Delta_g, A]\chi_I(H^2\Delta_g) \geq C\chi_I(H^2\Delta_g)^2, \tag{1.1}$$

where $H \uparrow \infty$ is a *large* parameter, I is a compact interval in $(0,\infty)$, and χ_I its indicator function, and where A is a differential operator supported outside a compact set and equal to $(1/2)(rD_r + (rD_r)^*)$ near infinity. The estimate (1.1) is thus a low-energy version of the positive commutator construction that is ubiquitous in scattering theory; we remark that the analogous *high*-energy estimate would not be true with this choice of A, supported outside a compact set: by standard results in microlocal analysis, the symbol of A would have to be strictly increasing along all geodesics, lifted to the cotangent bundle. Indeed, on a manifold with

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trapped geodesics, the construction of such a high-energy commutant is manifestly impossible.

In [1], the estimate (1.1) is proved by a multi-step process involving a sequence of perturbation arguments, starting from flat $\mathbb{R}^n$. It is thus a priori unclear whether such estimates continue to hold if we vary the topology of our space and its end structure. In this paper we show that (1.1) (as well as a related estimate for $\sqrt{\Delta}$) does indeed continue to hold on any long-range metric perturbation of a *scattering manifold*, and further holds even if a short range (in a suitable sense) non-negative potential is added. The class of scattering manifolds, introduced by Melrose [8], consists of all manifolds with ends that look asymptotically like the large ends of cones. The topology of interior and of the cross sections of the ends is unrestricted. Our methods are nonperturbative and simple, involving only commutator estimates for differential operators and a sharp Poincaré-type inequality on these manifolds. We anticipate that these methods will prove quite flexible in the investigation of energy decay in a variety of other asymptotic geometries.

We do not explore the applications of our estimate in detail here, as the methods of [1] apply, mutatis mutandis, directly to our situation. We content ourselves with restating the energy decay estimate of [1] for solutions to the wave equation in the final section of the paper and sketching the main ingredients in its proof, adapted to our setting. This estimate applies on scattering manifolds with no trapped geodesics.[1]

We point out here that Guillarmou and Hassell started an extensive and very detailed study of the Laplacian on scattering manifolds near the bottom of the spectrum, [4], with a particular emphasis on the Schwartz kernel of the resolvent of the Laplacian on a resolved space. Our methods give the estimates we need more quickly, but naturally the results of [4] give more detail on the resolvent kernel, which in principle implies for instance results on the energy decay[2]. We also remark that Bouclet [3] has recently proved weighted low-energy estimates generalized those of [1] for *powers* of the resolvent on an asymptotically Euclidean space.

Our paper is structured as follows. In Section 2 we recall the background material concerning b- (or totally characteristic) and scattering differential operators. In Section 3 we obtain Poincaré inequalities and in Section 4 weighted diferential estimates that we use in Section 5 to prove our positive commutator estimate. Finally, in Section 6 we show how these results can be applied to study energy decay for the wave equation, following the method of Bony and Häfner [1].

2. b- and scattering geometry

We very briefly recall the basic definitions of the b- and scattering structures on an n-dimensional manifolds with boundary, denoted X; we refer to [8] for more detail. A boundary defining function x on X is a non-negative $\mathcal{C}^\infty$ function on X whose zero set is exactly ∂X, and whose differential does not vanish there. We recall that $\dot{\mathcal{C}}^\infty(X)$, which may also be called the set of Schwartz functions, is the subset

[1]Such a manifold must in fact be contractible, but we note that even $\mathbb{R}^n$ can be equipped with scattering metrics different from the round metric on the sphere at infinity, so this result remains broader than that of [1].

[2]Note, however, that L^2-based estimates are not always easy to get from precise description of the Schwartz kernel!

of $\mathcal{C}^\infty(X)$ consisting of functions vanishing at the boundary with all derivatives, the dual of $\dot{\mathcal{C}}^\infty(X)$ is tempered distributional densities $\mathcal{C}^{-\infty}(X;\Omega X)$; tempered distributions $\mathcal{C}^{-\infty}(X)$ are elements of the dual of Schwartz densities, $\dot{\mathcal{C}}^\infty(X;\Omega X)$.

Let $\mathcal{V}(X)$ be the Lie algebra of all $\mathcal{C}^\infty$ vector fields on X; thus $\mathcal{V}(X)$ is the set of all $\mathcal{C}^\infty$ sections of TX. In local coordinates $(x, y_1, \ldots, y_{n-1})$,

$$\partial_x, \partial_{y_1}, \ldots, \partial_{y_{n-1}}$$

form a local basis for $\mathcal{V}(X)$, i.e. restrictions of elements of $\mathcal{V}(X)$ to the coordinate chart can be expressed uniquely as a linear combination of these vector fields with $\mathcal{C}^\infty$ coefficients. We next define $\mathcal{V}_{\mathrm{b}}(X)$ to be the Lie algebra of $\mathcal{C}^\infty$ vector fields tangent to ∂X; in local coordinates

$$x\partial_x, \partial_{y_1}, \ldots, \partial_{y_{n-1}}$$

form a local basis in the same sense. Thus, $\mathcal{V}_{\mathrm{b}}(X)$ is the set of all $\mathcal{C}^\infty$ sections of a bundle, called the b-tangent bundle of X, denoted ${}^{\mathrm{b}}TX$. Finally, $\mathcal{V}_{\mathrm{sc}}(X) = x\mathcal{V}_{\mathrm{b}}(X)$ is the Lie algebra of scattering vector fields;

$$x^2\partial_x, x\partial_{y_1}, \ldots, x\partial_{y_{n-1}}$$

form a local basis now. Again, $\mathcal{V}_{\mathrm{sc}}(X)$ is the set of all $\mathcal{C}^\infty$ sections of a bundle, called the scattering tangent bundle of X, denoted ${}^{\mathrm{sc}}TX$.

The dual bundles of $TX, {}^{\mathrm{b}}TX, {}^{\mathrm{sc}}TX$ are $T^*X, {}^{\mathrm{b}}T^*X, {}^{\mathrm{sc}}T^*X$ respectively, with local bases

$$dx,\ dy_j,\ \text{resp.}\ \frac{dx}{x},\ dy_j,\ \text{resp.}\ \frac{dx}{x^2},\ \frac{dy_j}{x},\ j = 1, \ldots, n-1.$$

These induce form bundles and density bundles as usual. In particular, local bases of the density bundles are

$$|dx\,dy_1 \ldots dy_{n-1}|\ \text{resp.}\ x^{-1}|dx\,dy_1 \ldots dy_{n-1}|,\ \text{resp.}\ x^{-n-1}|dx\,dy_1 \ldots dy_{n-1}|.$$

If X is compact, the L^2-spaces relative to these classes of densities are well-defined as Banach spaces, up to equivalence of norms; they are denoted by $L^2(X)$, $L^2_{\mathrm{b}}(X)$, $L^2_{\mathrm{sc}}(X)$, respectively.

The classes of vector fields mentioned induce algebras of differential operators, consisting of locally finite sums of products of these vector fields and elements of $\mathcal{C}^\infty(X)$, considered as operators on $\mathcal{C}^\infty(X)$. These are denoted by $\mathrm{Diff}(X)$, $\mathrm{Diff}_{\mathrm{b}}(X)$ and $\mathrm{Diff}_{\mathrm{sc}}(X)$, respectively. These in turn give rise to (integer order) Sobolev spaces. Thus, for $m \geq 0$ integer,

$$H^m_\bullet(X) = \{u \in L^2_\bullet(X) :\ Qu \in L^2_\bullet(X)\ \forall Q \in \mathrm{Diff}^m_\bullet(X)\},$$

where $\bullet$ is either b or sc and where Qu is a priori defined as a (tempered) distribution.

