Derived Algebraic Geometry Over $\mathcal{E}_n$-Rings

John Francis

Contents

CHAPTER 1

Introduction

We begin the study of certain *less* commutative algebraic geometries. In usual algebraic geometry, spaces are built from certain affine building blocks, i.e., commutative rings. In these less commutative theories, the role of the affine building blocks is instead assumed by $\mathcal{E}_n$-rings, that is, rings whose multiplication is parametrized by configuration spaces of points in $\mathbb{R}^n$. $\mathcal{E}_n$-rings interpolate between the homotopy theories of associative and commutative rings, as n ranges from one to infinity, and likewise algebraic geometry over $\mathcal{E}_n$-rings can be thought of as interpolating between some derived theories of noncommutative and commutative algebraic geomery. As n increases, these $\mathcal{E}_n$-geometries converge to the derived algebraic geometry of Toën-Vezzosi [**TV2**] and Lurie [**L5**].

Every $\mathcal{E}_n$-ring has an underlying discrete ring given by the zeroth homotopy (or homology) group, and this ring is commutative for $n \geq 2$. As a consequence, we will see that classical algebraic geometry underlies algebraic geometry over $\mathcal{E}_n$-rings, in such a way that $\mathcal{E}_n$-geometry offers different derived generalizations of classical geometry.

Preparatory to $\mathcal{E}_n$-geometry, our major focus will be a treatment of $\mathcal{E}_n$-algebra suitable for geometric generalization. We develop the basics of this theory in the next chapter, using the setting of ∞-categories. This leads to our discussion of the deformation theory of algebras over an operad. An operadic version of Illusie's cotangent complex governs this deformation theory, and this operadic cotangent complex will serve as our avatar through much of this work. We will also consider certain operadic Hochschild cohomology type constructions in the linear and nonlinear settings. We compare these constructions with the cotangent complex in the linear setting for $\mathcal{E}_n$-algebras, and this establishes a claim made by Kontsevich [**Ko**].

Of particular interest to us will be both global aspects of deformation theory, the global cotangent complex and Hochschild homology, as well as infinitesimal structures. The two theories have a different flavor. For instance, the tangent complex of an infinitesimal operadic moduli functor has a certain algebra structure, and the moduli problem versus its tangent complex is a form of Koszul duality. Thus, one would expect to reconstruct a formal moduli problem from structure on its tangent complex, just as for the case of characteristic zero deformation theory in commutative algebraic geometry. Linear structures associated to global deformation problems, in contrast, have a different type of structure, generalizing the notion of a Lie (co)algebroid. One might not expect expect structure on the global cotangent complex to completely determine the global moduli functor, but rather to serve as a good first-order approximation.

Our treatment of these subjects will be homotopy-theoretic. For instance, a recurring theme will be that of stabilization, i.e., applying stable homotopy methods (such as analogues of suspension spectra) to unstable homotopy theory (such as the ∞-category of algebras over an operad). The notion of the cotangent complex fits perfectly with these ideas, and the mixture allows for the approximation of many structures by Postnikov tower techniques.

In the final chapter, we globalize our treatment of the homotopy theory of $\mathcal{O}$-algebras and their deformation theory. This leads to the question: how can one glue $\mathcal{O}$-algebras together? In maximum generality, one can allow gluings by all $\mathcal{O}$-algebra maps in a certain universal way. The resulting algebraic geometry of $\mathcal{O}$-algebras in $\mathcal{C}$ is essentially the study of moduli functors on $\mathcal{O}$-algebras, i.e, functors $\mathcal{F}$ from $\mathcal{O}\text{-alg}(\mathcal{C})$ to ∞-groupoids or spaces. Such a moduli functor will have an ∞-category of $\mathcal{O}$-quasicoherent sheaves. At this generality, several properties of a moduli functor $\mathcal{F}$ may make it more geometrically behaved, for instance, if $\mathcal{F}$ has a cotangent complex, has affine diagonal, or commutes with filtered colimits.

However, this theory lacks even such basic notions as that of a (non-affine) scheme, and the study of these moduli functors is not as geometric as might be wished. Some extra data is necessary to refine the theory of gluings and to introduce more geometric classes of objects. The extra data we will use is that of a Grothendieck topology on $\mathcal{O}$-alg$(\mathcal{C})$ and a t-structure on $\mathcal{C}$. These choices allow a more refined theory of gluings, allowing objects analogous to schemes, Deligne-Mumford stacks, or general Artin stacks. Further, the t-structure exhibits the derived algebraic geometry of $\mathcal{O}$-algebras in $\mathcal{C}$ as a derived version of a more classical theory of $\mathcal{O}$-algebras in the heart of $\mathcal{C}$. Under certain hypotheses, a derived scheme in this setting may be expressed as a classical scheme with a derived enhancement of its structure sheaf.

Additional refinements could be made: for instance, one could enforce a choice of $\mathcal{O}$-algebra to corepresent the affine line. For instance, the incorporation of such data distinguishes between algebraic geometry over $H\mathbb{Z}$-algebras, where $H\mathbb{Z}$ is the Eilenberg-MacLane spectrum, and algebraic geometry over simplicial commutative rings. However, we will not pursue this tact and refer to [**L5**] and [**TV2**] for more general treatments of derived algebraic geometry that may incorporate such features.

The data of a Grothendieck topology and t-structure are available in the case of particular focus for us, where $\mathcal{O}$ is one of the $\mathcal{E}_n$ operads $(1 < n \leq \infty)$ and $\mathcal{C}$ is the ∞-category of spectra or chain complexes. In particular, one can define versions of the étale, Zariski, and flat topologies on certain ∞-categories of $\mathcal{E}_n$-algebras. Certain sheaves with respect to this topology can be constructed more geometrically by an appropriate theory of $\mathcal{E}_n$-ringed spaces, which we will describe.

Algebraic geometry over $\mathcal{E}_n$-rings combines certain aspects of commutative algebraic geometry and noncommutative algebra, and thus has a special flavor. For instance, because for an $\mathcal{E}_n$-ring A the underlying $\pi_0 A$ is commutative, geometry over $\mathcal{E}_n$-rings is thus a derived version of usual commutative algebraic geometry, particularly in the theory of gluings. However, like associative algebras, there are two different choices of quasicoherent sheaves, which are analogues of left modules and bimodules. These two notions converge as n tends to infinity, but in the finite case their interplay provides the noncommutative flavor of the theory. For instance, for a moduli functor X, the ∞-category of $\mathcal{E}_n$-quasicoherent sheaves on X acts on the ∞-category of quasicoherent sheaves on X in a universal way: $\mathrm{QC}_X^{\mathcal{E}_n}$ is an $\mathcal{E}_n$ version of the Drinfeld center of QC_X. This allows for the geometric description of higher Drinfeld centers of $\mathcal{E}_n$-categories, explored in work with Ben-Zvi and Nadler [**BFN**].

Certain other features of the theory for $n < \infty$ differ from the case $n = \infty$. In particular, colimits in $\mathcal{E}_n$-algebras are less well behaved than in $\mathcal{E}_\infty$-algebras, making general non-transverse intersections of stacks in $\mathcal{E}_n$-algebras more wildly behaved.

Finally, we offer a brief word on why one might or might not have interest in geometry over $\mathcal{E}_n$-rings. The subject is not motivated by applications to classical algebraic geometry, such as the construction of fundamental classes for moduli spaces (better handled by geometry over simplicial commutative rings). Rather, our motivation is to offer a geometric counterpart to structures, such as braided monoidal categories, that have only been considered algebraically. For instance, the class of spaces obtained by gluing $\mathcal{E}_n$-rings form a geometric counterpart to $\mathcal{E}_n$-categories, and these spaces further provide a geometric language for the deformation theory of general $\mathcal{E}_n$ structures.

Acknowledgments 0.1. I have benefitted enormously by knowing my advisor, Michael Hopkins. His friendship and mathe- matics have made such an impact on me. I am indebted to Jacob Lurie for generously sharing his ideas, which have greatly shaped this thesis. Over the past four years at MIT many people have taught me many things, and I thank especially Dennis Gaitsgory, Andre Henriques, and Haynes Miller. I thank David Ben-Zvi and David Nadler for their collaboration, which I look forward to continuing. Mark Behrens and David Ben-Zvi both made helpful comments on an earlier draft of this thesis.

I am lucky to have as friends Joe Burkett, Scott Carnahan, Vasiliy Dolgushev, John Duncan, Owen Gwilliam, Omar Sultan Haque, Loren Heinold, Tony Onah, Cotton Seed, and Marina Teper. My special lady, Rebekka Takamizu, has added so much to my life. I might not be doing mathematics now without Mikhail Petrov, and I look forward to seeing him again. Finally, it is hard to know how to thank my family and my parents. I hope my parents know how much I love and admire them.

CHAPTER 2

Operad Algebras in ∞-Categories

In this chapter, we consider some of the basics of the theory of operads and of algebra over an operad. We work throughout in the setting of Joyal's quasi-categories, which following Lurie we term ∞-categories. This theory is essentially equivalent to the theory of topological categories but more technically convenient and conceptually appropriate for certain aspects of homotopy theory. Our primary reference for this theory will be [**L**].

Our study of operads will take two approaches. In the first, more standard, approach, we define operads as algebras in symmetric sequences in a symmetric monoidal ∞-category $\mathcal{C}$. Algebras over an operad $\mathcal{O}$ may be defined in one of usual manners, as objects on which the operad acts in an appropriate fashion (e.g., as a full subcategory of the ∞-category of left $\mathcal{O}$-modules). This approach to the theory is well-known, and it carries over the ∞-categorical setting as one might hope or expect.

The main difference, perhaps, with the ∞-categorical theory of operads is a general feature of the difference between doing homotopy theory in an ∞-category versus a model category or topological category: in an ∞-category, one always gets the most homotopy-theoretic answer. For instance, in a model category one has both ordinary limits and homotopy limits, and one can distinguish between commutative algebras and $\mathcal{E}_\infty$-algebras, while in an ∞-category the natural notions of limit or algebra automatically correspond to the more homotopy-theoretic versions.

A second approach to the theory of operad algebras in an ∞-category is available for operads that comes from topological spaces. Here, we make use of the fact that ∞-categories are tensored over spaces, i.e., a space can parametrize a family of functors. Taking advantage of this, one can articulate the notion of an *$\mathcal{O}$-monoidal ∞-category* for $\mathcal{O}$ a topological operad. Roughly speaking, an $\mathcal{O}$-monoidal structure on an ∞-category $\mathcal{C}$ is exactly the structure necessary to define $\mathcal{O}$-algebras in $\mathcal{C}$. When $\mathcal{O}$ is the $\mathcal{E}_1$ or $\mathcal{E}_\infty$ operad this recovers notions equivalent to usual monoidal or symmetric monoidal ∞-category. Other interesting examples are given by the $\mathcal{E}_n$ operads, which interpolate between these associative and commutative structures. The notion of an $\mathcal{E}_2$-monoidal ∞-category is a homotopy-theoretic version of a braided monoidal category, a structure arising from the algebraic study of conformal field theory (i.e., quantum groups, affine Lie algebras, or chiral algebras).

This second approach may be useful when one want to do algebra in an ∞-category $\mathcal{C}$ that does not quite have a symmetric monoidal structure. For instance, to do associative algebra, $\mathcal{C}$ need only be monoidal. More generally, one can ask what structure on $\mathcal{C}$ is necessary in order to make sense of $\mathcal{O}$-algebras. The necessary structure is exactly an $\mathcal{O}$-monoidal structure on $\mathcal{C}$.

1. Operads

In this section, we develop the theory of operads in a symmetric monoidal ∞-category $\mathcal{C}$. Many approaches to this theory are possible. We will define operads as algebras in the ∞-category of symmetric sequences in $\mathcal{C}$, but it is also possible to consider them as symmetric monoidal functors from a category of rooted forests into $\mathcal{C}$, or as $\mathcal{C}$-valued sheaves on a poset of rooted trees with gluing conditions, after [**GK**]. These approaches should all be equivalent under reasonable conditions, such as $\mathcal{C}$ being presentable (or even compactly generated) and its monoidal structure distributing over colimits. We will not feel confined in restricting to these conditions, since they apply in the situations of greatest interest for us, e.g., when $\mathcal{C}$ is the ∞-category of spaces, chain complexes, or spectra.

1.1. Algebras in Symmetric Sequences. The theory of operads is very similar to the theory of monads. See [**L2**] for a treatment of the theory of monads in the ∞-categorical setting. Briefly, the ∞-category of functors $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ has a monoidal structure $\circ$ given by the composition of functors. The ∞-category of monads in $\mathcal{C}$ is then defined to be algebras in this monoidal category, $\mathrm{Monads}(\mathcal{C}) = \mathrm{Alg}(\mathrm{Fun}(\mathcal{C}, \mathcal{C}))$. The ∞-category of operads in $\mathcal{C}$ may be constructed in parallel, where symmetric sequences play the role of endofunctors.

Definition 1.1. The ∞-category of symmetric sequences in $\mathcal{C}$, $\mathcal{C}^{\Sigma}$, is the ∞-category of functors $\mathrm{Fun}(\Sigma, \mathcal{C})$, where Σ is the ∞-category of all finite sets with isomorphisms (including the empty set).

Given the additional structure on $\mathcal{C}$ of a symmetric monoidal product $\otimes$, and such that $\otimes$ distributes over colimits, we can construct a functor $\mathcal{C}^{\Sigma} \to \mathrm{Fun}(\mathcal{C}, \mathcal{C})$, assigning to a symmetric sequence M a corresponding split analytic functor whose value on X is $M(X) := \coprod_{n \geq 0} M(n) \otimes_{\Sigma_n} X^{\otimes n}$. There is a (non-symmetric) monoidal structure $\circ$ on $\mathcal{C}^{\Sigma}$ making this functor monoidal.

Definition 1.2. The ∞-category of operads in $\mathcal{C}$ is the ∞-category of algebras in $(\mathcal{C}^{\Sigma}, \circ)$.

This definition gives the following commutative diagram comparing operads and monads in $\mathcal{C}$.

$$\begin{tikzcd}
\mathrm{Operads}(\mathcal{C}) = \mathrm{Alg}(\mathcal{C}^{\Sigma}) \arrow[r] \arrow[d] & \mathrm{Alg}(\mathrm{Fun}(\mathcal{C}, \mathcal{C})) = \mathrm{Monads}(\mathcal{C}) \arrow[d] \\
\mathcal{C}^{\Sigma} \arrow[r] & \mathrm{Fun}(\mathcal{C}, \mathcal{C})
\end{tikzcd}$$

It will also be convenient for many purposes to consider the ∞-category of monads whose underlying endofunctors commute with sifted colimits, i.e., with both geometric realizations and filtered colimits.

Definition 1.3. Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories that admit filtered colimits and geometric realizations. Then $\mathrm{Fun}'(\mathcal{C}, \mathcal{D})$ is the full subcategory of functors from $\mathcal{C}$ to $\mathcal{D}$ that commute with geometric realizations and filtered colimits.

Since the property of preserving particular colimits is evidently closed under composition of functors, $\mathrm{Fun}'(\mathcal{C}, \mathcal{C})$ has the structure of a monoidal ∞-category under composition.

Definition 1.4. $\mathrm{Monads}'(\mathcal{C})$ is the ∞-category of algebras in $\mathrm{Fun}'(\mathcal{C}, \mathcal{C})$.

This subcategory of the usual ∞-category of monads has much better technical properties, as we will see later.

Lemma 1.5. *Let M be a symmetric sequence in $\mathcal{C}$, and assume the monoidal structure on $\mathcal{C}$ distributes over sifted colimits. Then the endofunctor of $\mathcal{C}$ defined by M commutes with sifted colimits.*

Thus, we obtain the following factorization of the diagram above:

$$\begin{tikzcd}
\mathrm{Operads}(\mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Monads}'(\mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Monads}(\mathcal{C}) \arrow[d] \\
\mathcal{C}^{\Sigma} \arrow[r] & \mathrm{Fun}'(\mathcal{C}, \mathcal{C}) \arrow[r] & \mathrm{Fun}(\mathcal{C}, \mathcal{C})
\end{tikzcd}$$

2. Algebras over an Operad

Operads are designed to parametrize multiplications, or operations. An object A in $\mathcal{C}$ with the structure of multiplications parametrized by an operad $\mathcal{O}$ is an $\mathcal{O}$-algebra. In this section, we make this definition precise and explore several basic properties. Throughout we assume that $\mathcal{C}$ is presentable and that its monoidal structure distributes over colimits. The cases where $\mathcal{C}$ is any of spaces, pointed spaces, spectra, or ∞-categories all satisfy these conditions.

DEFINITION 2.1. For $\mathcal{O}$ be an operad in $\mathcal{C}$, then $\mathrm{Mod}^l_{\mathcal{O}}(\mathcal{C}^\Sigma)$ (respectively, $\mathrm{Mod}^r_{\mathcal{O}}(\mathcal{C}^\Sigma)$) is the ∞-category of left $\mathcal{O}$-modules (respectively, right modules) in $\mathcal{C}^\Sigma$ with respect to the composition product.

There is a free module functor $F : \mathcal{C}^\Sigma \to \mathrm{Mod}^l_{\mathcal{O}}(\mathcal{C})$ that is calculated in $\mathcal{C}^\Sigma$ by the composition product $\mathcal{O} \circ X$. (This is a very general fact about modules over an algebra in a monoidal ∞-category, see Proposition 2.4.2 of [**L2**].)

DEFINITION 2.2. The ∞-category of $\mathcal{O}$-algebras in $\mathcal{C}$, $\mathcal{O}\text{-alg}(\mathcal{C})$, is the full subcategory of $\mathrm{Mod}^l_{\mathcal{O}}(\mathcal{C}^\Sigma)$ of objects whose underlying symmetric sequence is concentrated in degree zero.

REMARK 2.3. One might also wish to define the category of $\mathcal{O}$-algebras in $\mathcal{C}$ when $\mathcal{O}$ is an operad in a symmetric monoidal ∞-category $\mathcal{A}$ over which $\mathcal{C}$ is tensored. Under modest hypotheses, if $\mathcal{C}$ is tensored over $\mathcal{A}$ there results an adjunction $F : \mathcal{A} \leftrightarrows \mathcal{C} : G$, where F is a symmetric monoidal functor. Thus, F defines a functor from operads in $\mathcal{A}$ to operads in $\mathcal{C}$, and $\mathcal{O}$-algebras in $\mathcal{C}$ will be equivalent to $F\mathcal{O}$-algebras in $\mathcal{C}$.

The functor $\mathcal{C} \to \mathcal{C}^\Sigma$ assigns to $X \in \mathcal{C}$ the symmetric sequence with $X(0) = X$ and $X(n) = *$ for n nonempty. This is a fully faithful functor, and it preserves all limits and colimits. Since the monoidal structure on $\mathcal{C}$ has the property that there are equivalences $* \otimes X \simeq X \otimes * \simeq *$ for any X, it is then easy to check that the free module functor F preserves those symmetric sequences that are concentrated in degree zero. Since $\mathcal{O}$-algebras are a full subcategory of $\mathrm{Mod}^l_{\mathcal{O}}(\mathcal{C}^\Sigma)$, this implies that the free module functor F also calculates the left adjoint to the forgetful functor $\mathcal{O}\text{-alg}(\mathcal{C}) \to \mathcal{C}$. The following diagram summarizes the situation,

$$\begin{tikzcd}
\mathrm{Mod}_T(\mathcal{C}) & \mathcal{O}\text{-alg}(\mathcal{C}) \arrow[l, "\simeq"'] \arrow[r] \arrow[d, "G"] & \mathrm{Mod}^l_{\mathcal{O}}(\mathcal{C}^\Sigma) \arrow[r, "\simeq"] \arrow[d] & \mathrm{Mod}_{\tilde{T}}(\mathcal{C}^\Sigma) \\
& \mathcal{C} \arrow[r] \arrow[u, bend left, "F|_{\mathcal{C}}"] & \mathcal{C}^\Sigma \arrow[u, bend left, "F"] &
\end{tikzcd}$$

where T and $\tilde{T}$ are the monads on $\mathcal{C}$ and $\mathcal{C}^\Sigma$ obtained from $\mathcal{O}$.

We will be very interested in the general theory of building $\mathcal{O}$-algebras in the sequel, for the following lemma is useful.

LEMMA 2.4. *Let $\mathcal{D}$ be a monoidal ∞-category, and let A be an algebra in $\mathcal{D}$. If K is a small ∞-category such that the monoidal structure on $\mathcal{D}$ distributes over K-indexed colimits, then the forgetful functor $\mathrm{Mod}_A(\mathcal{D}) \to \mathcal{D}$ preserves K-indexed colimits.*

This has the following corollary.

COROLLARY 2.5. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category whose monoidal structure is compatible with small colimits. Then the forgetful functor $G : \mathcal{O}\text{-alg}(\mathcal{C}) \to \mathcal{C}$ preserves sifted colimits, e.g., filtered colimits and geometric realizations.*

PROOF. Using the previous lemma, the forgetful functor $\mathrm{Mod}^l_{\mathcal{O}}(\mathcal{C}^\Sigma) \to \mathcal{C}^\Sigma$ from left $\mathcal{O}$-modules to symmetric sequences preserves sifted colimits since composition monoidal structure on $\mathcal{C}^\Sigma$ does. Since the fully faithful functor $\mathcal{C} \to \mathcal{C}^\Sigma$ preserves sifted colimits, we thus obtain that the composite functor $\mathcal{O}\text{-alg}(\mathcal{C}) \to \mathrm{Mod}^l_{\mathcal{O}}(\mathcal{C}^\Sigma) \to \mathcal{C}^\Sigma$ preserves sifted colimits, and therefore G also preserves them. □

We will make liberal use of the following familiar result.

LEMMA 2.6. *Let $\mathcal{D}$ be a monoidal ∞-category for which the monoidal structure is compatible with geometric realizations, and let $\mathcal{M}$ be an ∞-category left-tensored over $\mathcal{D}$. Let $f : A \to A'$ be a map of algebras in $\mathcal{D}$, which induces a restriction functor $\mathrm{Res}_f : \mathrm{Mod}_{A'}(\mathcal{M}) \to \mathrm{Mod}_A(\mathcal{M})$ from A'-modules in $\mathcal{M}$ to A-modules in $\mathcal{M}$. Then the induction functor Ind_f left adjoint to restriction Res_f exists and is computed by the two-sided bar construction $\mathrm{Ind}_f\, M \simeq A' \otimes_A M$.*

PROOF. The two functors Ind_f and $A' \otimes_A (-)$ take the same values on free objects. Further, both functors are compatible with geometric realizations. Since an arbitrary A-module in $\mathcal{M}$ can be constructed as the geometric realization of free A-modules, the result follows. $\square$

The most obvious instance to apply the above is perhaps for algebras in a stable ∞-category, such as spectra or chain complexes. We list several other examples of interest.

EXAMPLE 2.7. Apply the above in the case of $\mathcal{D} = \mathrm{Fun}'(\mathcal{C}, \mathcal{C})$, equipped with the monoidal structure of composition, and $\mathcal{M} = \mathcal{C}$, which is left-tensored over endofunctors in the obvious way. Then the result says that for a map of monads $f : T \to T'$, the induction functor $\mathrm{Ind}_f : \mathrm{Mod}_T(\mathcal{C}) \to \mathrm{Mod}_{T'}(\mathcal{C})$ is calculated as the bar construction $T' \circ_T (-)$. In this example, it is necessary to assume T and T' preserve geometric realizations, otherwise the functor Ind_f need not exist.

EXAMPLE 2.8. In the lemma above, set $\mathcal{D}$ as symmetric sequences in $\mathcal{C}$ with the composition monoidal structure, and let $\mathcal{M}$ be either $\mathcal{C}$ or $\mathcal{C}^\Sigma$. Then the lemma implies that for a map of operads $f : \mathcal{O} \to \mathcal{O}'$ in $\mathcal{C}$, both the induction functors $\mathrm{Ind}_f : \mathcal{O}\text{-alg}(\mathcal{C}) \to \mathcal{O}'\text{-alg}(\mathcal{C})$ and $\mathrm{Mod}^l_{\mathcal{O}}(\mathcal{C}^\Sigma) \to \mathrm{Mod}^l_{\mathcal{O}'}(\mathcal{C}^\Sigma)$ are computed by the bar construction $\mathcal{O}' \circ_{\mathcal{O}} (-)$.

We now have the following conceptually helpful fact.

PROPOSITION 2.9. *Let $\mathcal{O} \to \mathcal{O}'$ be a map of operads whose underlying map of symmetric sequences is a homotopy equivalence, and as usual let $\mathcal{C}$ be a symmeric monoidal ∞-category whose monoidal structure distributes over colimits. Then the ∞-categories of $\mathcal{O}$- and $\mathcal{O}'$-algebras in $\mathcal{C}$ are equivalent.*

PROOF. Under the conditions above, both $\mathcal{O}$-algebras and $\mathcal{O}'$-algebras are modules over a monad, which we denote T and T'. The map of operads induces a map of monads $f : T \to T'$. The corresponding restriction functor $\mathrm{Res}_f : \mathrm{Mod}_{T'}(\mathcal{C}) \to \mathrm{Mod}_T(\mathcal{C})$ has a left adjoint computed by the bar constrution $T' \circ_T (-)$ as we saw in the previous lemma. To check that this adjunction gives an equivalence of ∞-categories, we can show the unit and counit of the adjunction are both equivalences. Or, since the forgetful functor to $\mathcal{C}$ is conservative, we can check just one of the maps. So for any $X \in \mathrm{Mod}_T(\mathcal{C})$, we show there is a equivalence $X \simeq \mathrm{Res}_f \mathrm{Ind}_f X$. Since $\mathrm{Res}_f \mathrm{Ind}_f X \simeq T' \circ_T X$, we consider the map $T \circ_T X \to T' \circ_T X$, which is induce by a map of simplicial objects $\mathrm{Bar}(T; T; X) \to \mathrm{Bar}(T'; T; X)$. Levelwise, this map of simplicial objects is an equivalence, since the monad map $f : T \to T'$ is an equivalence of endofunctors. Thus, the map is an equivalence on the realizations, proving the result. $\square$

REMARK 2.10. This result constrasts with the situation for the homotopy theory of $\mathcal{O}$-algebras in a topological category or a model category. For instance, the proposition above implies that theory of commutative and $\mathcal{E}_\infty$-algebras are equivalent in an ∞-category, since the operad map $\mathcal{E}_\infty \to \mathrm{Comm}$ satisfies the hypotheses of the proposition. However, this is true only in very restrictive circumstances in the setting of monoidal model categories, for instance under the hypothesis of being freely powered, [**L3**].

2.1. Abelianizing $\mathcal{E}_n$-Rings. In this section, we examine some of the behavior of the induction functor for a map of operads such as $\mathcal{E}_n \to \mathcal{E}_\infty$ in the case of spectra, though our discussion will apply equally in the case of any symmetric monoidal stable ∞-category with a compatible t-structure. Recall that the $\mathcal{E}_n$ operad, [**Ma**], models n-fold commutativity in a way similar to how the $\mathcal{E}_\infty$ operad models n-fold commutativity for all n, and that the ith space of the $\mathcal{E}_n$ operad is homotopy equivalent to the configuration space of i points in $\mathbb{R}^n$. The particular choice of model for the $\mathcal{E}_n$ operad, e.g., little n-disks or little n-cubes, will not be essential since they give rise to equivalent ∞-categories of algebras.

$\mathcal{E}_n$-rings and $\mathcal{E}_\infty$-rings are both derived generalizations of the same theory of commutative rings, and the difference between theories an be thought as being purely homotopy-theoretic. Let us be more precise. The maps of operads $\mathcal{E}_1 \xrightarrow{g} \mathcal{E}_n \xrightarrow{f} \mathcal{E}_\infty$ give rise to forgetful functors, e.g., $\mathrm{Res}_f : \mathcal{E}_\infty\text{-rings} \to \mathcal{E}_n\text{-rings}$. From the previous section, we know these functor have left adjoints, which is a form of abelianization. Our claim is that this abelianization agrees with the usual abelianization $\mathrm{Ab}(\pi_0 A)$ of the classical ring $\pi_0 A$, where $\mathrm{Ab}(\pi_0 A)$ is equivalent to $\pi_0 A/[\pi_0 A, \pi_0 A]$ the quotient by the commutator ideal. More precisely, we have the following proposition.

Proposition 2.11. *Let A be a $\mathcal{E}_n$-ring, $n \geq 2$, which is further assumed to be connective, $\pi_{i<0} A = 0$. Then there is a natural isomorphism of commutative rings $\pi_0 A \cong \pi_0(\mathrm{Ind}_f A)$. If A is a connective $\mathcal{E}_1$-ring, then there is a natural isomorphism of commutative rings $\mathrm{Ab}(\pi_0 A) \cong \pi_0 \mathrm{Ab}(A)$.*

Proof. Let $(\mathcal{E}_n\text{-rings})_{\geq 0}$ denote the ∞-category of connective $\mathcal{E}_n$-rings. The zeroth homotopy group of an $\mathcal{E}_n$-ring has the natural structure of a commutative ring, which gives a functor from $\mathcal{E}_n$-rings to the discrete ∞-category of classical commutative rings. We first claim that restriction of this functor to connective $\mathcal{E}_n$-rings has a right adjoint H, defined by sending a commutative ring R to the Eilenberg-MacLane spectrum HR, regarded as an $\mathcal{E}_n$-ring. This can be seen from the following: there is an adjunction $(\mathrm{Spectra})_{\geq 0} \overset{\tau_{\leq 1}}{\rightleftarrows} ((\mathrm{Spectra})_{\geq 0})_{\leq 1}$, between the ∞-category of connective spectra and the heart of its t-structure $((\mathrm{Spectra})_{\geq 0})_{\leq 1}$. Since the localization functor $\tau_{\leq 1}$ preserves products, we obtain an adjunction between the associated ∞-categories of $\mathcal{E}_n$-algebras. However, the right hand side is equivalent to the discrete category of abelian groups, so that the functor $\tau_{\leq 1}$ is just computing the zeroth homotopy group. This leads to the promised adjunction $\mathcal{E}_n\text{-rings} \leftrightarrows \mathrm{Com.\ Rings}$.

We thus obtain the following diagram of adjoint functors

$$\begin{tikzcd}
(\mathcal{E}_1\text{-rings})_{\geq 0} \arrow[r, bend left, "\mathrm{Ind}_g"] \arrow[d, bend right, "\pi_0"'] & (\mathcal{E}_n\text{-rings})_{\geq 0} \arrow[l, "\mathrm{Res}_g"] \arrow[r, bend left, "\mathrm{Ind}_f"] \arrow[d, bend right, "\pi_0"'] & (\mathcal{E}_\infty\text{-rings})_{\geq 0} \arrow[l] \arrow[dl, bend right, "\pi_0"'] \\
\mathrm{Assoc.\ Rings} \arrow[u, "H"'] \arrow[r, bend left, "\mathrm{Ab}"] & \mathrm{Com.\ Rings} \arrow[l, "\mathrm{forget}"] \arrow[u, "H"'] \arrow[ur, "H"'] &
\end{tikzcd}$$

in which the right adjoints (denoted by straight arrows) all commute. This implies that their left adjoints commute, which proves the proposition. $\square$

It is worth noting that the assumption of connectivity in the previous proposition was essential. More precisely, we have the following.

Lemma 2.12. *There exist nonconnective $\mathcal{E}_n$-rings A such that $\pi_0 A$ is not equivalent to $\pi_0 \mathrm{Ind}_f A$. I.e., without the assumption of connectivity the conclusion of the previous proposition is false.*

Proof. It suffices to consider the case where A is equivalent to FX, the free $\mathcal{E}_n$-ring on a spectrum X. The value of the induction functor Ind_f on FX is exactly the free $\mathcal{E}_\infty$-ring on X, i.e., $\mathrm{Ind}_f FX \simeq \mathrm{Sym}^* X$. So it suffices to check that the values of π_0 differ on the respective free algebas for some choice of nonconnective X. A representative example is a desuspension of the sphere spectrum, $X = \mathbb{S}^{-d} = \Sigma^\infty S^0[-d]$. A calculation in the homology of Thom spectra shows that the map $\coprod \mathcal{E}_n(i) \otimes_{\Sigma_i} (\mathbb{S}^{-d})^{\otimes i} \to \coprod (\mathbb{S}^{-d})^{\otimes i}_{\Sigma_i}$ does not induce an isomorphism on π_0 for any choice of positive d.

$\square$

3. Operadic ∞-Categories and Algebras

One motivation for this work is the study of operadic ∞-categories, arising from the topological study of categorical structures in conformal field theory. Several approaches present themselves in this theory. In the previous section, we gave a relatively conventional treatment of the theory of operadic algebras adopted to the ∞-categorical setting. This treatment suffices for most of our work on deformation theory and algebraic geometry in the next chapters, and we recommend the reader who is not already interested in operadic categories to skim the rest of this chapter.

We now develop operadic algebras from the point of view of fibered categories, analogous to Lurie's treatment of monoidal ∞-categories, [**L2**], [**L3**]. Intuitively, for any I-collection of objects $X_i \in \mathcal{C}$, an $\mathcal{O}$-algebra category structure on $\mathcal{C}$ provides an $\mathcal{O}(I)$-parametrization of possible tensor products of the X_i. If the $\mathcal{O}$-category structure on $\mathcal{C}$ was obtained by restriction from a symmetric monoidal structure $\otimes$, then this space of values should be exactly $\mathcal{O}(I) \otimes \bigotimes_I X_i$. If $\mathcal{O}$ was the commutative or $\mathcal{E}_\infty$-operad in spaces, then this notion would reproduce a notion equivalent to that of symmetric monoidal ∞-category as in [**L3**].

This intuition motivates an alternative candidate for the notion of an $\mathcal{O}$-category, a category fibered over the spaces in the $\mathcal{O}$-operad in an appropriate way. To make this precise, we first must formulate how to fiber and what to fiber over.

First, we recall the ∞-categorical analogue of a op-fibration of categories, a *coCartesian* fibration. Recall that for $p : \mathcal{X} \to \mathcal{S}$ a functor of discrete categories, a morphism $f : x \to y$ in $\mathcal{X}$ is coCartesian if for any $f' : x \to y'$ such that $p(f') = p(f)$, then there exists a unique map $y \to y'$ making the diagram in $\mathcal{X}$ commute. In ∞-category theory, it is instead natural to ask for a contractible space of diagram fillings, which leads to the following definition.

Definition 3.1. [**L**]. For $p : \mathcal{X} \to \mathcal{S}$ an inner fibration of ∞-cateogories, then $f : x \to y$ a morphism in $\mathcal{X}$ is p-coCartesian if the natural map

$$\mathcal{X}_{f/} \to \mathcal{X}_{x/} \times_{\mathcal{S}_{p(x)/}} \mathcal{S}_{p(f)/}$$

is a trivial Kan fibration.

Being an inner fibration means, in essence, that the fiber $p^{-1}(s)$ over an object s is an ∞-category. A morphism f being p-coCartesian over $p(f) : s \to s'$ defines a pushforward functor $f_! : p^{-1}(s) \to p^{-1}(s')$. Finally, being a coCartesian fibration means that for any morphism in the base, we have a pushforward functor for the fibers of the source and target. That is:

Definition 3.2. A map of ∞-categories $p : \mathcal{X} \to \mathcal{S}$ is a coCartesian fibration if every morphism in $\mathcal{S}$ is in the image of a p-coCartesian morphism in $\mathcal{X}$.

This details the manner in which we plan fiber. Returning to our second point, we now describe the object we wish to fiber over, whose morphism spaces should parametrize monoidal structures in the source:

Definition 3.3. Given $\mathcal{O}$ an operad in spaces, and for any J_* and I_* finite pointed sets, then define a topological category whose objects are finite pointed sets and with morphism spaces defined by

$$\mathrm{Hom}(J_*, I_*) = \coprod_{J_* \xrightarrow{\pi} I_*} \prod_I \mathcal{O}(J_i).$$

$\widetilde{\mathcal{O}}$ denotes the ∞-category obtained as the topological nerve of this topological category.