A similar construction leads to symbol classes: We let $S^k(X)$, the space of symbols of order k, consist of functions f such that

$$x^k Lf \in L^\infty(X)\ \text{for all}\ L \in \mathrm{Diff}_{\mathrm{b}}(X).$$

We note, in particular, that

$$x^\rho \mathcal{C}^\infty(X) \subset S^{-\rho}(X)$$

since $\mathrm{Diff}_{\mathrm{b}}(X) \subset \mathrm{Diff}(X)$. As $\mathrm{Diff}_{\mathrm{b}}(X)$ (a priori acting, say, on tempered distributions) preserves $S^k(X)$, and one can extend $\mathrm{Diff}_{\mathrm{b}}(X)$ and $\mathrm{Diff}_{\mathrm{sc}}(X)$ by 'generalizing the coefficients':

$$S^k\mathrm{Diff}_{\mathrm{b}}^m(X) = \{\sum_j a_j Q_j :\ a_j \in S^k(X),\ Q_j \in \mathrm{Diff}_{\mathrm{b}}^m(X)\},$$

with the sum being locally finite, and defining $S^k\mathrm{Diff}_{\mathrm{sc}}^m(X)$ similarly. In particular,

$$x^k\mathrm{Diff}_{\mathrm{b}}^m(X) \subset S^{-k}\mathrm{Diff}_{\mathrm{b}}^m(X),\ x^k\mathrm{Diff}_{\mathrm{sc}}^m(X) \subset S^{-k}\mathrm{Diff}_{\mathrm{sc}}^m(X).$$

Then $Q \in S^k\mathrm{Diff}_{\mathrm{sc}}^m(X)$, $Q' \in S^{k'}\mathrm{Diff}_{\mathrm{sc}}^{m'}(X)$ gives $QQ' \in S^{k+k'}\mathrm{Diff}_{\mathrm{sc}}^{m+m'}(X)$, and the analogous statement for $S^k\mathrm{Diff}_{\mathrm{b}}^m(X)$ also holds.

An example of particular interest is the radial, or geodesic, compactification of $\mathbb{R}^n$, which compactifies $\mathbb{R}^n$ as a closed ball, $X = \overline{\mathbb{B}^n}$; see [8, Section 1] for an extended discussion, with the compactification called stereographic compactification there. In this case, the set of Schwartz functions on $\mathbb{R}^n$ lifts to $\dot{\mathcal{C}}^\infty(X)$ (justifying the 'Schwartz' terminology for the latter), the set of 0th order classical symbols on $\mathbb{R}^n$, i.e. 0th order symbols a with an asymptotic expansion $a(r\omega) \sim \sum_{j=0}^\infty r^{-j} a_j(\omega)$ in polar coordinates, lifts to $\mathcal{C}^\infty(X)$, the translation invariant vector fields on $\mathbb{R}^n$ lift to a basis of $\mathcal{V}_{\mathrm{sc}}(X)$, and $H_{\mathrm{sc}}^m(X)$ is the standard Sobolev space $H^m(\mathbb{R}^n)$ (under the natural identification of functions), while $S^k(X)$ is the standard symbol space $S^k(\mathbb{R}^n)$. One way of seeing these statements is to introduce 'inverse polar coordinates' $z = x^{-1}\omega$, $x \in (0,1)$, $\omega \in \mathbb{S}^{n-1}$, in the exterior of a closed ball in $\mathbb{R}^n_z$, and use polar coordinates $(\rho,\omega) \in (1/2,1) \times \mathbb{S}^{n-1}$ near $\partial\mathbb{B}^n$, with $\mathbb{B}^n$ considered as the unit ball in $\mathbb{R}^n$; then one suitable identification of the exterior of the ball of radius 2 in $\mathbb{R}^n_z$ with the interior of a collar neighborhood of $\partial\mathbb{B}^n$ in $\overline{\mathbb{B}^n}$ is

$$(0,1/2) \times \mathbb{S}^{n-1} \ni (x,\omega) \mapsto (\rho,\omega) = (1-x,\omega) \in (1/2,1) \times \mathbb{S}^{n-1}.$$

3. Poincaré inequalities

Let g be an scattering metric on a compact manifold with boundary X of dimension n, and $L^2_g(X)$ the metric L^2-space. That is, as introduced by Melrose [8], we assume that g is a Riemannian metric on X°, and that ∂X has a collar neighborhood U and a boundary defining function x such that on U,

$$g = \frac{dx^2}{x^4} + \frac{h}{x^2},$$

where h is a symmetric 2-cotensor, $h \in \mathcal{C}^\infty(X; T^*X \otimes T^*X)$, which restricts to a metric on ∂X. Then with $h_0 = h|_{\partial X}$, and also extended to U using the product decomposition,

$$g = \frac{dx^2}{x^4} + \frac{h_0}{x^2} + g_1,\ g_1 \in x\mathcal{C}^\infty(X; {}^{\mathrm{sc}}T^*X \otimes {}^{\mathrm{sc}}T^*X). \tag{3.1}$$

Below we assume that g is of this form, with merely

$$g_1 \in S^{-\rho}(X; {}^{\mathrm{sc}}T^*X \otimes {}^{\mathrm{sc}}T^*X),\ \rho > 0. \tag{3.2}$$

Then the Laplacian $\Delta_g \in \mathrm{Diff}^2_{\mathrm{sc}}(X)$ satisfies $\Delta_g \in x^2\mathrm{Diff}^2_{\mathrm{b}}(X)$, namely $\Delta_g = x^2\Delta_{\mathrm{b}}$, $\Delta_{\mathrm{b}} \in \mathrm{Diff}^2_{\mathrm{b}}(X)$. Explicitly, as shown by Melrose [8, Proof of Lemma 3], in local coordinates (x,y) on a collar neighborhood of ∂X,

$$\Delta_{\mathrm{b}} = D_x x^2 D_x + i(n-1)xD_x + \Delta_0 + x^\rho R,\ R \in S^0\mathrm{Diff}^2_{\mathrm{b}}(X), \tag{3.3}$$

where Δ_0 is the Laplacian of the boundary metric. Moreover, the density $|dg| = x^{-n}|dg_{\mathrm{b}}|$, where $|dg_{\mathrm{b}}|$ is a non-degenerate b-density, so $L^2_g(X) = x^{n/2}L^2_{\mathrm{b}}(X)$.

It is also useful to have the Poincaré inequality at our disposal. This can be proved by b-techniques; we give an elementary proof.

Lemma 3.1. *Suppose $l > 1$, $l > l'$. Then for $u \in x^{l+1}H^1_{\mathrm{b}}(X)$,*

$$\|xu\|_{x^{l'}L^2_{\mathrm{b}}(X)} \leq C\|\nabla_g u\|_{x^l L^2_{\mathrm{b}}(X)}.$$

In particular, for $n \geq 3$, with $l = n/2$, $\epsilon = l - l' > 0$,

$$\|x^{1+\epsilon}u\|_{L^2_g(X)} \leq C\|\nabla_g u\|_{L^2_g(X)}.$$

Here the constant C can be taken independent of l, l', provided (l, l') is in a fixed compact subset of $\{(l,l') :\ l > 1,\ l > l'\}$.

Proof. It suffices to prove this for $u \in \dot{\mathcal{C}}^\infty(X)$ as both sides are continuous on $x^{l+1}H^1_{\mathrm{b}}(X)$. Moreover, it suffices to show that for such u,

$$\|\chi xu\|_{x^{l'}L^2_{\mathrm{b}}(X)} \leq C\|\nabla_g u\|_{x^l L^2_{\mathrm{b}}(X)},$$

$\chi \in \mathcal{C}^\infty_c(X)$ supported in a collar neighborhood of ∂X, which is then identified with $[0, x_0)_x \times \partial X$, for the rest will then follow by the standard Poincaré inequality on $H^1_0(K)$ where $K \subset X^\circ$ is compact. This in turn follows from

$$\|\chi xu\|_{x^{l'}L^2_{\mathrm{b}}(X)} \leq C\|x^2 D_x u\|_{x^l L^2_{\mathrm{b}}(X)},$$

i.e.

$$\int \chi^2|u|^2 x^{-2l'+1}\,dx\,dy \leq C^2\int |D_x u|^2 x^{-2l+3}\,dx\,dy. \tag{3.4}$$

But in local coordinates near ∂X, for $k < 1/2$, and for $x \leq x_0$

$$\begin{aligned}|u(x,y)| &= \Big|\int_0^x (\partial_x u)(s,y)\,ds\Big| = \Big|\int_0^x s^k(\partial_x u)(s,y)s^{-k}\,ds\Big| \\ &\leq \Big(\int_0^x s^{2k}|(\partial_x u)(s,y)|^2\,ds\Big)^{1/2}\Big(\int_0^x s^{-2k}\,ds\Big)^{1/2} \\ &\leq \Big(\int_0^{x_0} s^{2k}|(\partial_x u)(s,y)|^2\,ds\Big)^{1/2} C' x^{-k+1/2},\end{aligned}$$

and thus, provided $p - 2k + 1 > -1$,

$$\int_0^{x_0} x^p|u(x,y)|^2\,dx \leq C''\int_0^{x_0} s^{2k}|(\partial_x u)(s,y)|^2\,ds.$$

Integration with respect to y now gives

$$\int \chi^2|u|^2x^p\,dx\,dy \leq C^2\int |D_x u|^2 x^{2k}\,dx\,dy.$$

So take $k = -l + 3/2$, so $k < 1/2$ is satisfied for $l > 1$. Then let $p = -2l' + 1$, so $p - 2k + 1 = -2l' + 2l - 1$, and $p - 2k + 1 > -1$ is satisfied if $l' < l$.