Note that by construction we have a functor from the groupoid of finite pointed sets into $\widetilde{\mathcal{O}}$, $\Sigma_* \to \widetilde{\mathcal{O}}$, as well a projection of $\widetilde{\mathcal{O}}$ onto the category of finite pointed sets, $\widetilde{\mathcal{O}} \to \mathrm{Fin}_*$.

We are now formulate our main definition.

Definition 3.4. An $\mathcal{O}$-monoidal category consists of a coCartesian fibration of ∞-categories $p : \mathcal{C}^\otimes \to \widetilde{\mathcal{O}}$ satisfying the additional condition that the natural map $\mathcal{C}^\otimes_J \to (\mathcal{C}^\otimes_{[1]})^J$ is an equivalence of ∞-categories.

Remark 3.5. The coCartesian condition gives, for any $f : J_* \to I$ in $\widetilde{\mathcal{O}}$, $f_! : p^{-1}(J) \to p^{-1}(I)$. By the Segal condition, we may identify $p^{-1}(J)$ with $(\mathcal{C})^J$. Thus, in the case $I = [1]$, we get for any point $f \in \mathcal{O}(J)$ a functor $f_! : \mathcal{C}^J \to \mathcal{C}$, and by varying the point we get a family of functors parametrized by $\mathcal{O}(J)$. Thus, intuitively, this gives an $\mathcal{O}$-algebra structure on $\mathcal{C}$.

Definition 3.6. An $\mathcal{O}$-monoidal structure on an ∞-category $\mathcal{C}$ consists of an $\mathcal{O}$-monoidal ∞-category $\mathcal{C}^{\otimes}$ as above together with an identification $\mathcal{C} \cong \mathcal{C}^{\otimes}_{[1]}$.

Example 3.7. The identity map $\mathrm{id} : \widetilde{\mathcal{O}} \to \widetilde{\mathcal{O}}$ is a special coCartesian fibration and thus gives the trivial single-object category, which is the fiber over $\langle 1 \rangle$, the structure of $\widetilde{\mathcal{O}}$ an $\mathcal{O}$-monoidal ∞-category.

Example 3.8. Let $\mathcal{O} = \mathrm{Comm}$ be the commutative operad in spaces so that $\mathcal{O}(I) = *$ for any I. Then we have an equivalence $\widetilde{\mathcal{O}} \cong \mathrm{Fin}_*$, and the definition of an $\mathcal{O}$-monoidal ∞-category becomes verbatim Lurie's definition of a symmetric monoidal ∞-category in [**L3**].

Lemma 3.9. *Any symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$ acquires an $\mathcal{O}$-monoidal ∞-category structure on the underlying ∞-category $\mathcal{C}$, which is induced from restriction along $\widetilde{\mathcal{O}} \to \mathrm{Fin}_*$.*

Proof. Let $p : \mathcal{C}^{\otimes} \to \mathrm{Fin}_*$ present the symmetric monoidal structure on $\mathcal{C}$. We can form the following Cartesian square of ∞-categories:

$$\begin{tikzcd}
\mathcal{C}^{\otimes} \times_{\mathrm{Fin}_*} \widetilde{\mathcal{O}} \arrow[r] \arrow[d, "p'"] & \mathcal{C}^{\otimes} \arrow[d, "p"] \\
\widetilde{\mathcal{O}} \arrow[r, "f"'] & \mathrm{Fin}_*
\end{tikzcd}$$

To prove the proposition, we need to show that p' is a special coCartesian fibration, but this property is preserved under pullbacks.

$\square$

3.1. $\mathcal{O}$-Monoidal Functors. To make full use of the notion of an $\mathcal{O}$-category we will need the notions of $\mathcal{O}$-monoidal and lax $\mathcal{O}$-monoidal functors. This will allow a description of the notion of an $\mathcal{O}$-algebra in an $\mathcal{O}$-category; in a symmetric monoidal category $\mathcal{C}$, a commutative algebra may be viewed as a lax symmetric monoidal functor from the trivial category into $\mathcal{C}$. Likewise, we will see that an $\mathcal{O}$-algebra in an $\mathcal{O}$-category $\mathcal{C}$ is given by a lax $\mathcal{O}$-monoidal functor from the trivial category into $\mathcal{C}$. We will see further that if the $\mathcal{O}$-monoidal structure on $\mathcal{C}$ was inherited from a symmetric monoidal structure, then the theory of $\mathcal{O}$-algeras in the $\mathcal{O}$-category and in symmetric monoidal category are naturally equivalent.

Definition 3.10. An $\mathcal{O}$-monoidal functor between $\mathcal{O}$-categories $p : \mathcal{C}^{\otimes} \to \widetilde{\mathcal{O}}$ and $q : \mathcal{D}^{\otimes} \to \widetilde{\mathcal{O}}$ consists of a map F of ∞-categories such that the diagram

$$\begin{tikzcd}
\mathcal{C}^{\otimes} \arrow[rr, "F"] \arrow[dr, "p"] & & \mathcal{D}^{\otimes} \arrow[dl, "q"] \\
& \widetilde{\mathcal{O}} &
\end{tikzcd}$$

commutes, and satisfying the additional condition that F takes p-coCartesian morphisms to q-coCartesian morphisms.

In other words, F is a functor from $\mathcal{C}$ to $\mathcal{D}$ that sends $[X_1, \dots, X_i] \in \mathcal{C}^{\otimes}$ to $[FX_1, \dots, FX_i] \in \mathcal{D}^{\otimes}$, and F preserves the multiplication maps.

Intuitively, a lax $\mathcal{O}$-monoidal functor should satisfy the first property but not the second. To formulate this precisely we recall the definition of a collapsing map of finite pointed sets.

Definition 3.11. $\alpha : J_* \to I_*$ is collapsing if for any $i \in I$ the preimage $\alpha^{-1}(i)$ has at most a single element.

Definition 3.12. A lax $\mathcal{O}$-monoidal functor F is such the diagram commutes and F preserves coCartesian morphisms whose image in Fin_* under the composition $\mathcal{C}^{\otimes} \to \widetilde{\mathcal{O}} \to \mathrm{Fin}_*$ is a collapsing map.

Previously, we motivated the introduction of lax $\mathcal{O}$-monoidal functors as a means to define $\mathcal{O}$-algebras in an $\mathcal{O}$-monoidal category. We may now make good on that motivation:

Definition 3.13. Given $\mathcal{C}$ an ∞-category with an $\mathcal{O}$-monoidal structure, an $\mathcal{O}$-algebra A in $\mathcal{C}$ consists of a lax $\mathcal{O}$-monoidal functor $A : \langle 1\rangle \to \mathcal{C}$ from the trivial 1-object to $\mathcal{C}$. In other words, an $\mathcal{O}$-monoidal structure on an object $A(\langle 1\rangle) \in \mathcal{C}$ consists of a commutative diagram

$$\begin{tikzcd} \langle 1\rangle^{\otimes} = \widetilde{\mathcal{O}} \arrow[rr, "A"] \arrow[dr, "\mathrm{id}"'] & & \mathcal{C}^{\otimes} \arrow[dl, "p"] \\ & \widetilde{\mathcal{O}} & \end{tikzcd}$$

in which A sends collapsing morphisms in $\widetilde{\mathcal{O}}$ to p-coCartesian morphisms in $\mathcal{C}^{\otimes}$.

Note that the $\mathcal{E}_\infty$-operad is homotopy final in the ∞-category of operads in spaces, so any other such operad $\mathcal{O}$ has a natural homotopy terminal map to $\mathcal{E}_\infty$.

Proposition 3.14. *Let $\mathcal{O}$ topological operads, with $p : \mathcal{O} \to \mathcal{E}_\infty$ the natural map to the homotopy final object. Further, let $\mathcal{M}$ be a symmetric monoidal ∞-category, and $f^{-1}\mathcal{M}$ denote the corresponding $\mathcal{O}$-category. Then there is an equivalence*

$$\mathcal{O}\text{-alg}(\mathcal{M}) \simeq \mathcal{O}\text{-alg}(p^{-1}\mathcal{M}) := \mathrm{Fun}^{\mathrm{lax}}(\widetilde{\mathcal{O}}, p^{-1}\mathcal{M}^{\otimes})$$

More generally, all that is required to to make sense of an $\mathcal{O}$-algebra structure on an object in an ∞-category $\mathcal{M}$ is an $\mathcal{O}'$-category structure on $\mathcal{M}$ and a map $\mathcal{O} \to \mathcal{O}'$.

3.2. Straightening and Unstraightening. We might wish to compare the the approach above, in which an $\mathcal{O}$-category is viewed as a structure fibered over $\mathcal{O}$, with a 'straightened' version, as in the definition below.

Definition 3.15. An $\mathcal{O}$-algebra category $\mathcal{C}$ is an $\mathcal{O}$-algebra in the ∞-category of ∞-categories with the Cartesian symmetric monoidal structure, $\mathcal{C} \in \mathcal{O}\text{-alg}(\mathrm{Cat}_\infty)$.

Proposition 3.16. *Let $\mathcal{O}$ be an operad in the ∞-category of spaces. Then an $\mathcal{O}$-monoidal structure on an ∞-category $\mathcal{C}$ is equivalent to an $\mathcal{O}$-algebra structure on $\mathcal{C}$.*

Proof. We may first observe that an $\mathcal{O}$-algebra structure on $\mathcal{C}$ as above gives rise to a functor $F : \widetilde{\mathcal{O}} \to \mathrm{Cat}_\infty$ such that $F(I) \simeq \mathcal{C}^I$. That is, there is a fully-faithful embedding $\mathcal{O}\text{-alg}(\mathrm{Cat}_\infty) \to \mathrm{Fun}(\widetilde{\mathcal{O}}, \mathrm{Cat}_\infty)$, whose essential image is characterized by the Segal-type condition that the natural map $F(I) \to F([1])^I$ is an equivalence.

Here we apply Lurie's straightening-unstraightening theorem, [**L**], which allows the passage between coCartesian fibrations over an ∞-category $\mathcal{X}$ and functors from $\mathcal{X}$ to Cat_∞.

A coCartesian fibration $p : \mathcal{C}^{\otimes} \to \widetilde{\mathcal{O}}$ thereby gives rise to a functor $p' : \widetilde{\mathcal{O}} \to \mathrm{Cat}_\infty$ with several properties. First, for any $I \in \widetilde{\mathcal{O}}$, the ∞-category that p' assigns to I is exactly the fiber of p over I, i.e., $p^{-1}(I)$. Second, for any morphism $f : J \to I$ in $\widetilde{\mathcal{O}}$, the functor $p'(f) : p(J) \to p(I)$ is naturally equivalent to $f_! : p^{-1}(J) \to p^{-1}(I)$, using the identifications $p^{-1}(I) \cong p'(I)$.

Since the fibration p was subject to the additional property that the natural map $p^{-1}(J) \to p^{-1}([1])^J$ was an equivalence, we see that $p'(J) \cong \mathcal{C}^J$, and so the value of $p'([1]) \cong \mathcal{C}$ acquires an $\mathcal{O}$-algebra category structure as in the preceding definition. By unstraightening, we may return to the structure of $\mathcal{O}$-monoidal category.

□

4. Operadic Modules

In this section we discuss the theory of operadic modules for an algebra over an operad. Roughly speaking, an operadic module for an $\mathcal{O}$-algebra $A \in \mathcal{C}^{\otimes}$ should be some object equipped with a space of actions of A parametrized by the operad spaces $\mathcal{O}$. We will use the following terminology to formulate our definition.

Definition 4.1. The category Fin_*^+ consists of finite based sets I_* and $(I \amalg \{+\})_*$, and morphisms α of the underlying based sets such that the preimage of $+$ contains $+$, and either $\alpha(+) = +$ or $\alpha(+) = *$. The ∞-category $\widetilde{\mathcal{O}}^+$ is the fiber product of ∞-categories defined by the Cartesian square

$$\begin{tikzcd} \widetilde{\mathcal{O}}^+ \arrow[r] \arrow[d] & \mathrm{Fin}_*^+ \arrow[d] \\ \widetilde{\mathcal{O}} \arrow[r] & \mathrm{Fin}_* \end{tikzcd}$$

These categories are variants of the category of doubly based sets. The two objects $+$ and $+_*$ play very different roles, however. The fiber over the $+$ object will allow the extra data of including a module object, while the full subcategory of objects that do not include $+$ provide the structure of an algebra. This allows us to formulate the following definition.

Definition 4.2. An operadically tensored module category consists of a coCartesian fibration $p : \mathcal{M}^\otimes \to \widetilde{\mathcal{O}}^+$ such that the associated maps $p^{-1}(I_*) \to \prod_I p^{-1}(\langle 1\rangle_*)$ and $p^{-1}(I_*^+) \to p^{-1}(+_*) \times \prod_I p^{-1}(\langle 1\rangle_*)$ are equivalences.

In the situation above, the fiber $p^{-1}(\langle 1\rangle_*) =: \mathcal{C}$ obtains the structure of an $\mathcal{O}$-monoidal ∞-category, and we will speak of the fiber $p^{-1}(+_*) =: \mathcal{M}$ as having the structure of a $\mathcal{O}$-module category over $\mathcal{C}$. In this setting, we can now define the category of modules objects in $\mathcal{M}$ for $\mathcal{O}$-algebras in $\mathcal{C}$.

Definition 4.3. The ∞-category of operadic module objects in $\mathcal{M}$, $\mathrm{Mod}^{\mathcal{O}}(\mathcal{M})$, is the full subcategory of sections $\tilde{M}$ of p such that $\tilde{M}$ takes collapsing morphisms in $\widetilde{\mathcal{O}}^+$ to coCartesian morphisms in $\mathcal{M}^\otimes$. The subcategory of $\mathcal{O}$-A-modules in $\mathcal{M}$ for a fixed $\mathcal{O}$-algebra in $\mathcal{C}$, $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{M})$, consists of those sections whose restriction to $\widetilde{\mathcal{O}}$ is A.

The ∞-category $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{M})$ can often be described as an ∞-category of left modules over some other associative algebra, the universal enveloping algebra U_A of an $\mathcal{O}$-algebra A. It is convenient to assume that the $\mathcal{O}$-monoidal structure on $\mathcal{C}$ is obtained by restriction from a symmetric monoidal structure on $\mathcal{C}$. For simplicity, let us further assume that $\mathcal{C}$ is a presentable ∞-category whose monoidal structure is compatible with colimits and further that $\mathcal{C}$ is generated under colimits by the unit 1.

There exists a forgetful functor $G : \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}) \to \mathcal{C}$, and G preserves all limits. Since the monoidal structure of $\mathcal{C}$ is compatible with colimits, G also preserves colimits. We may therefore apply the adjoint functor theorem to conclude that G has a left adjoint F. Since G is conservative and colimit preserving, it satisfies Lurie's ∞-categorical Barr-Beck conditions [**L2**]. There is therefore a diagram

$$\begin{tikzcd} \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}) \arrow[r, "\sim"] \arrow[d, "G"] & \mathrm{Mod}_T(\mathcal{C}) \\ \mathcal{C} \arrow[u, bend left, "F"] & \end{tikzcd}$$

where T is the monad on $\mathcal{C}$ associated to the adjunction. Since $\mathcal{C}$ is generated under colimits by unit 1, for any $X \in \mathcal{C}$ there is an equivalence $T(X) \simeq T(1) \otimes X$. The monad structure on T provides an equivalent structure of an algebra on $T(1) \in \mathcal{C}$. We therefore obtain an equivalence between T-modules and left $T(1)$-modules.

Definition 4.4. The $\mathcal{O}$-enveloping algebra of A, U_A, is the algebra $U_A := T(1)$ in $\mathrm{Alg}(\mathcal{C})$ constructed above.

Under the assumptions on $\mathcal{C}$ above, there is a natural equivalence $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}) \simeq \mathrm{Mod}_{U_A}(\mathcal{C})$.

CHAPTER 3

The Operadic Cotangent Complex and Deformation Theory

1. The Cotangent Complex

An essential role in the classical study of a commutative ring is played by the module of Kähler differentials, which detects important properties of ring maps and governs aspects of deformation theory. Let A be a commutative ring, then the module of Kähler differentials, Ω_A, is defined as I/I^2, where I is the kernel of the multiplication $A \otimes A \to A$. It has the property that it corepresents derivations, i.e., that there is a natural equivalence $\mathrm{Hom}_A(\Omega_A, M) \simeq \mathrm{Der}(A, M)$. Grothendieck's insight was that this situation had a good derived enhancement, the cotangent complex, which Quillen fitted to a very general model category framework of taking the left derived functor of abelianization. We will now discuss the appropriate version of the cotangent complex for algebras over an operad.

1.1. The Absolute Cotangent Complex. In this section, we study the global, or absolute, cotangent complex of an $\mathcal{O}$-algebra A. In the following, $\mathcal{O}$ will be a unital operad and A will be a unital $\mathcal{O}$-algebra in $\mathcal{C}$. We will first specify our notion of a derivation, phrased in a manner that is sensible to this level of generality. Recall from the previous chapter that given an $\mathcal{O}$-A-module M, one can form an $\mathcal{O}$-algebra $A \oplus M$, the split square-zero extension of A by M, which is augmented over A.

The following discussion will require that $\mathcal{C}$ is a stable ∞-category for which the monoidal structure distributes over colimits.