□

We now prove a sharp version of the Poincaré inequality; this will follow from a weighted Hardy inequality, which can be found in the Appendix of [6]; we give a proof for completeness:[3]

[3]We are grateful to the referee for showing us the simple proof given here.

Lemma 3.2. *Let $u \in \dot{\mathcal{C}}_c^\infty([0,\infty))$, and let $d\mu = x^{-n-1}\,dx$ on $(0,\infty)$. If $s < (n-2)/2$, we have*

$$\left\|x^{1+s}u\right\|^2_{L^2(d\mu)} \leq \frac{4}{(n-2-2s)^2}\left\|x^{2+s}\partial_x u\right\|^2_{L^2(d\mu)}.$$

Proof. Observe that integration by parts yields

$$\begin{aligned}\left\|x^{1+s}\right\|^2_{L^2(d\mu)} &= \int x^{-n+1+2s}|u|^2\,dx\\ &= \frac{1}{(n-2-2s)}\int 2x^{-n+2+2s}\operatorname{Re}(\partial_x u)\overline{u}\,dx\\ &\leq \frac{2}{(n-2-2s)}\left|\left\langle x^{2+s}\partial_x u, x^{1+s}u\right\rangle_{L^2(d\mu)}\right|,\end{aligned}$$

and the result follows by Cauchy-Schwarz. □

Since in a collar neighborhood of ∂X,

$$\left|\nabla_g u\right|_g^2 \sim \left|x\partial_\theta u\right|^2 + \left|x^2\partial_x u\right|^2,$$

we can combine our Hardy inequality with the non-sharp Poincaré inequality above to get a sharp result:[4].

Proposition 3.4. *If $s < (n-2)/2$ and*

$$u \in x^{-s+(n-2)/2}H^1_{\mathrm{b}}(X),$$

then

$$\left\|x^{1+s}u\right\|^2_{L^2_g(X)} \leq C_s\|x^s\nabla u\|_{L^2_g(X)}.$$

In particular, the estimate holds for $u \in x^{-s}H^1_{\mathrm{sc}}(X)$, hence for $u \in H^1_{\mathrm{sc}}(X)$ for $s \geq 0$.

Here C_s can be taken independent of s as long as s is in a compact subinterval of $(-\infty,(n-2)/2)$.

Proof. By density of $\dot{\mathcal{C}}^\infty(X)$ and continuity of both sides in $x^{-s+(n-2)/2}H^1_{\mathrm{b}}(X)$ (recall that $L^2_g(X) = x^{n/2}L^2_{\mathrm{b}}(X)$), it suffices to consider $u \in \dot{\mathcal{C}}^\infty(X)$ when proving the estimate.

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[4]We remark that in sufficiently high dimension ($n \geq 5$), a standard result on mapping properties of Δ_g can be used instead to shorten parts of the later argument.

Lemma 3.3. *Suppose $n \geq 5$. Then*

$$\Delta_g : x^{n/2-2}H^2_{\mathrm{b}}(X) \to x^{n/2}L^2_{\mathrm{b}}(X) = L^2_g(X)$$

is an isomorphism. In particular, for any $Q \in \mathcal{V}_{\mathrm{b}}(X)$,

$$\|x^2u\|_{L^2_g(X)} + \|x^2Qu\|_{L^2_g(X)} \leq C\|\Delta_g u\|_{L^2(X)}.$$

We remark that this result can also be extended to apply to $\Delta_g + V$ with $V \geq 0$, $V \in S^{-2-\rho}(X)$ with $\rho > 0$:

A proof proceeds as follows. The statement of this lemma with 'isomorphism' replaced by 'Fredholm of index 0' follows from [4, Lemma 2.1] (which in turn essentially quotes [7]), since, keeping in mind that $\Delta = x^{n/2+1}P_b x^{-n/2+1}$ with the notation of that paper, $P_b : x^{-1}H^2_{\mathrm{b}}(X) \to xL^2_{\mathrm{b}}(X)$ is shown to be Fredholm of index 0 there. By Lemma 2.2 of [4] elements of the nullspace of Δ would necessarily be in $x^{n/2-1}H^\infty_{\mathrm{b}}(X)$. One deduces that $du \in L^2_{\mathrm{sc}}(X; {}^{\mathrm{sc}}T^*X)$, and a regularization argument allows one to conclude from $\Delta u = 0$ that $du = 0$, and then that $u = 0$.

Let $\phi \in \mathcal{C}^\infty(X)$ equal 1 on a collar neighborhood of ∂X of the form $\{x < \epsilon\}$ and equal 0 on $\{x > 2\epsilon\}$. By integrating the inequality Lemma 3.2 in the angular variables, i.e. along ∂X in the collar neighborhood, we have

$$\begin{aligned}\left\|x^{1+s}\phi u\right\|^2 &\lesssim \left\|x^{2+s}\partial_x(\phi u)\right\|^2\\ &\lesssim \|x^s\nabla_g(\phi u)\|^2\\ &\lesssim \|x^s\nabla u\|^2 + \|\phi' u\|^2.\end{aligned}$$

(We use the notation $f \lesssim g$ to indicate that there exists $C > 0$ such that $|f| \le Cg$.) So overall we obtain

$$\left\|x^{1+s}u\right\|^2 \lesssim \|x^s\nabla u\|^2 + \|\phi' u\|^2 + \|(1-\phi)u\|^2.$$

Now by compact support we certainly have

$$\phi', (1-\phi) \lesssim x^{s+\epsilon},$$

for all $\epsilon > 0$, hence by Lemma 3.1 (with $l = n/2 - s > 1$, $l' = n/2 - s - \epsilon$),

$$\|\phi' u\|^2 + \|(1-\phi)u\|^2 \lesssim \|x^s\nabla u\|^2,$$

and the desired estimate follows. □

Interpolating between $\|x^s u\|_{L^2_g(X)} \le \|x^s u\|_{L^2_g(X)}$ and Proposition 3.4, we immediately deduce:

Corollary 3.5. *For $s < (n-2)/2$, $u \in x^{-s}H^1_{\mathrm{sc}}(X)$,*

$$\|x^{s+\theta}u\|_{L^2_g(X)} \le C\|x^s\nabla_g u\|^{\theta}_{L^2_g(X)}\|x^s u\|^{1-\theta}_{L^2_g(X)},\ 0 \le \theta \le 1. \tag{3.5}$$

In particular, if $n \ge 3$, $s = 0$, then for $u \in H^1_{\mathrm{sc}}(X)$,

$$\|x^{\theta}u\|_{L^2_g(X)} \le C\|\nabla_g u\|^{\theta}_{L^2_g(X)}\|u\|^{1-\theta}_{L^2_g(X)},\ 0 \le \theta \le 1. \tag{3.6}$$

Of course, we can estimate $\nabla_g u$ with a right side of similar form: as $\Delta_g = \nabla_g^*\nabla_g$,

$$\|\nabla_g u\|^2_{L^2_g(X)} = \langle \Delta_g u, u\rangle \le \|\Delta_g u\|_{L^2_g(X)}\|u\|_{L^2_g(X)}. \tag{3.7}$$

Note also that if $Q \in x\mathcal{V}_{\mathrm{b}}(X) = \mathcal{V}_{\mathrm{sc}}(X)$ then

$$\|Qu\|_{L^2_g(X)} \le C\|\nabla_g u\|_{L^2_g(X)}. \tag{3.8}$$

We can also consider $P = \Delta_g + V$, $V \in S^{-2-\rho}(X)$, $V \ge 0$, $\rho > 0$ as beforehand. Then

$$\|\nabla_g u\|^2_{L^2_g(X)} = \langle \Delta_g u, u\rangle \le \langle (\Delta_g + V)u, u\rangle \le \|(\Delta_g + V)u\|_{L^2_g(X)}\|u\|_{L^2_g(X)}. \tag{3.9}$$

4. Weighted estimates for $\Delta_g + V$

We assume throughout this section that $n \ge 3$, g is a scattering metric in the sense of (3.1) with g_1 satisfying (3.2), $V \in S^{-2-\rho}(X)$, $V \ge 0$, $\rho > 0$. However, there exists $c > 0$ such that if $V \ge -cx^2$ then our arguments go through with minor modifications (indicated in a footnote).[5] As below only $L^2_g(X)$ is of interest, we will write $L^2(X) = L^2_g(X)$ henceforth.

[5] We need c such that for some $\delta \in (0,1)$, $c\|x^{1+s}u\|^2 \le (1-\delta)\|x^s\nabla_g u\|^2$ for $s \ge 0$ sufficiently close to 0. The existence of such $c > 0$ follows immediately from the Poincaré inequality, and that the constant in it can be chosen uniformly in s. Note that as $\rho > 0$, $V \ge -cx^2$ is only a constraint on V in a compact subset of X°.