Definition 1.1. For M an $\mathcal{O}$-A-module in $\mathcal{C}$, and $B \to A$ a map of $\mathcal{O}$-algebras, then the module of A-derivations of B into M is the mapping object

$$\mathrm{Der}(B, M) := \mathrm{Map}_{\mathcal{O}\text{-alg}_{/A}}(B, A \oplus M).$$

If the monoidal structure of $\mathcal{C}$ is closed, then it is evident from the definition that derivations defines a bifunctor with values in $\mathcal{C}$

$$\mathrm{Der} : (\mathcal{O}\text{-alg}(\mathcal{C})_{/A})^{\mathrm{op}} \times \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}) \longrightarrow \mathcal{C}$$

or alternatively with values in whatever ∞-category $\mathcal{C}$ is tensored over. Under modest hypotheses on the ∞-category $\mathcal{C}$, the functor of derivations out of A preserves small limits. Thus, one could ask that it be corepresented by a specific A-module. This allows us to formulate the definition of the cotangent complex.

Definition 1.2. The absolute cotangent complex of an $\mathcal{O}$-algebra $A \in \mathcal{O}\text{-alg}(\mathcal{C})$ consists of an $\mathcal{O}$-A-module L_A together with a derivation $d : A \to A \oplus L_A$ such that the induced natural transformation of functors

$$\mathrm{Map}_{\mathrm{Mod}_A^{\mathcal{O}}}(L_A, -) \longrightarrow \mathrm{Der}(A, -)$$

is an equivalence, where the map $\mathrm{Map}_{\mathrm{Mod}_A^{\mathcal{O}}}(L_A, M) \to \mathrm{Der}(A, M)$ is defined by sending a map $\alpha : L_A \to M$ to the derivation $\alpha \circ d$.

In other words, the absolute cotangent complex of A is the module corepresenting the functor of A-derivations $\mathrm{Der}(A, -) : \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}) \longrightarrow \mathcal{C}$. From the definition, it is direct that if L_A exists, then it is defined up to a natural equivalence. We now describe this object more explicitly.

Lemma 1.3. *The functor $A \oplus -$ that assigns to a module M the corresponding split square-zero extension $A \oplus M$, $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}) \to \mathcal{O}\text{-alg}(\mathcal{C})_{/A}$, is conservative and preserves small limits.*

Proof. As established earlier, the forgetful functor $G : \mathcal{O}\text{-alg}(\mathcal{C}) \to \mathcal{C}$ preserves limits, and therefore the functor $G : \mathcal{O}\text{-alg}(\mathcal{C})_{/A} \to \mathcal{C}_{/A}$ is also limit preserving. This gives us the following commutative diagram

$$\begin{tikzcd} \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}) \arrow[r, "A\oplus -"] \arrow[d] & \mathcal{O}\text{-alg}(\mathcal{C})_{/A} \arrow[d] \\ \mathcal{C} \arrow[r, "A\times -"] & \mathcal{C}_{/A} \end{tikzcd}$$

Since the bottom and vertical arrows are all limit preserving and conservative, therefore the functor on the top must be limit preserving and conservative. ☐

Proposition 1.4. *If $\mathcal{C}$ is presentable and the monoidal structure on $\mathcal{C}$ distributes over colimits, then the functor $A \oplus - : \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}) \longrightarrow \mathcal{O}\text{-alg}(\mathcal{C})_{/A}$ has a left adjoint, which we will denote L_A.*

Proof. As proved earlier, both $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})$ and $\mathcal{O}\text{-alg}(\mathcal{C})_{/A}$ are presentable ∞-categories under the hypotheses above. The functor $A\oplus -$ is therefore a limit preserving functor between presentable ∞-categories. To apply the ∞-categorical adjoint functor theorem, [**L**], it suffices to show that $A \oplus -$ additionally preserves filtered colimits. However, as proved earlier, the forgetful functor $\mathcal{O}\text{-alg}(\mathcal{C}) \to \mathcal{C}$ preserves filtered colimits, so in both the source and target of $A \oplus -$ filtered colimits are computed in $\mathcal{C}$. ☐

As a consequence we obtain the existence of the cotangent complex of A as the value of the left adjoint L_A on A. In other words, since there is an equivalence $L_A \simeq \mathsf{L}_A(\mathrm{id}_A)$ and the functor L_A exists, therefore the cotangent complex L_A exists.

We now consider the following picture that results from an $\mathcal{O}$-algebra map $f : B \to A$.

$$\begin{tikzcd} \mathcal{O}\text{-alg}(\mathcal{C})_{/B} \arrow[rr, bend left, "f"] \arrow[d, bend right, "\mathsf{L}_B"'] & & \mathcal{O}\text{-alg}(\mathcal{C})_{/A} \arrow[ll, "B\times_A -"] \arrow[d, bend right, "\mathsf{L}_A"'] \\ \mathrm{Mod}_B^{\mathcal{O}}(\mathcal{C}) \arrow[u, "B\oplus -"'] \arrow[rr, bend left, "f_!"] & & \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}) \arrow[u, "A\oplus -"'] \arrow[ll, "f^!"] \end{tikzcd}$$

(Above, the curved arrows are left adjoint and straight arrows are right adjoints.) It is evident that the compositions of right adjoints commute, i.e., that for any $\mathcal{O}$-A-module M there is an equivalence $B \times_A (A \oplus M) \simeq B \oplus f^! M$, where $f^!$ denotes the forgetful functor from A-modules to B-modules.

As a consequence, we obtain that the value of the L_A on $f \in \mathcal{O}\text{-alg}(\mathcal{C})_{/A}$ can be computed in terms of the absolute cotangent complex of B and the corresponding induction functor on modules. That is:

Proposition 1.5. *Let $f : B \to A$ be an $\mathcal{O}$-algebra over A, as above. Then there is a natural equivalence of $\mathcal{O}$-A-modules*

$$\mathsf{L}_A(f) \simeq f_! L_B.$$

Proof. The statement follows from the commutativity of the left adjoints in the above diagram, which commute because their right adjoints commute. ☐

We now turn to the question of describing more concretely what the cotangent complex L_A actually looks like. For starters, the functor $A \oplus - : \mathrm{Mod}_A^{\mathcal{O}} \to \mathcal{O}\text{-alg}_{/A}$ factors through the ∞-category of augmented A-algebra. We thus obtain a corresponding factorization of L_A through a relative cotangent complex $\mathsf{L}_{A|A}$. We will discuss relative cotangent complexes in more detail in the next section, but in the meantime it suffices to say that $\mathsf{L}_{A|A}$ is a functor from the ∞-category of $\mathcal{O}$-algebras augmented over A to $\mathcal{O}$-A-modules fitting into the following picture

$$\begin{tikzcd}
\mathcal{O}\text{-alg}(\mathcal{C})_{/A} \arrow[rr, bend left, "A\amalg_{\mathcal{O}}-"] \arrow[dr, bend right, "\mathsf{L}_A"'] & & \mathcal{O}\text{-alg}(\mathcal{C})^{A/}_{/A} \arrow[ll, "\mathsf{L}_{A|A}"] \arrow[dl, bend right] \\
& \mathrm{Mod}^{\mathcal{O}}_A(\mathcal{C}). \arrow[ul, "A\oplus-"] \arrow[ur, "A\oplus-"'] &
\end{tikzcd}$$

The functor $\mathsf{L}_{A|A}$ is closely related to the notion of the indecomposables of a non-unital algebra. In the case of a discrete commutative non-unital ring J, the indecomposables $\mathrm{Indec}(J)$ are defined as the kernel of the multiplication map of J. Thus, there is a left exact sequence $\mathrm{Indec}(J) \to J \otimes J \to J$. In the ∞-categorical setting, it is just as convenient to define the functor of indecomposables in terms of the cotangent complex. I.e., the $\mathcal{O}$-indecomposables $\mathrm{Indec}(J)$ of a non-unital $\mathcal{O}$-A-algebra J is given as $\mathrm{Indec}(J) = \mathsf{L}_{A|A}(A \oplus J)$, where $A \oplus J$ is the split extension of A by J (and is not square-zero).

The formula $L_A \simeq \mathsf{L}_{A|A}(A \amalg A) \simeq \mathrm{Indec}(\mathrm{Ker}(A \amalg A \to A))$, however, is not an especially convenient description. For instance, the coproduct $A\amalg A$ in $\mathcal{O}$-algebras is potentially wild. Although the coproduct of $\mathcal{E}_\infty$-algebras is very well-behaved, since it is just given by the tensor product, the coproduct of associative or $\mathcal{E}_n$-algebras is more complicated. Further the indecomposables functor Indec is similarly inconvenient, since it cannot be computed as just a kernel of a multiplication map as in the commutative case.

However, in the case of $\mathcal{E}_n$-algebras we will see that the composition cancels out some of this extra complication, and that for n finite the $\mathcal{E}_n$-cotangent complexes have a slightly simpler description not enjoyed by $\mathcal{E}_\infty$-cotangent complexes.

We will now give a more explicit description of the cotangent complex in the case of a free $\mathcal{O}$-algebra $A \simeq FX$. Recall that the enveloping algebra U_A of an $\mathcal{O}$-algebra A has the property that the ∞-category of left U_A-modules is equivalent to the ∞-category of left U_A-modules.

Lemma 1.6. *For A a free $\mathcal{O}$-algebra on an object X in $\mathcal{C}$, the cotangent complex of A is equivalent to $U_A \otimes X$.*

Proof. The proof is obtained by tracing adjunctions:

$$\mathrm{Map}_{\mathcal{O}\text{-alg}_{/A}}(A, A \oplus M) \simeq \mathrm{Map}_{\mathcal{C}_{/A}}(X, A \oplus M) \simeq \mathrm{Map}_{\mathcal{C}}(X, M) \simeq \mathrm{Map}_{\mathcal{O},A}(U_A \otimes X, M).$$

We thus obtain that $U_A \otimes X$ naturally corepresents derivations, and thus that $U_A \otimes X \simeq L_A$.

□

This reduces problem the problem of describing the cotangent complex of a free algebra to that of describing the enveloping algebra of a free algebra. The enveloping algebra of a free algebra has a very concrete description shown, for instance, in Fresse [**Fr**]:

Lemma 1.7. *Let A be the free $\mathcal{O}$-algebra on X, as above, then the universal enveloping algebra U_A is equivalent to $\coprod_{n\geq 0} \mathcal{O}(n+1) \otimes_{\Sigma_n} X^{\otimes n}$.*

We now specialize to the case of $\mathcal{O}$ an $\mathcal{E}_n$ operad, for $n < \infty$, in which case we have a certain splitting result further simplifying this description of the enveloping algebra of a free algebra. The $\mathcal{E}_n$ operads are built from configuration spaces that model mapping spaces. We will make use of the following result on stable splittings of mapping spaces. It is in large part contained in McDuff [**Mc**], but we use a formulation of Bödigheimer's.

Theorem 1.8. [**Bö**]. *Let Z be a pointed connected space, and let P be a parallelizable n-manifold that is the interior of a compact manifold with boundary. Then there is a natural homotopy* *equivalence of spectra*

$$\Sigma^\infty_* \operatorname{Map}_*(P^+,\ \Sigma^n Z) \simeq \Sigma^\infty \bigvee_{i\geq 0} \operatorname{Conf}_i(P)_* \wedge_{\Sigma_i} Z^{\wedge i},$$

where P^+ is the one point compactification of P, $\operatorname{Conf}_i(P)$ is the ordered configuration space of i disjoint points in P, and $\wedge$ is the usual smash product of pointed spaces.

This result reduces to the Snaith splitting $\Sigma^\infty(\Omega^n\Sigma^n Z) \simeq \Sigma^\infty \bigvee \mathcal{E}_n(i)_* \wedge_{\Sigma_i} Z^{\wedge i}$, in the case where the manifold P is $\mathbb{R}^n$.

Again, assume $\mathcal{C}$ is a stable presentable symmetric monoidal ∞-category, but let us assume further that $\mathcal{C}$ is generated under colimits by the unit 1 of the monoidal structure.. Denote by $F_{\mathcal{E}_1}$ the free $\mathcal{E}_1$-algebra functor.

Proposition 1.9. *Let $\mathcal{C}$ be as above and A be the free $\mathcal{E}_n$-algebra on an object X in $\mathcal{C}$. There is a natural equivalence*

$$U_A \simeq A \otimes F_{\mathcal{E}_1}(X[n-1]).$$

Proof. We first consider the case where $\mathcal{C}$ is the ∞-category of spectra. By the previous lemma, the enveloping algebra U_A is equivalent to $\coprod_{i\geq 0} \mathcal{E}_n(i+1) \otimes_{\Sigma_i} X^{\otimes i}$. All spectra are built under colimits out of suspension spectra of based spaces, and the construction of the enveloping algebra U_A preserves colimits in X. It therefore follows that checking the result for suspension spectra will imply it for all spectra. So we assume that X is the suspension spectrum of Z a pointed connected space, i.e., that $X \simeq \Sigma^\infty Z$. There is thus an equivalence

$$U_A \simeq \coprod_{i\geq 0} \mathcal{E}_n(i+1) \otimes_{\Sigma_i} (\Sigma^\infty Z)^{\otimes i} \simeq \Sigma^\infty \bigvee_{i\geq 0} \mathcal{E}_n(i+1)_* \wedge_{\Sigma_i} Z^{\wedge i},$$

where the second equivalence follows from the fact that the suspension spectrum functor Σ^∞ sends colimits of based spaces to colimits of spectra. The space $\mathcal{E}_n(i+1)$ can be taken to be the ordered configuration space of $i+1$ disjoint disks in the unit disk D^n. There is thus a homotopy equivalence $\mathcal{E}_n(i+1) \to \operatorname{Conf}_i(\mathbb{R}^n - \{0\})$, defined on a configuration $x \in \mathcal{E}_n(i+1)$ by collapsing each inscribed little disk to its center and scaling so that the distinguished $i+1$ point lies at the origin. This homotopy equivalence is Σ_i-equivariant, which allows the expression above to be rewritten as

$$\Sigma^\infty \bigvee_{i\geq 0} \mathcal{E}_n(i+1)_* \wedge_{\Sigma_i} Z^{\wedge i} \simeq \Sigma^\infty \bigvee_{i\geq 0} \operatorname{Conf}_i(\mathbb{R}^n - \{0\})_* \wedge_{\Sigma_i} Z^{\wedge i}.$$

This expression is now recognizable as a stable model for a mapping space, applying the above theorem for the case of the parallelizable n-manifold $P = \mathbb{R}^n - \{0\}$. The one point compactification of $\mathbb{R}^n - \{0\}$ is homotopy equivalent to the wedge $S^n \vee S^1$. Thus, we have the following chain of equivalences

$$\begin{aligned}\Sigma^\infty \bigvee_{i\geq 0} \operatorname{Conf}_i(\mathbb{R}^n - \{0\})_* \wedge_{\Sigma_i} Z^{\wedge i} &\simeq \Sigma^\infty_* \operatorname{Map}_*(S^n \vee S^1, \Sigma^n Z)\\ &\simeq \Sigma^\infty_*(\Omega^n\Sigma^n Z \wedge \Omega\Sigma^n Z) \simeq \Sigma^\infty_*(\Omega^n\Sigma^n Z) \otimes \Sigma^\infty_*(\Omega\Sigma\Sigma^{n-1} Z).\end{aligned}$$

At this point, we have essentially obtained our goal. The suspension spectrum $\Sigma^\infty_*(\Omega^n\Sigma^n Z)$ is homotopy equivalent to the free $\mathcal{E}_n$-ring on $\Sigma^\infty Z$, which is A. Likewise, $\Sigma^\infty_*(\Omega\Sigma\Sigma^{n-1}Z)$ is equivalent to the free $\mathcal{E}_1$-ring on $\Sigma^\infty\Sigma^{n-1}Z \simeq \Sigma^\infty X[n-1]$. Thus, tracing through our equivalences, we obtain the equivalence $U_A \simeq \coprod_{i\geq 0} \mathcal{E}_n(i+1) \otimes_{\Sigma_i} X^{\otimes i} \simeq A \otimes F_{\mathcal{E}_1}(X[n-1])$, which proves the result in the case of spectra. For the case of $\mathcal{C}$ subject to the conditions above, $\mathcal{C}$ is automatically tensored over the ∞-category of spectra. In particular, there is an adjunction $f : \mathrm{Spectra} \leftrightarrows \mathcal{C} : g$, and since $\mathcal{C}$ is generated under colimits by objects in the image of f the result for spectra implies the result for $\mathcal{C}$. $\square$

This brings us to the main result of this section, which in the stable setting gives a description of the absolute cotangent complex of an $\mathcal{E}_n$-algebra.

THEOREM 1.10. *Let A be an $\mathcal{E}_n$-algebra in $\mathcal{C}$, a stable presentable symmetric monoidal ∞-category whose monoidal structure is compatible with small colimits and such that $\mathcal{C}$ is generated under colimits by the unit. Then there is a cofiber sequence*

$$U_A \longrightarrow A \longrightarrow L_A[n]$$

in the ∞-category of $\mathcal{E}_n$-A-modules.

REMARK 1.11. This result has a more familiar form in the particular case of $\mathcal{E}_1$-algebras, where the enveloping algebra U_A is equivalent to $A \otimes A^{\mathrm{op}}$. The statement above then becomes that there homotopy fiber sequence $L_A \to A \otimes A^{\mathrm{op}} \to A$, which is a description of the associative algebra cotangent complex dating back to Quillen for simplicial rings and Lazarev [**La**] for A_∞-ring spectra.

PROOF. We will prove the theorem as a consequence of an equivalent statement formulated in terms of the ∞-category of all $\mathcal{E}_n$-algebras and their $\mathcal{E}_n$-modules, $\mathrm{Mod}^{\mathcal{E}_n}(\mathcal{C})$. We first define the following functors, L, U, and i, from $\mathcal{E}_n\text{-alg}(\mathcal{C})$ to $\mathrm{Mod}^{\mathcal{E}_n}(\mathcal{C})$: L is the cotangent complex functor, assigning the pair (A, L_A) in $\mathrm{Mod}^{\mathcal{E}_n}(\mathcal{C})$ to an $\mathcal{E}_n$-algebra A. U is the composite $\mathcal{E}_n\text{-alg}(\mathcal{C}) \times \{1\} \to \mathcal{E}_n\text{-alg}(\mathcal{C}) \times \mathcal{C} \to \mathrm{Mod}^{\mathcal{E}_n}(\mathcal{C})$, where the functor $\mathcal{E}_n\text{-alg}(\mathcal{C}) \times \mathcal{C} \to \mathrm{Mod}^{\mathcal{E}_n}(\mathcal{C})$ sends an object $(A, X) \in \mathcal{E}_n\text{-alg}(\mathcal{C}) \times \mathcal{C}$ to $(A, U_A \otimes X)$, the free $\mathcal{E}_n$-A-module generated by X; finally, the functor $i : \mathcal{E}_n\text{-alg}(\mathcal{C}) \to \mathrm{Mod}^{\mathcal{E}_n}(\mathcal{C})$ sends A to the pair (A, A), where A is regarded as an $\mathcal{E}_n$-A-module in the canonical way. We will now show that there is a cofiber sequence of functors $U \to i \to \Sigma^n L$.