For $0 \le s \le 1$, $u \in \dot{\mathcal{C}}^\infty(X)$, we now compute

$$\begin{aligned}\|x^s\nabla_g u\|^2_{L^2(X)} &= \langle \nabla_g u, x^{2s}\nabla_g u\rangle = \langle \Delta_g u, x^{2s}u\rangle + \langle \nabla_g u, [\nabla_g, x^{2s}]u\rangle \\ &= \langle (\Delta_g + V)u, x^{2s}u\rangle - \langle Vu, x^{2s}u\rangle + \langle \nabla_g u, [\nabla_g, x^{2s}]u\rangle.\end{aligned} \tag{4.1}$$

Now, for $0 \le s \le 1/2$,

$$\begin{aligned}|\langle (\Delta_g + V)u, x^{2s}u\rangle| &\le \|(\Delta_g + V)u\|_{L^2(X)}\|x^{2s}u\|_{L^2(X)} \\ &\le C\|(\Delta_g + V)u\|_{L^2(X)}\|\nabla_g u\|^{2s}_{L^2(X)}\|u\|^{1-2s}_{L^2(X)},\end{aligned} \tag{4.2}$$

where we used (3.6). On the other hand

$$[\nabla_g, x^{2s}] = x^{2s+1}f,\ f \in \mathcal{C}^\infty(X; TX),$$

and $\sup|f| \le C_0 s$, so using Proposition 3.4 and $n \ge 3$

$$\begin{aligned}|\langle \nabla_g u, [\nabla_g, x^{2s}]u\rangle| &\le C_0 s\|x^s\nabla_g u\|_{L^2(X)}\|x^{s+1}u\|_{L^2(X)} \\ &\le C_0 C s\|x^s\nabla_g u\|_{L^2(X)}\|x^s\nabla_g u\|_{L^2(X)} = C_0 C s\|x^s\nabla_g u\|^2_{L^2(X)},\end{aligned}$$

and for s sufficiently small this can be absorbed into the left hand side of (4.1). Since[6] $\langle Vu, x^{2s}u\rangle \ge 0$, we deduce from (4.1) that there exists $s_0 > 0$ such that for $0 \le s \le s_0$,

$$\|x^s\nabla_g u\|^2_{L^2(X)} \le C\|(\Delta_g + V)u\|_{L^2(X)}\|\nabla_g u\|^{2s}_{L^2(X)}\|u\|^{1-2s}_{L^2(X)}; \tag{4.3}$$

indeed this holds even with $\langle Vu, x^{2s}u\rangle$ added to the left hand side. Although we had assumed $u \in \dot{\mathcal{C}}^\infty(X)$, by density and continuity, the estimate holds for $u \in H^2_{\mathrm{sc}}(X)$, i.e. for u in the domain of $\Delta_g + V$. Using the Poincaré inequality, Proposition 3.4, we deduce [7]:

Proposition 4.1. *There exists $s_0 > 0$ such that for $0 \le s \le s_0$*

$$\begin{aligned}&\|x^{s+1}u\|_{L^2(X)} + \|x^s\nabla_g u\|_{L^2(X)} \\ &\quad\le C_s\|(\Delta_g + V)u\|^{1/2}_{L^2(X)}\|\nabla_g u\|^s_{L^2(X)}\|u\|^{1/2-s}_{L^2(X)},\ u \in H^2_{\mathrm{sc}}(X).\end{aligned} \tag{4.7}$$

In particular, for $L \in S^{-1-s}\mathrm{Diff}^1_{\mathrm{b}}(X)$,

$$\|Lu\|_{L^2(X)} \le C_s\|(\Delta_g + V)u\|^{1/2}_{L^2(X)}\|\nabla_g u\|^s_{L^2(X)}\|u\|^{1/2-s}_{L^2(X)},\ u \in H^2_{\mathrm{sc}}(X). \tag{4.8}$$

Since any $L \in S^{-2-2s}\mathrm{Diff}^2_{\mathrm{b}}(X)$ can be rewritten as $L = \sum Q_i^* R_i$, $Q_i, R_i \in S^{-1-s}\mathrm{Diff}^1_{\mathrm{b}}(X)$, with the sum finite, we immediately deduce

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[6] If only $V \ge -cx^2$ is assumed, then

$$\langle Vu, x^{2s}u\rangle \ge -c\langle x^{2+2s}u, u\rangle \ge -(1-\delta)\|x^s\nabla_g u\|^2$$

allows us to absorb $\langle Vu, x^{2s}u\rangle$ into the left hand side of (4.1); as a result, s_0 is reduced in this case.

[7] If $n \ge 5$, one can use Lemma 3.3 (or its analogue if $V \ge 0$) to obtain an estimate that slightly shortens some of the arguments that follow; one then needs to rely on the lemma, i.e. on b-machinery. Namely, by Lemma 3.3, if $n \ge 5$,

$$\|xQ_iu\|_{L^2(X)} \le C\|\Delta_g u\|_{L^2(X)},\ \|x^2u\|_{L^2(X)} \le C\|\Delta_g u\|_{L^2(X)}. \tag{4.4}$$

On the other hand,

$$\|Q_iu\|_{L^2(X)} \le C\|\nabla_g u\|_{L^2(X)}. \tag{4.5}$$

Interpolating between the first inequality of (4.4) and (4.5) gives for $n \ge 5$

$$\|x^sQ_iu\|_{L^2(X)} \le C\|\nabla_g u\|^{1-s}_{L^2(X)}\|\Delta_g u\|^s_{L^2(X)},\ 0 \le s \le 1. \tag{4.6}$$

Corollary 4.2. *Let* $s_0 > 0$ *be as in Proposition 4.1. For* $0 \le s \le s_0$, $L \in S^{-2-2s}\mathrm{Diff}^2_{\mathrm{b}}(X)$,

$$|\langle Lu, u\rangle| \le C_s \|(\Delta_g + V)u\|_{L^2(X)} \|\nabla_g u\|^{2s}_{L^2(X)} \|u\|^{1-2s}_{L^2(X)},\ u \in H^2_{\mathrm{sc}}(X). \tag{4.9}$$

In fact we can improve upon these results, allowing, among other things, estimates for $\|x^s \nabla u\|_{L^2(X)}$ the full range $0 \le s < (n-2)/2$. In our applications to Mourre estimates we only require the $s = 0$ case, so we confine the proof of the following strengthening of Proposition 4.1 to an appendix. (See (A.5) for the estimate of $\|x^s \nabla u\|_{L^2(X)}$.)

Proposition 4.3. *For* $0 \le s < 1/2$

$$\begin{aligned} &\|x^{s+1}u\|_{L^2(X)} + \|x^s \nabla_g u\|_{L^2(X)} \\ &\qquad \le C_s \|(\Delta_g + V)u\|^{1/2}_{L^2(X)} \|\nabla_g u\|^{s}_{L^2(X)} \|u\|^{1/2-s}_{L^2(X)} \\ &\qquad \le C_s \|(\Delta_g + V)u\|^{(1+s)/2}_{L^2(X)} \|u\|^{(1-s)/2}_{L^2(X)},\ u \in H^2_{\mathrm{sc}}(X). \end{aligned} \tag{4.10}$$

In particular, for $L \in S^{-1-s}\mathrm{Diff}^1_{\mathrm{b}}(X)$,

$$\begin{aligned} \|Lu\|_{L^2(X)} &\le C_s \|(\Delta_g + V)u\|^{1/2}_{L^2(X)} \|\nabla_g u\|^{s}_{L^2(X)} \|u\|^{1/2-s}_{L^2(X)} \\ &\le C_s \|(\Delta_g + V)u\|^{(1+s)/2}_{L^2(X)} \|u\|^{(1-s)/2}_{L^2(X)},\ u \in H^2_{\mathrm{sc}}(X). \end{aligned} \tag{4.11}$$

Using again that any $L \in S^{-2-2s}\mathrm{Diff}^2_{\mathrm{b}}(X)$ can be rewritten as $L = \sum Q_i^* R_i$, $Q_i, R_i \in S^{-1-s}\mathrm{Diff}^1_{\mathrm{b}}(X)$, with the sum finite, we conclude

Corollary 4.4. *For* $0 \le s < 1/2$, $L \in S^{-2-2s}\mathrm{Diff}^2_{\mathrm{b}}(X)$,

$$\begin{aligned} |\langle Lu, u\rangle| &\le C_s \|(\Delta_g + V)u\|_{L^2(X)} \|\nabla_g u\|^{2s}_{L^2(X)} \|u\|^{1-2s}_{L^2(X)} \\ &\le C_s \|(\Delta_g + V)u\|^{1+s}_{L^2(X)} \|u\|^{1-s}_{L^2(X)},\ u \in H^2_{\mathrm{sc}}(X). \end{aligned} \tag{4.12}$$