The first map in the sequence can be defined as follows: For A an algebra over any operad $\mathcal{O}$, there always exists a naturally defined map of $\mathcal{O}$-A-modules $U_A \to A$. This can be expressed in terms of the adjunction $F : \mathcal{C} \leftrightarrows \mathrm{Mod}^{\mathcal{O}}_A(\mathcal{C}) : G$. The counit of this adjunction, $FG \to \mathrm{id}$, applied to $A \in \mathrm{Mod}^{\mathcal{O}}_A(\mathcal{C})$, gives the desired map $U_A \simeq FG(A) \to A$. The functoriality of the counit map thus defines a natural transformation of functors $U \to i$. We will identify $\Sigma^n L$ as the cokernel of this map. We first prove this in the case that A is the free algebra on an object X, so that we have $A \simeq \coprod \mathcal{E}_n(i) \otimes_{\Sigma_i} X^{\otimes i}$ and $U_A \simeq \coprod \mathcal{E}_n(i+1) \otimes_{\Sigma_i} X^{\otimes i}$. The map $U_A \to A$ defined above is concretely realized by the operad structure maps $\mathcal{E}_n(i+1) \xrightarrow{\circ_{i+1}} \mathcal{E}_n(i)$ given by plugging the $i+1$ input of $\mathcal{E}_n(i+1)$ with the unit of $\mathcal{C}$. The map $\circ_{i+1}$ is Σ_i-equivariant, since it respects the permutations of the first i inputs of $\mathcal{E}_n(i+1)$, so this gives an explicit description of the map

$$U_A \simeq \coprod_{i\geq 0} \mathcal{E}_n(i+1) \otimes_{\Sigma_i} X^{\otimes i} \longrightarrow \coprod_{i \geq 0} \mathcal{E}_n(i) \otimes_{\Sigma_i} X^{\otimes i} \simeq A.$$

Using the previous result that $U_A \simeq A \otimes \mathrm{Free}_{\mathcal{E}_1}(X[n-1])$, we may rewrite this as

$$\coprod_{j\geq 0} \mathcal{E}_n(j) \otimes_{\Sigma_j} X^{\otimes j} \otimes \coprod_{k \geq 0} (X[n-1])^{\otimes k} \simeq \coprod_{i\geq 0} \mathcal{E}_n(i+1) \otimes_{\Sigma_i} X^{\otimes i} \longrightarrow \coprod_{i\geq 0} \mathcal{E}_n(i) \otimes_{\Sigma_i} X^{\otimes i}.$$

The kernel of this map exactly consists of the direct sum of all the terms $\mathcal{E}_n(j) \otimes_{\Sigma_j} X^{\otimes j} \otimes (X[n-1])^{\otimes k}$ for which k is greater than zero. So we obtain a fiber sequence

$$\coprod_{j\geq 0} \mathcal{E}_n(j) \otimes_{\Sigma_j} X^{\otimes j} \otimes \coprod_{k\geq 1} (X[n-1])^{\otimes k} \longrightarrow \coprod_{j\geq 0} \mathcal{E}_n(j) \otimes_{\Sigma_j} X^{\otimes j} \otimes \coprod_{k\geq 0} (X[n-1])^{\otimes k} \longrightarrow \coprod_{i\geq 0} \mathcal{E}_n(i) \otimes_{\Sigma_i} X^{\otimes i}$$

of $\mathcal{E}_n$-A-modules. It is now convenient to note the equivalence $\coprod_{k\geq 1} (X[n-1])^{\otimes k} \simeq X[n-1] \otimes \coprod_{k\geq 0} (X[n-1])^{\otimes k}$. That is, the fiber in the sequence above is equivalent to $U_A \otimes X[n-1]$. Thus, whenever A is the free $\mathcal{E}_n$-algebra on an object X of $\mathcal{C}$, we obtain a fiber sequence

$$U_A \otimes X[n-1] \longrightarrow U_A \longrightarrow A.$$

However, we can now recognize the appearance of the cotangent complex, since we saw previously that the cotangent complex of a free algebra A is equivalent to $U_A \otimes X$. Thus, we now obtain the statement of the theorem, that there is a fiber sequence $L_A[n-1] \to U_A \to A$, in the special case where A is a free $\mathcal{E}_n$-algebra.

We now turn to the general case. Denote the functor $J : \mathcal{E}_n\text{-alg}(\mathcal{C}) \to \mathrm{Mod}^{\mathcal{E}_n}(\mathcal{C})$ defined as objectwise as the cokernel of the map $U \to i$. We will show that the functor J is colimit preserving, a property which we will then use to construct a map from L to J. To show a functor preserves all colimits, it suffices to verify the preservation of geometric realizations and coproducts. Since geometric realizations commute with taking cokernels, we may show that J preserves geometric realizations by showing that both the functor U and i preserve them.

First, consider the functor U: The inclusion $\mathcal{E}_n\text{-alg}(\mathcal{C}) \to \mathcal{E}_n\text{-alg} \times \mathcal{C}$ preserves geometric realizations; additionally, the free $\mathcal{E}_n$-A-module functor $\mathcal{E}_n\text{-alg} \times \mathcal{C} \to \mathrm{Mod}^{\mathcal{E}_n}(\mathcal{C})$ is a left adjoint. U is thus the composite of a left adjoint and a functor that preserves geometric realizations, hence U preserves geometric realizations. Secondly, consider the functor i. Given a simplicial object $A_\bullet$ in $\mathcal{E}_n\text{-alg}(\mathcal{C})$, the realization of $|iA_\bullet|$ is equivalent to $(|A_\bullet|, |U_A \otimes_{U_{A_\bullet}} A_\bullet|)$. We now use the general result: For $R_\bullet$ a simplicial algebra, $M_\bullet$ an $R_\bullet$-module, and $R_\bullet \to S$ a algebra map, then there is an equivalence $|S \otimes_{R_\bullet} M_\bullet| \simeq S \otimes_{|R_\bullet|} |M_\bullet|$. Applying this in our example gives that $|U_A \otimes_{U_{A_\bullet}} A_\bullet|$ is equivalent to $U_A \otimes_{|U_{A_\bullet}|} |A_\bullet|$. The geometric realization $|U_{A_\bullet}|$ is equivalent to U_A, since by the description of U_A as a left Kan extension it preseves these geometric realizations. Thus, we obtain that i does preserve geometric realizations and as a consequence J does as well.

Now, we show that J preserves coproducts. First, if a functor $F : \mathcal{E}_n\text{-alg}(\mathcal{C}) \to \mathcal{D}$ preserves geometric realizations and coproducts of free $\mathcal{E}_n$-algebras, then F also preserves arbitrary coproducts. We see this as follows: Let A_i, $i \in I$, be a collection of $\mathcal{E}_n$-algebras in $\mathcal{C}$, and let $C_\bullet A_i$ be the functorial simplicial resolution of A_i by free $\mathcal{E}_n$-algebras, where $C_n A_i := \mathrm{Free}_{\mathcal{E}_n}^{\circ(n+1)}(A_i)$. Since geometric realizations commute with coproducts, there is a natural equivalence of $F(\coprod_I A_i) \simeq F(\coprod_I |C_\bullet A_i|)$ with $F(|\coprod_I C_\bullet A_i|)$. Applying our assumption that F preserves coproducts of free algebras and geometric realizations, we thus obtain equivalences

$$F(|\coprod_I C_\bullet A_i|) \simeq |\coprod_I F(C_\bullet A_i)| \simeq \coprod_I F(|C_\bullet A_i|) \simeq \coprod_I F(A_i)$$

where the second equivalence again follows from F preserving geometric realizations. Thus, we obtain that F preserves arbitrary coproducts given the previous assumption. We now demonstrate that J preserves coproducts of free $\mathcal{E}_n$-algebras, which will consequently imply that J preserves all colimits. Note that the functor L is a left adjoint, hence it preserves all colimits. We showed, above, that for free algebras $A = \mathrm{Free}_{\mathcal{E}_n}(X)$, there is an equivalence $J(A) \simeq L_A[n]$. Let $\{A_i\}$ be a collection of free $\mathcal{E}_n$-algebras; since the coproduct of free algebras is again a free algebra, we obtain that $J(\coprod_I A_i) \simeq L_{\coprod_I A_i}[n] \simeq \coprod_I L_{A_i}[n] \simeq \coprod_I J(A_i)$. Thus, J preserves coproducts of free algebras, hence J preserves all colimits.

The universal property of the cotangent complex functor L now applies to produce a map from L to J: The stabilization functor $\mathsf{L}_A : \mathcal{E}_n\text{-alg}(\mathcal{C})_{/A} \to \mathrm{Mod}_A^{\mathcal{E}_n}(\mathcal{C})$, from Theorem 4.2, has the property that for any colimit preserving functor F from $\mathcal{E}_n\text{-alg}(\mathcal{C})_{/A}$ to a stable ∞-category $\mathcal{D}$, there exists an essentially unique functor $F' : \mathrm{Mod}_A^{\mathcal{E}_n}(\mathcal{C}) \to \mathcal{D}$ factorizing $F' \circ \mathsf{L}_A \simeq F$. Choose F to be the composite $J_A : \mathcal{E}_n\text{-alg}(\mathcal{C})_{/A} \to \mathrm{Mod}^{\mathcal{E}_n}(\mathcal{C})_{/A} \to \mathrm{Mod}_A^{\mathcal{E}_n}(\mathcal{C})$, where the first functor is J and the second functor sends a pair $(B \xrightarrow{f} A, M)$, where M is an $\mathcal{E}_n$-B-module, to the $\mathcal{E}_n$-A-module $U_A \otimes_{U_B} M$. J_A preserves colimits, since it is a composite of two functors each of which preserve colimits. The universal property now applies to show that there is an equivalence of functors $j \circ \mathsf{L}_A \simeq J_A$. However, we have shown there is also an equivalence $J_A(B) \simeq \mathsf{L}_A(B)[n]$ whenever B is a free $\mathcal{E}_n$-algebra. Since cotangent complexes of free algebras generate $\mathrm{Mod}_A^{\mathcal{E}_n}(\mathcal{C})$ under colimits, we may conclude that the functor j is therefore the n-fold suspension functor. Thus, we obtain the equivalence of functors $J_A \simeq \Sigma^n \mathsf{L}_A$. Since this equivalence holds for every A, we finally have an equivalence of functors $J \simeq L[n]$ and a cofiber sequence of functors $U \to i \to \Sigma^n L$.

$\square$

One may think of the result above as saying that the $\mathcal{E}_n$-A-module A is very close to being its own cotangent complex.

Remark 1.12. Let A be an $\mathcal{E}_\infty$-algebra in $\mathcal{C}$. Then the $\mathcal{E}_\infty$-cotangent complex of A is equivalent to the filtered colimit of its $\mathcal{E}_n$-cotangent complexes, i.e., $L_A \simeq \varinjlim \mathrm{Ind}(L_A^{\mathcal{E}_n})$. One can also see from the Goodwillie derivative description of the cotangent complex that $L_A \simeq \varinjlim \Omega^n_A \Sigma^n_A(A \otimes A)$, where the loop and suspension functors are calculated in the ∞-category of $\mathcal{E}_\infty$-algebras augmented over A. Since $\Sigma^n_A(A \otimes A) \simeq S^n \otimes A$, we obtain the equivalence $L_A \simeq \varinjlim (S^n \otimes A)[-n]$, which we can see coincides with the description of L_A as a filtered colimit of $\mathcal{E}_n$-cotangent complexes.

1.2. The Relative Cotangent Complex. In this section, we will consider the cotangent complex $\mathsf{L}_{B|A}$ in a relative setting for a map $f : A \to B$, in which we might view the $\mathcal{O}$-B-module $L_{B|A}$ as an approximation to the difference between B and A. This reduces to the case of the absolute cotangent complex already discussed when A the unit of $\mathcal{C}$.

Definition 1.13. For A an $\mathcal{O}$-algebra over B, the relative cotangent complex $L_{B|A}$ is an $\mathcal{O}$-B-module corepresenting the functor of derivations $\mathrm{Mod}^{\mathcal{O}}_B \to$ Spaces sending M to the space of A-linear B-derivations from B to M, $\mathrm{Der}_{B|A}(B, M) := \mathrm{Map}_{\mathcal{O}\text{-alg}^{A/}_{/B}}(B, B \oplus M)$.

As with the absolute cotangent complex, the relative cotangent complex $L_{B|A}$ is a value of a linearization functor $\mathsf{L}_{B|A}$ on the ∞-category of $\mathcal{O}$-algebras over B and under A. $\mathsf{L}_{B|A}$ is the left adjoint to the functor $\mathrm{Mod}^{\mathcal{O}}_B(\mathcal{C}) \to \mathcal{O}\text{-alg}(\mathcal{C})^{A/}_{/B}$ that assigns to an $\mathcal{O}$-B-module the square zero algebra $B \oplus M$, equipped with a map from A and a map to B. This obtains the following diagram

$$\begin{tikzcd}
& \mathcal{O}\text{-alg}(\mathcal{C})^{A/}_{/B} \arrow[dl, bend right, "B\amalg_A -"'] \arrow[dd, bend right, "\mathsf{L}_{B|A}"'] \\
\mathcal{O}\text{-alg}(\mathcal{C})^{B/}_{/B} \arrow[ur] \arrow[dr, bend right, "\mathsf{L}_{B|B}"'] & \\
& \mathrm{Mod}^{\mathcal{O}}_B(\mathcal{C}). \arrow[uu, "B\oplus -"'] \arrow[ul]
\end{tikzcd}$$

So for any $C \in \mathcal{O}\text{-alg}(\mathcal{C})^{A/}_{/B}$, the value of the relative cotangent complex on C is $\mathsf{L}_{B|A}(C) \simeq L_{B|B}(B \amalg_A C)$. The $\mathcal{O}$-B-module $L_{B|A}$ is obtained as the value $\mathsf{L}_{B|A}(B)$.

Proposition 1.14. *There is a cofiber sequence $f_! L_A \to L_B \to L_{B|A}$ in the ∞-category of $\mathcal{O}$-B-modules.*

Proof. To check that $L_{B|A}$ is the cofiber of the natural map $f_! L_A \to L_B$, it suffices to check, for any M in $\mathcal{O}$-B-modules, that $\mathrm{Map}_{\mathrm{Mod}^{\mathcal{O}}_B}(L_{B|A}, M)$ is the fiber of the natural map $\mathrm{Map}_{\mathrm{Mod}^{\mathcal{O}}_B}(L_B, M) \to \mathrm{Map}_{\mathrm{Mod}^{\mathcal{O}}_B}(f_! L_A, M)$. Note that using that $f_!$ is the left adjoint to the forgetful functor $\mathrm{Mod}^{\mathcal{O}}_B(\mathcal{C}) \to \mathrm{Mod}^{\mathcal{O}}_A(\mathcal{C})$, we obtain the equivalence $\mathrm{Map}_{\mathrm{Mod}^{\mathcal{O}}_B}(f_! L_A, M) \simeq \mathrm{Map}_{\mathrm{Mod}^{\mathcal{O}}_A}(L_A, M) \simeq \mathrm{Map}_{\mathcal{O}\text{-alg}_{/A}}(A, A \oplus M)$. We have thereby reduced to evaluating the fiber of

$$\mathrm{Map}_{\mathcal{O}\text{-alg}_{/B}}(B, B \oplus M) \to \mathrm{Map}_{\mathcal{O}\text{-alg}_{/A}}(A, A \oplus M)$$

which is exactly $\mathrm{Map}_{\mathcal{O}\text{-alg}^{A/}_{/B}}(B, B \oplus M)$.

□

More generally, we have the following:

Proposition 1.15. *There is a natural cofiber sequence $f_! L_{A|k} \to L_{B|k} \to L_{B|A}$ for any sequence of $\mathcal{O}$-algebras $k \to A \xrightarrow{f} B$.*

A particularly interesting case of the relative cotangent complex functor is that where both A and B are the unit of $\mathcal{C}$, that is, the relative cotangent complex $\mathsf{L}_{1|1}$ of augmented $\mathcal{O}$-algebras in $\mathcal{C}$. The value $\mathsf{L}_{1|1}(D) \simeq L_{1|D}[-1]$ of an augmented $\mathcal{O}$-algebra D might also be called the *infinitesimal* cotangent complex, and its shift $L_{1|D}$ can be termed the cotangent *space* of the $\mathcal{O}$-algebra D at the point of D given by the augmentation $\epsilon : D \to 1$. This is equivalent to the case of the absolute cotangent complex of the non-unital $\mathcal{O}$-algebra $\mathrm{Ker}(\epsilon)$, which is the $\mathcal{O}$-indecomposables functor.

Remark 1.16. Let X be a topological space with a specified basepoint. The n-fold loop space of X, $\Omega^n X := \mathrm{Map}_*(S^n, X)$, has the structure of an nonunital $\mathcal{E}_n$-algebra in the ∞-category of pointed spaces with respect to the smash product monoidal structure. Thus, $\Sigma^\infty \Omega^n X$ acquires the structure of an $\mathcal{E}_n$-algebra in spectra. Let us first examine the case in which X is in fact the n-fold suspension of some other connected based space Y, i.e., that $X \simeq \Sigma^n Y$. In this case, we obtain an instance of the classical Snaith splitting

$$\Sigma^\infty \Omega^n(\Sigma^n Y) \simeq \coprod_{n \geq 1} \mathcal{E}_n(i) \otimes_{\Sigma_i} Y^{\otimes i}$$

which expresses the suspension spectrum on this particular n-fold loop space as the free $\mathcal{E}_n$-ring spectrum on Y. Let L denote the infinitesimal cotangent complex functor. Then for $A = \Sigma^\infty \Omega^n(\Sigma^n Y)$, the for the infinitesimal cotangent complex we have the equivalence $\mathsf{L}(A) \simeq \Sigma^\infty Y \simeq \Sigma^\infty X[n]$. (More generally, the infinitesimal $\mathcal{E}_n$-cotangent complex deloops the suspension spectrum of an n-fold loop space.)

Let us apply the previous analysis of the absolute cotangent complex in the $\mathcal{E}_n$ setting to obtain a similar description of the infinitesimal cotangent complex of an augmented $\mathcal{E}_n$-algebra A.

Proposition 1.17. *Let A be an augmented $\mathcal{E}_n$-algebra in $\mathcal{C}$, as above, with augmentation $f : A \to 1$. Then there exists a cofiber sequence in $\mathcal{C}$, $1 \to 1 \otimes_{U_A} A \to L_{1|A}[n-1]$, where $L_{1|A}$ is the cotangent space of A at f.*

Proof. Recall from the previous theorem the cofiber sequence $U_A \to A \to L_A[n]$ of $\mathcal{E}_n$-A-modules. Given an $\mathcal{E}_n$-ring map $f : A \to B$, we can apply the induction functor to obtain $U_B \to f_! A \to f_! L_A[\mathrm{n}]$, a cofiber sequence of $\mathcal{E}_n$-B-modules. Specializing to where $f : A \to 1$ is the augmentation of A, this cofiber sequence becomes $1 \to f_! A \to f_! L_A[n]$. Note that since there is an equivalence between $\mathcal{E}_n$-1-modules and $\mathcal{C}$, the enveloping algebra of the unit is equivalent to 1. So we have an equivalence $f_! A \simeq 1 \otimes_{U_A} A$.

Finally, we can specialize the cofiber sequence $f_! L_{A|k} \to L_{B|k} \to L_{B|A}$ to the case of $k = B = 1$, to obtain a cofiber sequence $f_! L_A \to L_{1|1} \to L_{1|A}$. Since $L_{1|1}$ is contractible, this gives an equivalence $L_{1|A}[-1] \simeq f_! L_A$. Substituting into $1 \to 1 \otimes_{U_A} A \to f_! L_A[n]$, we obtain a cofiber sequence $1 \to 1 \otimes_{U_A} A \to L_{1|A}[n-1]$ as desired. □

Remark 1.18. The object $1 \otimes_{U_A} A$ may be thought as the infinitesimal $\mathcal{E}_n$-Hochschild homology of A, or the $\mathcal{E}_n$-Hochschild homology with coefficients in the augmentation, i.e., $1 \otimes_{U_A} A = \mathrm{HH}_*^{\mathcal{E}_n}(A; 1)$. This result is then saying that, modulo the unit, the infinitesimal cotangent complex is equivalent to a shift of the infinitesimal $\mathcal{E}_n$-Hochschild homology.

2. Connectivity, Square-Zero Extensions, and Deformations

We now turn to the of study square-zero extensions of $\mathcal{O}$-algebras, which we first motivate with a discussion of the following basic question: how can one build $\mathcal{O}$-algebras? In the setting of classical commutative rings, one can build rings by either of two standard procedures, as a sequence of quotients of polynomial rings, or as a sequence of square-zero extensions. For instance, the ring $R \cong \mathbb{Z}[t]/t^n$ can be build from the polynomial ring $\mathbb{Z}[t]$ by imposing by the relation $t^n = 0$, or it could be build from the integers $\mathbb{Z}$ by a sequence of square-zero extensions $\mathbb{Z} \leftarrow \mathbb{Z}[t]/t^2 \leftarrow \ldots \leftarrow \mathbb{Z}[t]/t^n$. The first procedure is good for studying ring maps out of R; the second procedure is good for studying ring maps into R. The first procedure is perhaps more obvious and generally available, but it is often difficult practically to understand even basic properties of a ring from a presentation. The second procedure, when available, is often more informative because the pieces involved are simpler and fit together in a more comprehensible manner.

Both of these approaches generalize to homotopy theory. In the first approach, given any $\mathcal{O}$-algebra A in $\mathcal{C}$, one can realize A as a colimit of free algebras. There is even a canonical way to do this: the composite functor $C : \mathcal{O}\text{-alg}(\mathcal{C}) \xrightarrow{G} \mathcal{C} \xrightarrow{F} \mathcal{O}\text{-alg}(\mathcal{C})$ gives rise to a simplicial object $C_\bullet A$ whose geometric realization is equivalent to A. The drawback is that the terms in this resolution are enormous. Further, even a small presentation of an $\mathcal{O}$-algebra by some diagram of relations could be difficult to analyze due to the potentially wild behavior of colimits in the ∞-category of $\mathcal{O}$-algebras.

It is thus appealing look for an analogue of the second procedure and attempt to build an $\mathcal{O}$-algebra as a limit of "square-zero" extensions rather than as a colimit of free algebras. This has the advantage that $\mathcal{O}$-algebras with trivial algebra structure might be more manageable than free $\mathcal{O}$-algebras. Further, limits of $\mathcal{O}$-algebras are simpler to analyze since, unlike colimits, they are computed in the underlying ∞-category $\mathcal{C}$. This motivation leads us to look for a method to construct square-zero extensions of $\mathcal{O}$-algebras.

Given an $\mathcal{O}$-algebra A, an initial supply of square-zero extensions of A arise from $\mathcal{O}$-A-modules. As we saw previously, given an $\mathcal{O}$-A-module M one can form an extension of A by M, denoted $A\oplus M$. This extension $A\oplus M \to A$ is square-zero, in that the restriction of the multiplication maps $\mathcal{O}(n)\otimes(A\oplus M)^{\otimes n} \to A\oplus M$ from the factors in which M appears with multiplicity greater than one are equivalent to zero maps. Furthermore, the extension is split, in that there exists a section of the projection map $s : A \to A\oplus M$.

We are interested in the larger class of square-zero extensions which do not split, just as the extension $\mathbb{Z}[t]/t^3 \to \mathbb{Z}[t]/t^2$ does not split. We will define these as extensions of A that arise from split extensions by the following construction: let $d : A \to M$ be a derivation, i.e., the restriction of $\mathcal{O}$-algebra map $d \in \mathrm{Map}_{\mathcal{O}\text{-alg}_{/A}}(A, A\oplus M)$ to the factor of M in the target. Then we define an extension $\widetilde{A}$ of A by forming the pullback diagram

$$\begin{tikzcd} \widetilde{A} \arrow[r] \arrow[d] & A \arrow[d, "s"] \\ A \arrow[r, "d"] & A\oplus M \end{tikzcd}$$

$\widetilde{A}$ sits in a exact triangle $M[-1] \to \widetilde{A} \to A \to M$ in $\mathcal{C}$, and thus $\widetilde{A}$ is a square-zero extension of A by $M[-1]$. Further, given any other extension $N \to \widetilde{A} \to A$, we may take the structure of a derivation on the resulting map $A \to N[1]$ as the definition of the extension being square-zero. I.e., the data of a square-zero extension of A by N is exactly an $\mathcal{O}$-A-module map $L_A \to N[1]$. We have the following comparisons:

$$\begin{tikzcd} \{\text{split square–zero extensions of } A\} \arrow[r, equal] \arrow[d] & \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}) \arrow[d] \\ \{\text{square–zero extensions of } A\} \arrow[r, equal] & \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})^{L_A/} \end{tikzcd}$$

where the functor $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}) \to \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})^{L_A/}$ equips an $\mathcal{O}$-A-module M with the zero map $L_A \xrightarrow{0} M$. On the left hand side, this corresponds to forgetting the data of a splitting in the extension $M[-1] \to \widetilde{A} \leftrightarrows A$.

Remark 2.1. Unlike in the classical theory of discrete rings and modules, being square-zero is not a property of an extension but rather an additional structure. For instance, the functor $A\oplus - : \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}) \to \mathcal{O}\text{-alg}(\mathcal{C})_A$ is not at all a fully faithful functor of ∞-categories, even though classically it is.

Let us imagine the following scenario: we are provided with an $\mathcal{O}$-algebra map $f : A' \to A$, where A is something simple and A' is something we want to understand in terms of A. First, the map $A' \to A$ has a relative cotangent complex $L_{A|A'} \in \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})$, together with a universal derivation $d : A \to L_{A|A'}$. We can thereby make a first approximation to A' in terms of this linear data determined by the map f, and define a square-zero extension A_1 of A by the pullback diagram

$$\begin{tikzcd}
A' \arrow[dr, dashed] \arrow[drr, "f"] \arrow[ddr, "f"'] & & \\
& A_1 \arrow[r] \arrow[d] & A \arrow[d, "s"] \\
& A \arrow[r, "d"] & A \oplus L_{A|A'}
\end{tikzcd}$$

If A' was a square-zero extension to begin with, then the resulting map $A' \to A_1$ will be an equivalence. Otherwise, we can interate this process to produce a sequence $A \leftarrow A_1 \leftarrow A_2 \leftarrow \ldots$, in which A_{i+1} is the square-zero extension of A_i defined by limit of the diagram $A_i \xrightarrow{d} A_i \oplus L_{A_i|A'} \xleftarrow{s} A_i$. There is a natural map $A' \to \varprojlim A_i$, which one might one might think should be an equivalence if A' can be obtained from A by square-zero extensions. This will not always be the case, for instance, if $A = 0$ is the zero object and A' is anything nonzero.

Remark 2.2. Some of the discussion of this section may be illuminated by our later treatment of stabilization and costabilization. The question of what class of $\mathcal{O}$-algebras over A can be realized by a limit of square-zero extensions of A is like asking for a subcategory of $\mathcal{O}\text{-alg}(\mathcal{C})_{/A}$ on which the stabilization functor is conservative. For a general ∞-category $\mathcal{X}$, this is like asking what subcategory of objects can be realized as the limit of a filtered diagram of extensions of the final object in which the fibers of the diagram maps are infinite loop objects. Applying these considerations to the ∞-category of spaces, the process outlined above would converge to the usual Postnikov tower of a space X if X was 1-connected; more generally, one could apply this process to the ∞-category of spaces over BG, to obtain the Postnikov tower of a non-simply connected space X with fundamental groupoid G. Similar application of these considerations to the ∞-category of ∞-categories reproduces the obstruction theory of Dwyer-Kan-Smith. That is, for a given ∞-category $\mathcal{C}$, one can apply the process the ∞-category of ∞-categories over $\tau_{\leq 2}\mathcal{C}$ to obtain a Postnikov tower for $\mathcal{C}$. Over each hom space this reduces to the usual Postnikov tower of a non-simply connected space.

In order to realize A' as the filtered limit of the A_i, one would like to say in some precise fashion how the successive A_i better approximate A', that is, one would like to say that the map $A' \to A_i$ becomes highly connected. However, the notion of connectedness does not come for free in an ∞-category: in the stable setting this additional structure is formalized in the definition of a t-structure [**BBD**].

Definition 2.3. A t-structure on a stable ∞-category $\mathcal{C}$ consists of a two full subcategories, $\mathcal{C}_{\geq 0}$ (connective objects) and $\mathcal{C}_{\leq 0}$ (coconnective objects) such that:

- $\mathcal{C}_{\geq 0}$ is closed under suspension, and $\mathcal{C}_{\leq 0}$ is closed under desuspension;
- $\mathrm{Hom}_{\mathcal{C}}(X, Y) \simeq 0$ if $X \in \mathcal{C}_{\geq 0}$ and $Y[1] \in \mathcal{C}_{\leq 0}$, i.e., $Y \in \mathcal{C}_{\leq -1}$;
- any object X lies in a cofiber sequence $X' \to X \to X''$ with X' connective and X'' coconnected (i.e., $X' \in \mathcal{C}_{\geq 0}$ and $X'' \in \mathcal{C}_{\leq -1}$).

The cofiber sequence above can be constructed functorially in X. That is, for any n the inclusion $\mathcal{C}_{\leq n} \to \mathcal{C}$ admits a left adjoint $\tau_{\leq n}$ making $\mathcal{C}_{\leq n}$ a localization of $\mathcal{C}$. Likewise, the inclusion $\mathcal{C}_{\geq n} \to \mathcal{C}$ admits a right adjoint $\tau_{\geq n}$ making $\mathcal{C}_{\geq n}$ a colocalization of $\mathcal{C}$.

Remark 2.4. Equivalently, the homotopy category $\mathrm{h}\mathcal{C}$ of a stable ∞-category $\mathcal{C}$ has the structure of a triangulated category, and a t-structure on $\mathcal{C}$ is equivalent to a t-structure on the homotopy category. See [**L1**] for a fuller treatment of this subject.

In order to better apply some notion of connectedness in studying algebra structures, some compatibility between the t-structure and the monoidal structure on $\mathcal{C}$ is desirable. Let $\mathcal{C}$ be a stable monoidal ∞-category for which the monoidal structure $\otimes$ is compatible with colimits.

Definition 2.5. [**L2**]. A t-structure on $\mathcal{C}$ is compatible with the monoidal structure if the subcategory of connective objects $\mathcal{C}_{\geq 0}$ contains the unit and is closed under tensor products, i.e., the connective objects form a tensor subcategory of $\mathcal{C}$.

Use of the t-structure can often divide some analysis of connective $\mathcal{E}_n$-algebras into two parts, the first involving questions of classical commutative rings, and the second involving the higher homotopy theory of $\mathcal{E}_n$ with fixed underlying classical data, which is controlled by the cotangent complex. That is, for A a connective $\mathcal{E}_n$-algebra, there is a map $A \to \tau_{\leq 0}A$, where $\tau_{\leq 0}A$ is a commutative algebra in the heart of $\mathcal{C}$ (e.g., $\tau_{\leq 0}A \cong \pi_0 A$, if $\mathcal{C}$ is chain complexes or spectra). The approximation of A as the filtered limit of square-zero extensions $\tau_{\leq 0}A \leftarrow \tau_{\leq 1}A \leftarrow \ldots$ is analogous to the Postnikov tower of a non-simply connected space.

Assume that $\mathcal{C}$ has a t-structure, which allows any object X in $\mathcal{C}$ to be described by a sequence of truncations $\tau_{\leq n}X$: there is a map $X \to \tau_{\leq n}X$ and thus a map $X \to \varprojlim \tau_{\leq n}X$. Under these assumptions, we now consider some more technical connectivity results that facilitate the analysis discussed above for the case of $\mathcal{E}_n$-algebras.

Proposition 2.6. *Let A be a connective $\mathcal{E}_n$-algebra in $\mathcal{C}$. Then the natural map $U_A \to A$ is $(n-2)$-connected. I.e., the map $\tau_{\leq n-2}U_A \to \tau_{\leq n-2}A$ is an equivalence.*

Proof. We first discuss the case where A is the free $\mathcal{E}_n$-algebra on a connective object $X \in \mathcal{C}_{\geq 0}$. By the assumption of compatibility with the t-structure, the free $\mathcal{E}_n$-algebra on a connective object will remain connective. Writing out the terms in the map $U_A \to A$, we have $\coprod_{j\geq 0} \mathcal{E}_n(j+1) \otimes_{\Sigma_j} X^{\otimes j} \to \coprod_{j\geq 0} \mathcal{E}_n(j) \otimes_{\Sigma_j} X^{\otimes j}$, and we reduce to considering the connectivity of each individual map $\mathcal{E}_n(j+1) \otimes_{\Sigma_j} X^{\otimes j} \to \mathcal{E}_n(j) \otimes_{\Sigma_j} X^{\otimes j}$. All of the spaces $\mathcal{E}_n(k)$ in the $\mathcal{E}_n$ operad are $(n-2)$-connected, and therefore we obtain that the Σ_j-equivariant map $\mathcal{E}_n(j+1) \otimes X^{\otimes j} \to \mathcal{E}_n(j) \otimes X^{\otimes j}$ is $(n-2)$-connected. Taking the homotopy orbits of the Σ_j-action preserves this connectivity, since the truncation functor $\tau_{\geq n-2}\mathcal{C} \to \mathcal{C}_{\geq n-2}$ is a localization and preserves colimits. Therefore the map $\mathcal{E}_n(j+1) \otimes_{\Sigma_j} X^{\otimes j} \to \mathcal{E}_n(j) \otimes_{\Sigma_j} X^{\otimes j}$ is $(n-2)$-connected, as is the map $U_A \to A$.

We now argue for general A. A is equivalent to the geometric realization of the simplicial $\mathcal{E}_n$-algebra $C_\bullet A$ with terms are given by iterated free algebras, $C_i A = (FG)^{i+1}A$. Further, the functor $U : \mathcal{E}_n\text{-alg}(\mathcal{C}) \to \mathrm{Alg}(\mathcal{C})$ sending an $\mathcal{E}_n$-algebra to its enveloping algebra is compatible with geometric realizations. As before we may then construct U_A for general A as the realization of simplicial algebra whose terms are enveloping algebras of free $\mathcal{E}_n$-algebras. That is, the natural map $|U_{C_\bullet A}| \to U_A$ is an equivalence. We now have a map of simplicial objects in $\mathcal{C}$, $U_{C_\bullet A} \to C_\bullet A$, each of whose terms is $(n-2)$-connected. We thus obtain that the induced map between the geometric realizations is $(n-2)$-connected, again since the functor $\tau_{\leq n-2}$ is a localization. ☐

We can use similar argumentation to address how the connectivity of the relative cotangent complex of a map f of $\mathcal{O}$-algebras depends on the connectivity of f.