Now, suppose that $u = \psi(H^2(\Delta_g + V))v$, $v \in L^2(X)$, where $\psi \in L^\infty_c(I)$, $I \subset (0, \infty)$ compact, $0 \le \psi \le 1$, $H > 0$. Then $u \in H^2_{\mathrm{sc}}(X)$ and

$$C_I' \|u\|_{L^2(X)} \le \|H^2(\Delta_g + V)u\|_{L^2(X)} \le C_I \|u\|_{L^2(X)}$$

and

$$C_I' \|u\|^2_{L^2(X)} \le \langle H^2(\Delta_g + V)u, u\rangle \le C_I \|u\|^2_{L^2(X)}.$$

Combining these with Corollary 4.4 we deduce that for $L \in S^{-2-\sigma}\mathrm{Diff}^2_{\mathrm{b}}(X)$ with $0 \le \sigma < 1$,

$$|\langle Lu, u\rangle| \le C' C_I^{1+\sigma/2} H^{-2-\sigma} \|u\|^2_{L^2(X)}. \tag{4.13}$$

Note that $|\langle Vu, u\rangle|$ satisfies the same estimate as $|\langle Lu, u\rangle|$. If $\sigma > 0$ this gives a gain of $H^{-\sigma}$ over e.g. $\langle(\Delta + V)u, u\rangle$ as $H \to \infty$; ultimately, this gain arose due to the Poincaré estimate in (4.2). We also remark that (4.11) yields for $L \in S^{-1-s}\mathrm{Diff}^1_{\mathrm{b}}(X)$, $0 \le s < 1/2$,

$$\|Lu\|_{L^2(X)} \le C C_I^{(1+s)/2} H^{-1-s} \|u\|_{L^2(X)}. \tag{4.14}$$

The estimates (4.13)–(4.14) are analogues of Lemma B.12 of [1], with $\lambda = H^2$ in their notation: one can trade powers of x for negative powers of H (within limits), i.e. in the notation of [1], one can trade negative powers of $\langle x\rangle$ for powers of $\lambda^{-1/2}$ (see the exponent γ in [1]).

Below, we also need an analogous result in which the resolvent appears in place of the compactly supported functions of $P = \Delta_g + V$. Thus, for $L \in S^{-1-s}\mathrm{Diff}^1_{\mathrm{b}}(X)$, $u \in L^2(X)$, replacing u by $(\Delta_g + V - z)^{-1}u$, and using $\|(\Delta_g + V - z)^{-1}\|_{\mathcal{L}(L^2(X))} \leq |\operatorname{Im} z|^{-1}$ (for $\operatorname{Im} z \neq 0$), we deduce that

$$\begin{aligned}
&\|L(\Delta + V - z)^{-1}u\|_{L^2(X)} \\
&\leq C\|(\mathrm{Id} + z(\Delta_g + V - z)^{-1})u\|^{(1+s)/2}_{L^2(X)} \|(\Delta_g + V - z)^{-1}u\|^{(1-s)/2}_{L^2(X)} \\
&\leq C(1 + |z|/|\operatorname{Im} z|)^{(1+s)/2}\|u\|^{(1+s)/2}_{L^2(X)} |\operatorname{Im} z|^{-(1-s)/2}\|u\|^{(1-s)/2}_{L^2(X)} \\
&\leq 2C(|z|/|\operatorname{Im} z|)^{(1+s)/2}|\operatorname{Im} z|^{-(1-s)/2}\|u\|_{L^2(X)}.
\end{aligned} \tag{4.15}$$

In addition, using the positivity of $\Delta_g + V$, we have for z with $\operatorname{Re} z < 0$,

$$\|(\Delta_g + V - z)^{-1}\|_{\mathcal{L}(L^2(X))} \leq |z|^{-1},$$

so in fact

$$\|L(\Delta + V - z)^{-1}u\|_{L^2(X)} \leq 2C|z|^{-(1-s)/2}\|u\|_{L^2(X)}, \ \operatorname{Re} z < 0. \tag{4.16}$$

Replacing z by $z = w/H^2$, we deduce the following:

Proposition 4.5. *Suppose $L \in S^{-1-s}\mathrm{Diff}^1_{\mathrm{b}}(X)$, $0 \leq s < 1/2$. Then there exists $C > 0$ such that for all $u \in L^2(X)$ we have*

(4.17)

$$\begin{aligned}
&\|L(H^2(\Delta + V) - w)^{-1}u\|_{L^2(X)} \\
&\qquad \leq 2CH^{-1-s}(|w|/|\operatorname{Im} w|)^{(1+s)/2}|\operatorname{Im} w|^{-(1-s)/2}\|u\|_{L^2(X)}, \ \operatorname{Im} w \neq 0, \\
&\|L(H^2(\Delta + V) - w)^{-1}u\|_{L^2(X)} \leq 2CH^{-1-s}|w|^{-(1-s)/2}\|u\|_{L^2(X)}, \ \operatorname{Re} w < 0.
\end{aligned}$$

In particular, this gives uniform bounds on $L(H^2(\Delta + V) - w)^{-1}$ in $\mathcal{L}(L^2(X))$ for H large.

5. Low frequency Mourre estimate

We now prove the low frequency Mourre estimate.

Let $\phi \in \mathcal{C}^\infty_c(X)$ be chosen as above, i.e. let it be identically 1 near ∂X, supported in a collar neighborhood of ∂X, on which xD_x is thus defined, and let

$$A = -\frac{1}{2}\big((\phi x D_x) + (\phi x D_x)^*\big).$$

Since (cf. (3.3))

$$\Delta_g = \sum Q_i^* G_{ij} Q_j = (x^2 D_x)^*(x^2 D_x) + x^2 d^*_{\partial X} d_{\partial X} + x^{2+\rho} R,$$

where

$$Q_i \in \mathcal{V}_{\mathrm{sc}}(X), \ G_{ij} \in S^0(X), \ R \in S^0\mathrm{Diff}^2_{\mathrm{b}}(X),$$

we have

$$[\Delta_g + V, A] = -2i(\Delta_g + L), \ L \in S^{-2-\rho}\mathrm{Diff}^2_{\mathrm{b}}(X), \tag{5.1}$$

at first as a quadratic form on $\dot{\mathcal{C}}^\infty(X)$, but then noting that the right hand side extends (by density) to a continuous map from $H^2_{\mathrm{sc}}(X)$ to $L^2(X)$. For $u = \psi(H^2(\Delta_g + V))v$, $v \in L^2(X)$, we now use Corollary 4.4. Thus, without loss of generality taking $\rho < 1$, (4.13) gives

$$|\langle Lu, u\rangle| \leq C' C_I^{1+\rho/2} H^{-2-\rho}\|u\|^2_{L^2(X)}.$$

Note that $|\langle Vu, u\rangle|$ satisfies the same estimate as $|\langle Lu, u\rangle|$.

In summary,

$$\begin{aligned}\langle \frac{i}{2}[\Delta_g + V, A]u, u\rangle &= \Big\langle \Big(\Delta_g + V + L - V\Big)u, u\Big\rangle \\ &\geq \langle(\Delta_g + V)u, u\rangle - CH^{-2-\rho}\|u\|^2_{L^2(X)} = H^{-2}\langle (H^2(\Delta_g + V) - CH^{-\rho})u, u\rangle.\end{aligned}$$

We thus deduce that there exist $H_0 > 0$ and $C' > 0$ such that for $H > H_0$,

$$\langle \frac{i}{2}[H^2(\Delta_g + V), A]u, u\rangle \geq C'\|u\|^2,\ u = \psi(H^2(\Delta_g + V))v.$$

Now let $\psi = \chi_I$, the characteristic function of I, we deduce the following:

Theorem 5.1. *Suppose $n \geq 3$, g is a scattering metric in the sense of* (3.1) *with g_1 satisfying* (3.2)*, $V \in S^{-2-\rho}(X)$, $\rho > 0$, $V \geq 0$, $P = \Delta_g + V$. Let $I \subset (0, \infty)$ be a compact interval, and χ_I the characteristic function of I. Then there exist $H_0 > 0$ and $C > 0$ such that for $H > H_0$,*

$$\chi_I(H^2P)\frac{i}{2}[H^2P, A]\chi_I(H^2P) \geq C\chi_I(H^2P).$$

In particular for $\psi \in \mathcal{C}^\infty((0, \infty))$,

$$\psi(H^2P)\chi_I(H^2P)\frac{i}{2}[H^2P, A]\chi_I(H^2P)\psi(H^2P) \geq C(\inf_I \psi)^2\chi_I(H^2P). \tag{5.2}$$

Remark 5.2. The commutator is defined here as a quadratic form on $\dot{\mathcal{C}}^\infty(X)$, which extends to $H^2_{\mathrm{sc}}(X)$ continuously. If $\psi \in \mathcal{C}^\infty_c(I)$ then for $v \in \dot{\mathcal{C}}^\infty(X)$ one has $u \in \dot{\mathcal{C}}^\infty(X)$ by the functional calculus in the algebra of scattering pseudodifferential operators – the main point here is that the decay properties are preserved, see [5, Theorem 11]. (One can also obtain this decay without using the full ps.d.o. algebra, working with the Helffer-Sjöstrand formula and commutators directly, if one so desires.) Thus, for such v and ψ, one can expand the commutator and manipulate it directly, which is important in applications.

This at once implies the corresponding estimate with H^2P replaced by $H\sqrt{P}$, which is the main content of [1, Proposition 3.1] when $X = \mathbb{R}^n$ equipped with a metric asymptotic to the standard Euclidean metric. In order to do this recall that $H^{m,l}_{\mathrm{sc}}(X) = x^l H^m_{\mathrm{sc}}(X)$ is the scattering Sobolev space of Melrose [8], which for X the radial compactification of $\mathbb{R}^n$ is just the standard weighted Sobolev space $H^{m,l}(\mathbb{R}^n)$, and one has the high energy estimate that $(P + \lambda)^{-1} : H^{m,l}_{\mathrm{sc}}(X) \to H^{m,l}_{\mathrm{sc}}(X)$, $P = \Delta_g + V$, is bounded by $C\lambda^{-1}$ in $\lambda > 1$ from the semiclassical scattering calculus; this is of course very easy to see for $l = 0$, which is what we need below. (Recall that we are using the nonnegative Laplace operator.) Now, one has by the functional calculus

$$\sqrt{P} = \pi^{-1}\int_0^\infty \lambda^{-1/2}P(P + \lambda)^{-1}\,d\lambda,$$

so

$$H\sqrt{P} = \pi^{-1}\int_0^\infty \lambda^{-1/2}H^2P(H^2P + \lambda)^{-1}\,d\lambda;$$

using the above observation, the integral converges for any m as a bounded operator in $\mathcal{L}(H^{m,0}_{\mathrm{sc}}(X), H^{m-2,0}_{\mathrm{sc}}(X))$. We now evaluate the commutator $[H\sqrt{P}, A]$:

$H^{m,1}_{\mathrm{sc}}(X) \to H^{m-3,-1}_{\mathrm{sc}}(X)$; the integral for the products $H\sqrt{P}A$ and $AH\sqrt{P}$ converges in this sense. As

$$[H^2P(H^2P+\lambda)^{-1}, A] = \lambda(H^2P+\lambda)^{-1}[H^2P, A](H^2P+\lambda)^{-1},$$

using $(t+\lambda)^{-1} \geq (\sup I+\lambda)^{-1}$ on I, we deduce from (5.2) that for $H > H_0$,

(5.3)

$$\begin{aligned} &i\chi_I(H^2P)[H\sqrt{P}, A]\chi_I(H^2P) \\ &\quad = \pi^{-1}\int_0^\infty \lambda^{1/2}(H^2P+\lambda)^{-1}\chi_I(H^2P)i[H^2P, A]\chi_I(H^2P)(H^2P+\lambda)^{-1}\,d\lambda \\ &\quad \geq \pi^{-1}\int_0^\infty C\lambda^{1/2}(\sup I+\lambda)^{-2}\chi_I(H^2P)^2\,d\lambda = C'\chi_I(H^2P)^2,\ C'>0, \end{aligned}$$

on $H^{3,1}_{\mathrm{sc}}(X)$, hence by density of $H^{3,1}_{\mathrm{sc}}(X)$ and continuity of both sides on $L^2(X)$, on $L^2(X)$. Thus, the analogue of the low energy Mourre estimate of Bony and Häfner in this more general setting follows immediately.

Theorem 5.3. *Suppose $n \geq 3$, g is a scattering metric in the sense of* (3.1) *with g_1 satisfying* (3.2)*, $V \in S^{-2-\rho}(X)$, $\rho > 0$, $V \geq 0$, $P = \Delta_g + V$. Let $I \subset (0,\infty)$ be a compact interval, and χ_I the characteristic function of I. Then there exist $H_0 > 0$ and $C > 0$ such that for $H > H_0$,*

$$\chi_I(H^2P)\frac{i}{2}[H\sqrt{P}, A]\chi_I(H^2P) \geq C\chi_I(H^2P).$$

6. Energy decay for the wave equation

If the metric on X is additionally assumed to be non-trapping, we have a finite- and high-energy Mourre estimate due to Vasy-Zworski [9] (or can re-use the construction employed in [1]). Putting these ingredients together as in Theorem 1.3 of [1] we obtain by the same means the analogous energy decay result for solutions to the wave equation; for brevity, we confine our discussion of these results to the case of (unperturbed) scattering metrics, i.e. those given by (3.1) near infinity.

Theorem 6.1. *Let (X, g) be a scattering manifold having no trapped geodesics, and let $V \in S^{-3}(X)$ be a nonnegative potential. If*

$$\big(D_t^2 - (\Delta_g + V)\big)u = 0$$

on $\mathbb{R} \times X$, then for all $\epsilon > 0$ and $\mu \in (0,1]$,

$$\|x^\mu u'\|_{L^2([0,T]\times X)} \lesssim \big\langle F^\epsilon_\mu(T)\big\rangle^{1/2}\|u'(0,\cdot)\|_{L^2(X)}$$

where $u' = (\partial_t u, \nabla_g u)$. and

$$F^\epsilon_\mu(T) = \begin{cases} T^{1-2\mu-2\epsilon}, & \mu \leq 1/2 \\ 1 & \mu > 1/2 \end{cases}.$$

Proof. As indicated above, the relevant medium and high energy estimates are well known in this setting, and it will suffice, following the strategy of [1], to demonstrate that the low energy commutant that we have constructed above satisfies all of the hypotheses of the Mourre theory. As discussed in Proposition 3.1 of [1], it remains for us to verify, in our notation, the following estimates on the operator

$$\mathcal{A}_H \equiv \psi(H^2P)A\psi(H^2P):$$

$$\Big\|[\mathcal{A}_H, HP^{1/2}]\Big\| \lesssim 1, \tag{6.1}$$

$$\Big\|\,[\mathcal{A}_H,[\mathcal{A}_H, HP^{1/2}]]\Big\| \lesssim 1, \tag{6.2}$$

$$\||\mathcal{A}_H|^\mu x^\mu\| \lesssim H^{-\mu}, \quad \mu \in [0,1] \tag{6.3}$$

$$\big\|\langle \mathcal{A}_H\rangle^\mu \psi(H^2P)x^\mu\big\| \lesssim H^{-\mu}, \quad \mu \in [0,1]. \tag{6.4}$$

We begin by proving a lemma allowing us to commute powers of x with spectral projections:

Lemma 6.2. *Let $L \in x\,\mathrm{Diff}^1_{\mathrm{sc}}(X)$. The operators*

$$x^{-1}\psi(H^2P)L \text{ and } L\psi(H^2P)x^{-1}$$

are uniformly L^2-bounded as $H \uparrow \infty$.

Proof of lemma: As the two types of operator in question are adjoints of one another, it suffices to consider the latter. Moreover, for the desired boundedness it suffices to estimate $L[\psi(H^2P), x^{-1}]$.

Letting $\tilde\psi$ be a compactly supported almost-analytic extension of ψ. Let $R(z)$ denote the resolvent

$$R(z) = (H^2P - z)^{-1}.$$

We have

$$\begin{aligned} L[\psi(H^2P), x^{-1}] &= \frac{1}{2\pi}\int_{\mathbb{C}} \overline{\partial}\tilde\psi(z) L[R(z), x^{-1}]\, dz d\overline{z} \\ &= -\frac{H^2}{2\pi}\int_{\mathbb{C}} \overline{\partial}\tilde\psi(z) LR(z)[P, x^{-1}]R(z)\, dz d\overline{z}. \end{aligned}$$

As $[P, x^{-1}] = Q \in x\,\mathrm{Diff}^1_b(X)$, (4.17) gives (with $s = 0$) that

$$\begin{aligned} \|LR(z)\|_{\mathcal{L}(L^2(X))} &\le CH^{-1}|\operatorname{Im} z|^{-1}|z|^{1/2}, \\ \|QR(z)\|_{\mathcal{L}(L^2(X))} &\le CH^{-1}|\operatorname{Im} z|^{-1}|z|^{1/2}, \end{aligned}$$

so we can estimate the integral by a multiple of

$$H^2 \int_{\mathbb{C}} \big|\overline{\partial}\tilde\psi(z)\big| H^{-2} |\operatorname{Im} z|^{-2}|z|\, dz d\overline{z}.$$

This concludes the proof of the lemma.