Proposition 2.7. *Let $f : A \to B$ be a map of connective $\mathcal{O}$-algebras. If f is i-connected, then the induced map of cotangent complexes $f_! L_A \to L_B$ is i-connected, or, equivalently, the relative cotangent complex $L_{B|A}$ is $(i+1)$-connected.*

Proof. We first consider the special case where $A \simeq FX$ and $B \simeq FY$ are free $\mathcal{O}$-algebras on $X,\ Y \in \mathcal{C}$, and the map f is induced from a map $g : X \to Y$. If g is i-connected then both of the maps $\mathcal{O}(i) \otimes_{\Sigma_i} X^{\otimes i} \to \mathcal{O}(i) \otimes_{\Sigma_i} Y^{\otimes i}$ and $\mathcal{O}(i+1) \otimes_{\Sigma_i} X^{\otimes i} \to \mathcal{O}(i+1) \otimes_{\Sigma_i} Y^{\otimes i}$ are i-connected. As a consequence, therefore the maps $FX \to FY$ and $U_{FX} \to U_{FY}$ are both i-connected. Since the absolute cotangent complex of a free algebra FX is given by $L_{FX} \simeq U_{FX} \otimes X$, we thus conclude that the map $L_{FX} \to L_{FY}$ is i-connected, since the map $U_{FX} \otimes X \to U_{FY} \otimes Y$ is i-connected on both terms and the t-structure is compatible with the monoidal structure.

For the case of general i-connected map $f : A \to B$ of connective $\mathcal{O}$-algebras, both A and be B can be constructed as the geometric realizations of simplicial $\mathcal{O}$-algebras $C_\bullet A$ and $C_\bullet B$ which are termwise free. The map f is likewise obtained as the realization of a map $f_\bullet : C_\bullet A \to C_\bullet B$, where $f_\bullet$ is i-connected termwise. Similarly, we obtain maps of simplicial objects $U_{C_\bullet A} \to U_{C_\bullet B}$

and $L_{C_\bullet A} \to L_{C_\bullet B}$ that are termwise i-connected by the arguments above, and the realizations of these maps produce the natural maps $U_A \to U_B$ and $L_A \to L_B$.

We now make use the fact that for a general colimit of a diagram Z_α in $\mathcal{C}$, there is a natural equivalence $\tau_{\leq i}(\operatorname{colim} Z_\alpha) \simeq \tau_{\leq i}(\operatorname{colim} \tau_{\leq i} Z_\alpha)$, since the truncation functor $\tau_{\leq i}$ is a localization. Thus, we can conclude that the maps $U_A \to U_B$ and $L_A \to L_B$ are both i-connected under the conditions above.

To complete our proof, we consider the map of $\mathcal{O}$-B-modules $f_! L_A \to L_B$ adjoint the map $L_A \to L_B$. We reapply our trick: the module $f_! L_A$ is computed by $U_B \otimes_{U_A} L_A$, which as the realization of the simplicial bar construction with terms $\mathrm{Bar}(U_B; U_A; L_A)_j = U_B \otimes (U_A)^{\otimes j} \otimes L_A$. The map $f_! L_A \to L_B$ is then realized by a map of simplicial objects $\mathrm{Bar}(U_B; U_A; L_A) \to \mathrm{Bar}(U_B; U_B; L_B)$ such that the maps termwise are i-connected. Therefore the induced map on realizations is i-connected, and the result follows.

$\square$

The result above has the following important special case.

Corollary 2.8. *Let A be an $\mathcal{O}$-algebra in $\mathcal{C}$, as above. If A is connective, then its absolute cotangent complex L_A is also connective.*

Proof. Apply the above proposition to the case where the map f is the unit of A, $f : 1 \to A$. This map is at least -1-connective, since by assumption the unit of $\mathcal{C}$ is connective. Therefore the relative cotangent complex $L_{A|1}$ is 0-connective, but since there is an equivalence $L_{A|1} \simeq L_A$, the result follows. $\square$

3. $\mathcal{E}_n$-Hochschild Cohomology

We now consider the notion of the operadic Hochschild cohomology of $\mathcal{E}_n$-algebras. The following definitions are sensible for general operads, but in this work we will only be concerned with the $\mathcal{E}_n$-operads.

Definition 3.1. Let A be an $\mathcal{O}$-algebra in $\mathcal{C}$, and let M be an $\mathcal{O}$-A-module. Then the $\mathcal{O}$-Hochschild cohomology of A with coefficients in M is

$$\mathrm{HH}^*_{\mathcal{O}}(A; M) = \mathrm{Hom}_{\mathrm{Mod}^{\mathcal{O}}_A}(A, M).$$

When the coefficient module M is the algebra A itself, we will abbreviate $\mathrm{HH}^*_{\mathcal{O}}(A; A)$ to $\mathrm{HH}^*_{\mathcal{O}}(A)$.

Remark 3.2. The preceding definition does not require that $\mathcal{C}$ is stable. A particular case of interest in when $\mathcal{C} = \mathrm{Cat}_\infty$, the ∞-category of ∞-categories, in which case this notion of Hochschild cohomology categories offers derived analogues to the classical theory of Drinfeld centers, a topic also developed in [**BFN**].

In the case that $\mathcal{C}$ is stable, the $\mathcal{E}_n$-Hochschild cohomology is closely related to our previously defined notion of $\mathcal{E}_n$-derivations and the cotangent complex. We have the following corollary of the main theorem above.

Corollary 3.3. *Let M be an $\mathcal{E}_n$-A-module in $\mathcal{C}$, with A and $\mathcal{C}$ as above. There is then a natural fiber sequence in $\mathcal{C}$*

$$\mathrm{HH}^*_{\mathcal{E}_n}(A, M) \longrightarrow M \longrightarrow \mathrm{Der}(A, M)[1-n].$$

Proof. Mapping the cofiber sequence $L_A[n-1] \to U_A \to A$ into M, we obtain fiber sequences

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Mod}^{\mathcal{E}_n}_A}(L_A, M)[1-n] \arrow[d, "\simeq"] & \mathrm{Hom}_{\mathrm{Mod}^{\mathcal{E}_n}_A}(U_A, M) \arrow[l] \arrow[d, "\simeq"] & \mathrm{Hom}_{\mathrm{Mod}^{\mathcal{E}_n}_A}(A, M) \arrow[l] \arrow[d, "\simeq"] \\
\mathrm{Der}(A, M)[1-n] & M \arrow[l] & \mathrm{HH}^*_{\mathcal{E}_n}(A; M) \arrow[l]
\end{tikzcd}$$

which obtains the stated result.

$\square$

Before proceeding, it is convenient to first provide a definition of the $\mathcal{E}_n$-tangent complex .

DEFINITION 3.4. The relative tangent complex $T_{B|A}$ of an $\mathcal{E}_n$-ring map $A \to B$ is the dual of $L_{B|A}$ in ∞-category of $\mathcal{E}_n$-B-modules.

In other words, $T_{B|A}$ can be computed by the mapping object $T_{B|A} \simeq \mathrm{Hom}_{\mathrm{Mod}_B^{\mathcal{E}_n}}(L_{B|A}, B)$. We will abbreviate $T_{B|1}$ by T_B, and refer to it simply as the (global) tangent complex of B.

REMARK 3.5. A particular case of the corollary above establishes a claim of Kontsevich in [**Ko**]. Kontsevich suggested that for an $\mathcal{E}_n$-algebra A in chain complexes, that there was an equivalence between the quotient of the tangent complex T_A quotiented by $A[n-1]$ and an $\mathcal{E}_n$-version of Hochschild cohomology of A shifted by $n-1$. This follows from the above by setting $M = A$, since the tangent complex of A is equivalent to $\mathrm{Der}(A, A)$, we thus obtain

$$\mathrm{HH}^*_{\mathcal{E}_n}(A)[n-1] \longrightarrow A[n-1] \longrightarrow \mathrm{Der}(A, A) \simeq T_A,$$

implying the quasi-isomorphism of complexes $\mathrm{HH}^*_{\mathcal{E}_n}(A)[n] \simeq T_A/(A[n-1])$. This is the statement of the second claim of [**Ko**], where Kontsevich refers to what we term the tangent complex as the deformation complex, denoted $\mathrm{Def}(A)$.

We also have an infinitesimal version of the statement above.

COROLLARY 3.6. *Let A be an augmented $\mathcal{E}_n$-algebra in $\mathcal{C}$. Then there is a cofiber sequence in $\mathcal{C}$ given by $T_{1|A}[1-n] \to \mathrm{HH}^*_{\mathcal{E}_n}(A; 1) \to 1$, where $T_{1|A}$ denotes the tangent space at the augmentation $f : A \to 1$.*

PROOF. As in previous corollary, we obtain this result by dualizing a corresponding result for the infinitesimal cotangent complex. From a previous proposition, we have a cofiber sequence $1 \to f_!A \to L_{1|A}[n-1]$. We now dualize This produces a fiber sequence $\mathrm{Hom}_{\mathcal{C}}(L_{1|A}[n-1], 1) \to \mathrm{Hom}_{\mathcal{C}}(f_!A, 1) \to \mathrm{Hom}_{\mathcal{C}}(1, 1)$. Since $\mathcal{C}$ is presentable and the monoidal structure is compatible with colimits, thus $\mathcal{C}$ is closed implying the equivalence $1 \simeq \mathrm{Hom}_{\mathcal{C}}(1, 1)$. Also, since $f_!$ is the left adjoint to the functor $\mathcal{C} \to \mathrm{Mod}_A^{\mathcal{E}_n}(\mathcal{C})$ given by restriction along the augmentation f, we have an equivalence $\mathrm{Hom}_{\mathcal{C}}(f_!A, 1) \simeq \mathrm{Hom}_{\mathrm{Mod}_A^{\mathcal{E}_n}}(A, 1)$. This is the infinitesimal $\mathcal{E}_n$-Hochschild cohomology of A, $\mathrm{HH}^*_{\mathcal{E}_n}(A; 1)$, by definition. Thus, we can rewrite our sequence as $\mathrm{Hom}_{\mathcal{C}}(L_{1|A}, 1)[1-n] \to \mathrm{HH}^*_{\mathcal{E}_n}(A; 1) \to 1$, which proves the result. $\square$

REMARK 3.7. The previous corollary directly generalizes a result of Po Hu, [**Hu**], from the case where $\mathcal{C}$ is chain complexes in characteristic zero. In the terminology of [**Hu**], the result above says that the based $\mathcal{E}_n$-Hochschild cohomology is equivalent to a shift of the based Quillen homology of augmented $\mathcal{E}_n$-algebras.

4. Stabilization of $\mathcal{O}$-Algebras

In this section, we will see that the $\mathcal{O}$-algebra cotangent complex is part of a more general theory of the cotangent complex in the context of stabilization. The theory of stabilization and costabilization is an ∞-categorical analogue of the study of abelian group and abelian cogroup objects in ordinary categories.

DEFINITION 4.1. [**L1**]. Let $\mathcal{C}$ be a ∞-category that has finite limits, and let $\mathcal{C}_*$ be the pointed envelope of $\mathcal{C}$. The stabilization of $\mathcal{C}$ is a stable ∞-category $\mathrm{Stab}(\mathcal{C})$ with a functor $\Omega^\infty : \mathrm{Stab}(\mathcal{C}) \to \mathcal{C}_*$ such that Ω^∞ is limit preserving and universal among limit preserving functors to $\mathcal{C}_*$ from a stable ∞-category.

Note that objects in the image of Ω^∞ attain the structure of infinite loop objects in $\mathcal{C}_*$, hence the notation.

The rest of this section will establish the following result on the stabilization of $\mathcal{O}$-algebras. Our discussion will mirror that of Lurie's in [**L4**], where these results are established in the commutative algebra setting. Similar results were also discussed previously by Basterra and Mandell in [**BM**].

Theorem 4.2. *Let $\mathcal{C}$ be a symmetric monoidal stable ∞-category whose monoidal structure is compatible with small colimits. For A an $\mathcal{O}$-algebra in $\mathcal{C}$, the stabilization of the ∞-category of $\mathcal{O}$-algebras over A is equivalent to the ∞-category of $\mathcal{O}$-A-modules in the stabilization of $\mathcal{C}$, i.e., there is a natural equivalence*

$$\mathrm{Stab}(\mathcal{O}\text{-alg}(\mathcal{C})_{/A}) \simeq \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})$$

and equivalences of functors $\Sigma^\infty \simeq \mathsf{L}_A$ and $\Omega^\infty \simeq A \oplus (-)$.

We will require the following lemma from the Goodwillie calculus, which is a familiar fact concerning derivatives of split analytic functors. For a further discussion of the Goodwillie calculus see [**Go**], which is the basic reference.

Lemma 4.3. *Let T be a split analytic functor on a stable monoidal ∞-category $\mathcal{C}$ defined by a symmetric sequence $\mathcal{T} \in \mathcal{C}^\Sigma$ with $\mathcal{T}(0) \simeq *$, so that $T(X) = \coprod_{n\geq 1} \mathcal{T}(n) \otimes_{\Sigma_n} X^{\otimes n}$. The first Goodwillie derivative DT is equivalent to $DT(X) \simeq \mathcal{T}(1) \otimes X$.*

Proof. We calculate the following,

$$DT(X) \simeq \varinjlim \Omega^i T(\Sigma^i X) \simeq \varinjlim \Omega^i (\coprod_{n\geq 1} \mathcal{T}(n) \otimes_{\Sigma_n} (\Sigma^i X)^{\otimes n}) \simeq$$

$$\varinjlim \Omega^i (\mathcal{T}(1) \otimes \Sigma^i X) \;\oplus\; \coprod_{n\geq 2} \varinjlim \Omega^i(\mathcal{T}(n) \otimes_{\Sigma_n} (\Sigma^i X)^{\otimes n}),$$

using the commutation of Ω with the infinite coproduct and the commutation of filtered colimits and infinite coproducts. However, we can now note that the higher terms are n-homogeneous functors for $n > 1$, and hence they have trivial first Goodwillie derivative. This obtains that $DT(X) \simeq \varinjlim \Omega^i(\mathcal{T}(1) \otimes \Sigma^i X) \simeq \mathcal{T}(1) \otimes X$. $\square$

We will now prove the theorem above in the special case where A is just 1, the unit of the monoidal structure on $\mathcal{C}$. In this case, $\mathcal{O}$-algebras over A are literally the same as augmented $\mathcal{O}$-algebras in $\mathcal{C}$, $\mathcal{O}\text{-alg}^{\mathrm{aug}}(\mathcal{C}) \simeq \mathcal{O}\text{-alg}(\mathcal{C})_{/1}$. There is an adjunction between augmented and non-unital $\mathcal{O}$-algebras

$$\begin{tikzcd} \mathcal{O}\text{-alg}^{\mathrm{nu}}(\mathcal{C}) \arrow[d, bend right, "1\oplus(-)"'] \\ \mathcal{O}\text{-alg}^{\mathrm{aug}}(\mathcal{C}) \arrow[u, "I"'] \end{tikzcd}$$

where I denotes the augmentation ideal functor, with left adjoint given by adjoining a unit. The adjunction above is an equivalence of ∞-categories, since the unit and counit of the adjunction are equivalences when $\mathcal{C}$ is stable. We now formulate a special case of the theorem above.

Proposition 4.4. *There is a natural equivalence $\mathrm{Stab}(\mathcal{O}\text{-alg}^{\mathrm{nu}}(\mathcal{C})) \simeq \mathrm{Mod}_{\mathcal{O}(1)}(\mathcal{C})$.*

Proof. Let T denote the monad associated to non-unital $\mathcal{O}$-algebras, so that there is a natural equivalence $\mathcal{O}\text{-alg}^{\mathrm{nu}}(\mathcal{C}) \simeq \mathrm{Mod}_T(\mathcal{C})$. We may thus consider stabilizing this adjunction, to produce another adjunction:

$$\begin{tikzcd} \mathrm{Mod}_T(\mathcal{C}) \arrow[r, shift left, "\Sigma^\infty"] \arrow[d] & \mathrm{Stab}(\mathrm{Mod}_T\,\mathcal{C}) \arrow[l, shift left, "\Omega^\infty"] \arrow[r] \arrow[d, "g"] & \mathrm{Mod}_{gf}(\mathcal{C}) \\ \mathcal{C} \arrow[u, bend left] \arrow[r, "\simeq"] & \mathrm{Stab}(\mathcal{C}). \arrow[u, bend left, "f"] & \end{tikzcd}$$

The stabilization of $\mathrm{Mod}_T(\mathcal{C})$ is monadic over $\mathcal{C}$, [**L4**], since the right adjoint is conservative, preserves split geometric realizations, and hence satisfies the ∞-categorical Barr-Beck theorem. The resulting monad $g \circ f$ on $\mathcal{C}$ is the first Goodwillie derivative of T, which by the above lemma is computed by $\mathcal{O}(1) \otimes (-)$, with the monad structure of $g \circ f$ corresponding to the associative algebras structure on $\mathcal{O}(1)$. Thus, the result follows. $\square$

Note that if the operad $\mathcal{O}$ is such that $\mathcal{O}(1) \simeq 1$, where 1 is the unit of $\mathcal{C}$, then this implies the equivalence $\mathrm{Stab}(\mathcal{O}\text{-alg}) \simeq \mathcal{C}$. In particular, the functor Ind_η of induction along the augmentation $\eta : \mathcal{O} \to 1$ is equivalent to the stabilization functor Σ^∞.

To complete the proof of the main theorem, we will reduce it to the proposition above. Consider $\mathcal{O}_A$, the universal enveloping operad of A, defined by the property that $\mathcal{O}_A\text{-alg}(\mathcal{C})$ is equivalent to $\mathcal{O}$-algebras under A. Likewise, we have that non-unital $\mathcal{O}$-A-algebras is equivalent to non-unital $\mathcal{O}_A$-algebras. Since the ∞-category of $\mathcal{O}$-algebras augmented over A is again equivalent to $\mathcal{O}\text{-alg}^{\mathrm{nu}}(\mathcal{C})$, we reduce to considering this case.