We now sketch the proofs of (6.1)–(6.4). The estimate (6.1) follows from (5.3), as we may again write

$$\begin{aligned} &\psi(H^2P)[H\sqrt{P}, A]\psi(H^2P) \\ &\quad = \pi^{-1}\int_0^\infty \lambda^{1/2} R(-\lambda)\psi(H^2P)[H^2P, A]\psi(H^2P)R(-\lambda)\, d\lambda. \end{aligned} \tag{6.5}$$

By anti-self-adjointness of the commutator, it suffices to estimate the norm of

$$\begin{aligned} &\Big\langle [\mathcal{A}_H, HP^{1/2}]u, u\Big\rangle \\ &= \pi^{-1}\int_0^\infty \lambda^{1/2}\big\langle R(-\lambda)\psi(H^2P)[H^2P, A]\psi(H^2P)R(-\lambda)u, u\big\rangle\, d\lambda. \end{aligned}$$

Now as $[H^2P, A] \in x^2\,\mathrm{Diff}^2_b(X)$, we may rewrite this pairing in the form

$$H^2 \int_0^\infty \lambda^{1/2}\big\langle xM_1\psi(H^2P)R(-\lambda)u, xM_2\psi(H^2P)R(-\lambda)u\big\rangle\, d\lambda$$

with $M_i \in \mathrm{Diff}_b^1(X)$. Applying (4.14) (in the 'easy' case $s=0$) yields (6.1).

We can now prove (6.2) in the same manner (cf. Remark 3.5 in [1]): By (5.1) we have

$$[A, P] = 2iP + M,$$

where

$$M \in S^{-3}\,\mathrm{Diff}_b^2(X).$$

Thus,

$$[\mathcal{A}_H, HP] = 2iP + M,$$

Thus

(6.6)

$$\begin{aligned}
[\mathcal{A}_H, HP^{1/2}] &= \psi(H^2P)[A, H\sqrt{P}]\psi(H^2P)\\
&= \pi^{-1}\int_0^\infty \lambda^{1/2}R(-\lambda)\psi(H^2P)[A, H^2P]\psi(H^2P)R(-\lambda)\,d\lambda\\
&= \pi^{-1}\int_0^\infty \lambda^{1/2}R(-\lambda)\psi(H^2P)(2iP+M)\psi(H^2P)R(-\lambda)\,d\lambda\\
&= 2i\psi(H^2P)^2HP^{1/2} + \pi^{-1}\int_0^\infty \lambda^{1/2}R(-\lambda)\psi(H^2P)M\psi(H^2P)R(-\lambda)\,d\lambda\\
&= 2i\psi(H^2P)^2HP^{1/2} + \mathcal{B}
\end{aligned}$$

Hence to estimate

$$[\mathcal{A}_H, [\mathcal{A}_H, HP^{1/2}]],$$

by (6.1), it suffices to estimate

$$[\mathcal{A}_H, \mathcal{B}];$$

to do this, noting that $\mathcal{A}_H$ is self-adjoint, and $\mathcal{B}$ is anti-self-adjoint, we see that it suffices to obtain boundedness of

$$\mathcal{A}_H\mathcal{B} = (\mathcal{A}_H x)(x^{-1}\mathcal{B}).$$

Now application of Lemma 6.2 shows that

$$\mathcal{A}_H x = (\psi(H^2P)Ax)(x^{-1}\psi(H^2P)x)$$

is uniformly bounded. By similar considerations, $x^{-1}\mathcal{B}$ is bounded: we write the integrand for $x^{-1}\mathcal{B}$ as

$$\begin{aligned}
&\lambda^{1/2}x^{-1}R(-\lambda)\psi(H^2P)M\psi(H^2P)R(-\lambda)\\
&\qquad = \sum_j \lambda^{1/2}\big(x^{-1}R(-\lambda)\psi(H^2P)(xL_{j1})\big)\big(L_{j2}\psi(H^2P)R(-\lambda)\big).
\end{aligned}$$

where $M = \sum_j xL_{j1}L_{j2}$, $L_{ji} \in x\,\mathrm{Diff}_b^1(X)$. By (4.14) the last factor is norm bounded by a multiple of $H^{-1}(c+\lambda)^{-1}$, $c = \inf\operatorname{supp}\psi > 0$. Commuting the factor of x^{-1} across both $R(-\lambda)$ and $\psi(H^2P)$ yields an operator bounded by a multiple of $H\lambda^{-1}$ by (4.14), while the commutator terms involved in doing this have the same bound by Lemma 6.2 and the observation that

$$[x^{-1}, R(-\lambda)]\psi(H^2P) = -R(-\lambda)QR(-\lambda)\psi(H^2P)$$

with $Q \in x\,\mathrm{Diff}_b^1(X)$; this expression is bounded by a multiple of $H^{-1}(c+\lambda)^{-1}$ by (4.17). We thus obtain (6.2).

To prove (6.3), it suffices by interpolation to prove uniform boundedness as $H \uparrow \infty$ of

$$\|H\psi(H^2P)A\psi(H^2P)x\|_{\mathcal{L}(L^2(X))};$$

as above, this follows from Lemma 6.2. Likewise, (6.4) follows by interpolation with the $\mu = 1$ estimate

$$\|H\psi(H^2P)A\psi(H^2P)x\|^2_{\mathcal{L}(L^2(X))} + \|\psi(H^2P)x\|^2_{\mathcal{L}(L^2(X))}.$$

□

Appendix A. Proof of Proposition 4.3

Rather than working with $\langle x^{2s}(\Delta_g+V)u, u\rangle$, and rewriting it in terms of $\|x^s\nabla_g u\|^2_{L^2(X)}$ plus a commutator, we now work with a symmetric expression:

$$\langle f(\Delta_g + V)u, u\rangle + \langle u, f(\Delta_g + V)u\rangle$$

for some f which behaves like x^{2s} for small x. First we compute

$$f(\Delta_g + V) + (\Delta_g + V)f = 2\nabla_g^* f\nabla_g + ((\Delta_g + 2V)f),$$

where the last term on the right hand side is multiplication by the function $(\Delta_g + 2V)f$, which can be seen by observing that both sides are real self-adjoint second order scalar differential operators with the same principal symbol, so their difference is first order, hence by reality and self-adjointness zeroth order, and it vanishes on the constant function 1. Now if $f \geq 0$ then $Vf \geq 0$, so all terms on the right hand side are positive provided $\Delta_g f \geq 0$, and we have

$$\begin{aligned} 2\langle f\nabla_g u, \nabla_g u\rangle + \langle((\Delta_g f)u, u\rangle + \langle 2Vfu, u\rangle \\ = \langle f(\Delta_g + V)u, u\rangle + \langle u, f(\Delta_g + V)u\rangle, \end{aligned}$$

hence

$$\|f^{1/2}\nabla_g u\|^2 \leq \|(\Delta_g + V)u\| \, \|fu\|. \tag{A.1}$$

It remains to find $f \geq 0$ such that $\Delta_g f \geq 0$; we remind the reader that this is the *positive* Laplacian.

With $t_0 > 0$ to be fixed, we consider

$$\begin{aligned} &\chi(t) = e^{1/(t-t_0)}, \ t < t_0, \\ &\chi(t) = 0, \ t \geq t_0, \end{aligned}$$

and define f by

$$\begin{aligned} &f(p) = \mathsf{g}(p)^{2s}, \text{where} \\ &\mathsf{g}(p) = \chi(0) - \chi(x(p)/\epsilon), \ p \in X, \end{aligned} \tag{A.2}$$

where $\epsilon > 0$. For $\epsilon > 0$ is sufficiently small, $d\chi$ is supported in such a collar neighborhood of ∂X in which we can take x as one of the coordinates and g is of the form (3.1) with g_1 as in (3.2). Moreover, $\mathsf{g} \geq 0$ (hence $f \geq 0$), $\mathsf{g}'(0) > 0$, and $\partial_x \mathsf{g}(x = 0) = 0$, hence $f \sim x^{2s}$ for x near 0. As usual, we abuse notation and write $f = f(x)$. Recall that

$$\Delta_g = x^2\Delta_{\mathrm{b}}, \ \Delta_{\mathrm{b}} = -(x\partial_x)^2 + (n-2)(x\partial_x) + \Delta_0 + x^\rho R, \ R \in S^0\mathrm{Diff}^2_{\mathrm{b}}(X),$$

and R annihilates constants. We then compute, for $x/\epsilon < t_0$ (since $df = 0$ for $x/\epsilon \geq t_0$), i.e. with $t = x/\epsilon$ for $0 \leq t < t_0$, writing $f(p) = \mathsf{g}(p)^{2s}$, and primes denoting derivatives in t,

$$\begin{aligned}\big(-x^2\partial_x^2+(n-3)x\partial_x\big)f&=\big(-t^2\partial_t^2+(n-3)t\partial_t\big)\mathsf{g}^{2s}\\&=2s\mathsf{g}^{2s-2}\Big(-(2s-1)t^2(\mathsf{g}')^2+(n-3)t\mathsf{g}\mathsf{g}'-t^2\mathsf{g}\mathsf{g}''\Big).\end{aligned}$$

Now, for $0 \leq t < t_0$,

$$\begin{aligned}\mathsf{g}'&=(t-t_0)^{-2}e^{1/(t-t_0)}>0,\\ \mathsf{g}''&=(t-t_0)^{-4}\big(-1-2(t-t_0)\big)e^{1/(t-t_0)}.\end{aligned}$$

We deduce that for $t_0 < 1/2$, $\mathsf{g}'' < 0$ (on $[0, t_0)$). Thus, for $n \geq 3$, $0 < s < 1/2$,

$$\begin{aligned}&\big(-(x\partial_x)^2+(n-2)x\partial_x+\Delta_0\big)f\\&\qquad=2s\mathsf{g}^{2s-2}\Big(-(2s-1)t^2(\mathsf{g}')^2+(n-3)t\mathsf{g}\mathsf{g}'-t^2\mathsf{g}\mathsf{g}''\Big)\geq 0,\end{aligned}$$

i.e. the 'model Laplacian' of f is always non-negative provided $s \leq 1/2$.

If $n = 3$, we have obtained non-negativity of $\Delta_g f$ for the whole range $0 < s < (n-2)/2$. In general, however if $n > 3$ and $s \geq 1/2$, we need to estimate $t\mathsf{g}'$ relative to g. An estimate $t\mathsf{g}' \leq C\mathsf{g}$ is automatic for sufficiently large $C > 0$, as it is easily checked at 0, and g is bounded away from 0 elsewhere. However, we need a sharp constant, so we proceed as follows. A straightforward calculation gives

$$t\mathsf{g}'-\mathsf{g}=\Big(\frac{t}{(t-t_0)^2}+1\Big)e^{1/(t-t_0)}-e^{-1/t_0},$$

so $t\mathsf{g}' - \mathsf{g}$ vanishes at $t = 0$ and it is decreasing, as its derivative is

$$\frac{t}{(t-t_0)^4}\big(-1-2(t-t_0)\big)e^{1/(t-t_0)}\leq 0,\ 0\leq t<t_0,\ t_0<1/2,$$

so $t\mathsf{g}' \leq \mathsf{g}$ on $[0, t_0)$. In summary

$$\big(-t^2\partial_t^2+(n-3)t\partial_t\big)\mathsf{g}^{2s}=2s\mathsf{g}^{2s-2}\Big((n-2s-2)t\mathsf{g}'\mathsf{g}-t^2\mathsf{g}\mathsf{g}''\Big)\geq 0,$$

provided $1/2 \leq s < (n-2)/2$, so

$$\Big(-(x\partial_x)^2+(n-2)(x\partial_x)+\Delta_0\Big)f\geq 0$$

in this case.

We deduce that for any $0 < s < (n-2)/2$ we have

$$\Big(-(x\partial_x)^2+(n-2)(x\partial_x)+\Delta_0\Big)f\geq 0,$$

provided that we choose $0 < t_0 < 1/2$, and indeed we have the somewhat stronger estimate (useful for error terms below) that for $c > 0$ sufficiently small,

$$\Big(-(x\partial_x)^2+(n-2)(x\partial_x)+\Delta_0\Big)f\geq c\,\mathsf{g}^{2s-2}\Big(t^2(\mathsf{g}')^2-t^2\mathsf{g}\mathsf{g}''\Big),\tag{A.3}$$

where both summands on the right hand side are non-negative, and where we used $t\mathsf{g}' \leq \mathsf{g}$ in the case $s \geq 1/2$. Note that this estimate is valid for any choice of $\epsilon > 0$ provided it is sufficiently small (i.e. $\epsilon \leq \epsilon_1$, ϵ_1 suitably chosen) so that df is supported in the collar neighborhood of ∂X. We can also deal with the error term

$x^\rho R$ by letting $\epsilon \to 0$. Namely, on the support of Rf, $x \leq \epsilon$, so $x^\rho Rf \leq \epsilon^\rho Rf$, so $\Delta_g f \geq 0$ follows provided

$$Rf \leq C\Big(-(x\partial_x)^2 + (n-2)(x\partial_x)\Big) f \tag{A.4}$$

for some $C > 0$. But writing out Rf explicitly in terms of $x\partial_x$ and ∂_{y_j} in local coordinates (of which the latter annihilate f), using that R annihilates constants, we conclude that for $C' > 0$ sufficiently large Rf is bounded by

$$C'\mathsf{g}^{2s-2}(t^2(\mathsf{g}')^2 + t\mathsf{g}\mathsf{g}' - t^2\mathsf{g}\mathsf{g}''),$$

where we note that all terms in the parantheses are non-negative and C' is independent of $\epsilon \in (0, \epsilon_1]$. We now note that sufficiently close to 0, $t\mathsf{g}\mathsf{g}'$ can be absorbed into $t^2(\mathsf{g}')^2$ (uniformly in ϵ) for both are quadratic in t, and the latter is non-degenerate, while outside any neighborhood of 0, $t\mathsf{g}\mathsf{g}'$ can be absorbed in $-t^2\mathsf{g}\mathsf{g}''$, i.e. g' can be absorbed into g'', as is easy to check. Thus, for $C'' > 0$ sufficiently large, Rf is bounded by

$$C''\mathsf{g}^{2s-2}(t^2(\mathsf{g}')^2 - t^2\mathsf{g}\mathsf{g}''),$$

and this is bounded by $C'''\Big(-(x\partial_x)^2 + (n-2)(x\partial_x) + \Delta_0\Big) f$ for sufficiently large $C''' > 0$ by (A.3), i.e. (A.4) holds. This proves that for $\epsilon > 0$ sufficiently small $\Delta_g f \geq 0$. In summary we have proved:

Lemma A.1. *Let $0 < t_0 < 1/2$, $0 < s < (n-2)/2$. Then there exists $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$, with f as in* (A.2)*, $\Delta_g f \geq 0$.*

As immediate consequences of Lemma A.1 and (A.1) we deduce that

$$\|x^s \nabla_g u\|^2 \leq C_s \|(\Delta_g + V)u\|\, \|x^{2s}u\|,\ \ 0 < s < (n-2)/2 \tag{A.5}$$

which yields, in view of the Corollary 3.5, for $0 \leq s < 1/2$,

$$\|x^s \nabla_g u\|^2 \leq C'_s \|(\Delta_g + V)u\| \|\nabla_g u\|^{2s}_{L^2(X)} \|u\|^{1-2s}_{L^2(X)}. \tag{A.6}$$

Using the Poincaré inequality again, and applying (3.9) we therefore deduce Proposition 4.3.

References

[1] J.-F. Bony and D. Häfner, *The semilinear wave equation on asymptotically Euclidean manifolds*, preprint, 2008.

[2] J.-F. Bony and D. Häfner, *Low frequency resolvent estimates for long range perturbations of the Euclidean Laplacian*, preprint, 2009.

[3] J.-M. Bouclet, *Low energy behaviour of the resolvent of long range perturbations of the Laplacian*, preprint, 2009.

[4] C. Guillarmou and A. Hassell, *Resolvent at low energy and Riesz transform for Schrödinger operators on asymptotically conical manifolds. I*, Math. Ann. 341 (2008), no. 4, 859–896.

[5] A. Hassell and A. Vasy, *Symbolic functional calculus and N-body resolvent estimates*, J. Func. Anal. 173 (2000),257–283.

[6] R. Mazzeo and R. McOwen, *Singular Sturm-Liouville theory on manifolds*, J. Differential Equations 176 (2001), no. 2, 387–444.

[7] R. B. Melrose, *The Atiyah-Patodi-Singer index theorem*, A K Peters Ltd., 1993.

[8] R. B. Melrose, *Spectral and scattering theory for the Laplacian on asymptotically Euclidian spaces.* In *Spectral and scattering theory*, M. Ikawa, editor, Marcel Dekker, 1994.

[9] A. Vasy and M. Zworski, *Semiclassical estimates in asymptotically Euclidean scattering,* Comm. Math. Phys. 212 (2000), no. 1, 205–217.

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