Thus, we obtain that $\mathrm{Stab}(\mathcal{O}_A\text{-alg}^{\mathrm{nu}}(\mathcal{C}))$ is equivalent to $\mathrm{Mod}_{\mathcal{O}_A(1)}(\mathcal{C})$. Since the first term of the enveloping operad $\mathcal{O}_A(1)$ is equivalent to the enveloping algebra U_A, and $\mathrm{Mod}_{U_A}(\mathcal{C}) \simeq \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})$, this implies that the equivalence $\mathrm{Stab}(\mathcal{O}_A\text{-alg}^{\mathrm{nu}}(\mathcal{C})) \simeq \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})$.

By definition, the stabilization of an unpointed ∞-category $\mathcal{X}$ is the stabilization of its pointed envelope $\mathcal{X}_*$, the ∞-category of objects of $\mathcal{X}$ under $*$, the final object. Thus the pointed envelope of $\mathcal{O}$-algebras over A is $\mathcal{O}$-algebras augmented over and under A. This is the stabilization we have computed, which our proof the theorem.

5. Stabilization and Costabilization

In the previous section we studied the stabilization of the ∞-category of augmented $\mathcal{O}$-algebras, and we saw that the answer was highly nontrivial. This might lead us to the question of just how much the stabilization of an ∞-category knows about the original ∞-category. This question can be framed as part of a general question about the extent to which nonabelian structure can be analyzed as some type of algebra on an underlying abelian structure. We will see that there are two ways of doing this, from stabilization and costabilization, and the interplay between these leads a type of duality between coalgebras and algebras.

5.1. Digression: Cohomology Theories. The notion of stabilization is also closely related to the notion of cohomology theories, which we will briefly elaborate.

Definition 5.1. Let $\mathcal{C}$ be a pointed ∞-category. A cohomology theory on $\mathcal{C}$ is a contravariant functor from $\mathcal{C}$ to the ∞-category of spectra that sends colimits in $\mathcal{C}$ to limits of spectra. The ∞-category of cohomology theories is $\mathrm{Fun}^{\lim}(\mathcal{C}^{\mathrm{op}}, \mathrm{Spectra})$, the full subcategory of all contravariant spectra-valued functors on $\mathcal{C}$. If $\mathcal{C}$ is not pointed, an unreduced cohomology theory on $\mathcal{C}$ is a cohomology theory on $\mathcal{C}_*$, the pointed envelope of $\mathcal{C}$.

There exists a functor $\mathrm{Stab}(\mathcal{C}) \to \mathrm{Fun}^{\lim}(\mathcal{C}^{\mathrm{op}}, \mathrm{Spectra})$, assigning to a stable object E of $\mathcal{C}$ the corresponding cohomology theory with values $E(X) := \mathrm{Hom}_{\mathrm{Stab}\,\mathcal{C}}(\Sigma^\infty X, E)$. The following result says that this functor is very often an equivalence.

Proposition 5.2. *If $\mathcal{C}$ is a compactly generated presentable ∞-category, then the functor $\mathrm{Stab}(\mathcal{C}) \to \mathrm{Fun}^{\lim}(\mathcal{C}^{\mathrm{op}}, \mathrm{Spectra})$ is an equivalence of ∞-categories. That is, stable objects in $\mathcal{C}$ are equivalent to cohomology theories on $\mathcal{C}$.*

Proof. We have the following natural chain of equivalences:

$$\mathrm{Fun}^{\lim}(\mathcal{C}^{\mathrm{op}}, \mathrm{Spectra}) \simeq \mathrm{Fun}^{\mathrm{colim}}(\mathcal{C}, \mathrm{Spectra}^{\mathrm{op}})^{\mathrm{op}} \simeq \mathrm{Fun}^{\mathrm{colim}}(\mathrm{Stab}(\mathcal{C}), \mathrm{Spectra}^{\mathrm{op}})^{\mathrm{op}}.$$

We now use the universal property of the stabilization of spaces, that a limit preserving functor from a stable ∞-category to spaces can be canonically delooped to obtain a limit preserving functor to spectra. We thus obtain that

$$\mathrm{Fun}^{\mathrm{colim}}(\mathrm{Stab}(\mathcal{C}), \mathrm{Spectra}^{\mathrm{op}})^{\mathrm{op}} \simeq \mathrm{Fun}^{\lim}(\mathrm{Stab}(\mathcal{C})^{\mathrm{op}}, \mathrm{Spectra}) \simeq \mathrm{Fun}^{\lim}(\mathrm{Stab}(\mathcal{C})^{\mathrm{op}}, \mathrm{Spaces}).$$

For the final step we use that $\mathcal{C}$ is presentable, and hence that $\mathrm{Stab}(\mathcal{C})$ is presentable. It is then a general result, [**L**], that for a presentable ∞-category $\mathcal{X}$, representable presheaves of spaces on $\mathcal{X}$ are exactly those functors $\mathcal{X}^{\mathrm{op}} \to \mathrm{Spaces}$ that preserve limits. Thus, we obtain that $\mathrm{Fun}^{\lim}(\mathrm{Stab}(\mathcal{C})^{\mathrm{op}}, \mathrm{Spaces}) \simeq \mathrm{Stab}(\mathcal{C})$, which completes the proof.

$\square$

5.2. Abelian Groups and Abelian Cogroups. Many methods in mathematics use abelian structure to study nonabelian structure. The notion of abelianizing provides a certain universal way of doing this. The homotopy-theoretic version of this accounts for many instances of cohomology, including the cohomology of spaces or of groups. We will discuss this briefly in such a manner to setup our discussion of the ∞-categorical analogue.

The notion of stabilization provides an ∞-categorical analogue of abelian group objects in an ordinary category. Costabilization likewise provides an ∞-categorical analogue of abelian cogroup objects in ordinary categories.

Let us run through some examples. In the category of sets, abelian group objects are exactly abelian groups, while there are no nontrivial cogroups in finite sets. Similarly, in the category of spaces, abelian group objects correspond to the usual notion of topological abelian groups, while again there are no nontrivial cogroups in spaces.

In the category of augmented commutative rings, abelian group objects exactly correspond to rings of the form $\mathbb{Z} \oplus V$, split square-zero extensions of $\mathbb{Z}$ by an abelian group V. Augmented rings also have an abundance of cogroup objects. These cogroups arise as free algebras: the diagonal map of an abelian group $V \to V \oplus V$ gives rise to a cogroup map $\mathrm{Sym}^* V \to \mathrm{Sym}^*(V \oplus V) \cong \mathrm{Sym}^* V \otimes \mathrm{Sym}^* V$. So, unlike the category of spaces, the category of augmented rings has a good supply of both group objects and cogroup objects. Let us formulate this more precisely:

Definition 5.3. A category $\mathcal{X}$ *has enough abelian groups* if and only if for any map $f : X \to Y$ in $\mathcal{X}$ that is not an isomorphism, there then exists an abelian group A in $\mathcal{X}$ such that the map $\mathrm{Hom}_{\mathcal{X}}(Y, A) \to \mathrm{Hom}_{\mathcal{X}}(X, A)$ is not an isomorphism of abelian groups. A category $\mathcal{X}$ *has enough abelian cogroups* if and only if for every map $f : X \to Y$ that is not an isomorphism, there then exists an abelian cogroup C in $\mathcal{X}$ such that the map $\mathrm{Hom}_{\mathcal{X}}(C, X) \to \mathrm{Hom}_{\mathcal{X}}(C, Y)$ is not an isomorphism of abelian groups.

In other words, $\mathcal{X}$ has enough abelian groups if maps into abelian group objects detect equivalences, and it has enough cogroups if maps out of cogroups detect equivalences. The category of sets, for instance, has enough abelian groups. The category of augmented algebras has a lot but not quite enough: abelian group objects cannot detect that the map $\mathbb{Z}[\epsilon]/\epsilon^3 \to \mathbb{Z}[\epsilon]/\epsilon^2$ is not an equivalence. Maps out of free algebras can detect this, however, and thus the category of augmented algebras does have enough abelian cogroups.

Let us say this in a slightly different fashion. There often exists an abelianization functor $\mathcal{X} \to \mathrm{Ab}(\mathcal{X})$, which is left adjoint to the inclusion $\mathrm{Ab}(\mathcal{X}) \to \mathcal{X}$. Then $\mathcal{X}$ has enough abelian groups if and only if the abelianization functor is conservative. Likewise the inclusion $\mathrm{Coab}(\mathcal{X}) \to \mathcal{X}$ often has a right adjoint, coabelianization, and $\mathcal{X}$ has enough abelian cogroups if and only if the coabelianization functor is conservative.

Why might it be important whether a category has enough abelian groups or cogroups? If $\mathcal{X}$ does, then we can describe $\mathcal{X}$ in terms of abelian data. Any object in $\mathcal{X}$ will have an underlying abelian group, the (co)abelianization, and we can describe the additional data of X as being something like a coalgebra or algebra structure on this abelian group. This might then allow hard questions about $\mathcal{X}$ to be translated into easier algebra, and is thus worth consideration.

Let us consider the the failure of the category of groups to have enough abelian group objects: the problem is that the natural map $G \to G/[G, G]$ cannot be distinguished by maps into abelian groups. However, the situation improves if treated in homotopy-theory: group cohomology distinguishes between G and $G/[G, G]$.

5.3. Duality. We now revise the previous discussion for $\mathcal{X}$ an ∞-category. Here, the role of abelian groups is played by infinite loop objects in $\mathcal{X}$, and the role of abelian cogroups is played by infinite suspension objects. We would like to address the question of how effectively $\mathcal{X}$ can be studied in terms of its stable or costable objects. As we saw before, a stable object E in $\mathrm{Stab}(\mathcal{X})$ defines a cohomology theory on $\mathcal{X}$ valued in spectra, so this half of the question can also be expressed as asking how effectively $\mathcal{X}$ can be studied by cohomology theories.

The costabilization is defined by the opposite universal property as stabilization:

Definition 5.4. Let $\mathcal{X}$ be a pointed ∞-category that contains finite limits and colimits. The costabilization of $\mathcal{X}$ is a stable ∞-category $\mathrm{Costab}(\mathcal{X})$ with a functor $\Sigma_\infty : \mathrm{Costab}(\mathcal{X}) \to \mathcal{X}$ that preserves colimits and is universal among colimit preserving functors from stable ∞-categories to $\mathcal{X}$.

Thus, is it evident that there is a natural equivalence $\mathrm{Costab}(\mathcal{X}) \simeq \mathrm{Stab}(\mathcal{X}^{\mathrm{op}})^{\mathrm{op}}$. Just as the image of the functor Ω^∞ consists of infinite loop objects in $\mathcal{X}$, likewise the image of the functor Σ_∞ consists of infinite suspension objects in $\mathcal{X}$.

Under these assumptions above, both the stabilization and costabilization of $\mathcal{X}$ will exist. If we further assume that either $\mathcal{X}$ or $\mathcal{X}^{\mathrm{op}}$ is either presentable, in which case the natural functors $\Omega^\infty : \mathrm{Stab}(\mathcal{X}) \to \mathcal{X}$ and $\Sigma_\infty : \mathrm{Costab}(\mathcal{X}) \to \mathcal{X}$ will both have adjoints by the adjoint functor theorem.

Remark 5.5. Let us briefly place some of this discussion in the setting of prespectra, which may be more familiar to some topologists. Given an object $X \in \mathcal{X}$, we can form a prespectrum with $X(n) := \Sigma^n X$ and $X(-n) := \Omega^n X$ for $n \geq 0$. To form the suspension spectrum, we localize. If looping commutes with filtered colimits in $\mathcal{X}$, this localization produces a spectrum with terms $(\Sigma^\infty X)(n) := \varinjlim_i \Omega^i \Sigma^{i+n} X$. Likewise, to form the loop spectrum we perform the appropriate colocalization. If suspension commutes with filtered limits in $\mathcal{X}$, this resulting cospectrum has terms computed by the formula $(\Omega_\infty X)(-n) := \varprojlim_i \Sigma^i \Omega^{i+n} X$. The terms of $\Sigma^\infty X$ are infinite loop objects in $\mathcal{X}$, and the terms of the loop spectrum $\Omega_\infty X$ are infinite suspension objects in $\mathcal{X}$. We note that even without these commutation assumptions on $\mathcal{X}$ the stabilization and costabilization of $\mathcal{X}$ will still exist, but it will not admit as simple a description in the familiar terms of prespectra.

The functors Σ^∞ and Ω_∞ are analogues of the previously mentioned abelianization and coabelianization. As usual, the suspension spectrum functor $\Sigma^\infty : \mathcal{X} \to \mathrm{Stab}(\mathcal{X})$ is the left adjoint to the infinite loop functor Ω^∞. Likewise, the right adjoint $\Omega_\infty : \mathcal{X} \to \mathrm{Costab}(\mathcal{X})$ is the loop spectrum functor. This obtains the following diagram of adjunctions:

$$
\begin{tikzcd}
& \mathcal{X} \arrow[dl, "\Omega_\infty"'] \arrow[dr, bend left, "\Sigma^\infty"] & \\
\mathrm{Costab}(\mathcal{X}) \arrow[ur, bend left, "\Sigma_\infty"] \arrow[rr, bend left, "\Sigma^\infty \circ \Sigma_\infty"] & & \mathrm{Stab}(\mathcal{X}) \arrow[ul, "\Omega^\infty"'] \arrow[ll, "\Omega_\infty \circ \Omega^\infty"]
\end{tikzcd}
$$

The compositions above produce an adjunction between the stabilization and costabilization of a pointed ∞-category $\mathcal{X}$. We now turn to the question of the efficacy of stable or costable objects in describing $\mathcal{X}$. Previously, we had the notion of whether an ordinary category had enough abelian groups or cogroups. In the ∞-categorical setting, we can state the corresponding concepts, of whether an ∞-category $\mathcal{X}$ has enough infinite loop objects, or enough infinite suspension objects.

Let us consider the case of stabilization first. The adjunction between $\mathcal{X}$ and $\mathrm{Stab}(\mathcal{X})$ provides an automatic approximation to the ∞-category $\mathcal{X}$ as objects of $\mathrm{Stab}(\mathcal{X})$ equipped with extra structure: given $X \in \mathcal{X}$, we obtain the associated object $\Sigma^\infty X$ in $\mathrm{Stab}(\mathcal{X})$, and this object naturally has the additional structure of a comodule over the comonad $C := \Sigma^\infty \Omega^\infty$. Thus, we obtain a functor $\tilde{\Sigma}^\infty : \mathcal{X} \to \mathrm{Comod}_C(\mathrm{Stab}\,\mathcal{X})$. This functor is colimit preserving, and modulo some modest hypotheses it will have a right adjoint.

We thus obtain two approximations, as algebras and coalgebras, to the ∞-category $\mathcal{X}$ in terms of stable data: one approximation is like a ∞-category of comodules over the comonad $C = \Sigma^\infty \Omega^\infty$ in $\mathrm{Stab}(\mathcal{X})$, the other approximation is the ∞-category of modules over the monad $T = \Omega_\infty \Sigma_\infty$ in $\mathrm{Costab}(\mathcal{X})$. This leads to the following enhancement our prevous diagram:

$$
\begin{tikzcd}
\mathrm{Mod}_T(\mathrm{Costab}\,\mathcal{X}) \arrow[rr, bend left] \arrow[dd] \arrow[dr, bend right] & & \mathrm{Comod}_C(\mathrm{Stab}\,\mathcal{X}) \arrow[ll] \arrow[dl] \arrow[dd, bend right] \\
& \mathcal{X} \arrow[ul, "\widetilde{\Omega}_\infty"'] \arrow[ur, bend left, "\widetilde{\Sigma}^\infty"] \arrow[dl, "\Omega_\infty"] \arrow[dr, bend right, "\Sigma^\infty"'] & \\
\mathrm{Costab}(\mathcal{X}) \arrow[uu, bend left, "\mathrm{free}_T"] \arrow[ur, bend left, "\Sigma_\infty"] \arrow[rr, bend left] & & \mathrm{Stab}(\mathcal{X}) \arrow[ll] \arrow[uu, "\mathrm{cofree}_C"'] \arrow[ul, "\Omega^\infty"']
\end{tikzcd}
$$

Lurie's ∞-categorical Barr-Beck theorem, [**L2**], says when these two approximations will produce equivalences. The functor $\mathcal{X} \to \mathrm{Mod}_T(\mathrm{Costab}\,\mathcal{X})$ is an equivalence if and only if:

- the functor $\Omega_\infty : \mathcal{X} \to \mathrm{Costab}(\mathcal{X})$ is conservative, which is equivalent to $\mathcal{X}$ having enough infinite suspension objects;
- the functor Ω_∞ preserves Ω_∞-split geometric realizations. E.g., it suffices for Ω_∞ to preserve all geometric realizations.

Likewise, the functor $\mathcal{X} \to \mathrm{Comod}_C(\mathrm{Stab}\,\mathcal{X})$ is an equivalence if and only if:

- the functor $\Sigma^\infty : \mathcal{X} \to \mathrm{Stab}(\mathcal{X})$ is conservative, which is equivalent to $\mathcal{X}$ having enough infinite loop objects (or enough cohomology theories);
- the functor Σ^∞ preserves Σ^∞-split totalizations. E.g., it suffices for Σ^∞ to preserve all totalizations.

Proposition 5.6. *Let $\mathcal{C}$ be a stable ∞-category, and let $\mathcal{X}$ be the ∞-category of S-modules in $\mathcal{C}$ for a monad S, $\mathcal{X} := \mathrm{Mod}_S(\mathcal{C})$. Then the natural functor $\widetilde{\Omega}_\infty : \mathcal{X} \to \mathrm{Mod}_T(\mathrm{Costab}\,\mathcal{X})$ is an equivalence.*

Proof. The forgetful functor $G : \mathrm{Mod}_S(\mathcal{C}) \to \mathcal{C}$ is a limit preserving functor to a stable ∞-category. Thus, by the universal property of the functor $\Omega_\infty : \mathrm{Mod}_S(\mathcal{C}) \to \mathrm{Costab}(\mathrm{Mod}_S\,\mathcal{C})$, there exists a natural factorization of G through $\mathrm{Costab}(\mathrm{Mod}_S\,\mathcal{C})$ via a functor $\widetilde{G} : \mathrm{Costab}(\mathrm{Mod}_S\,\mathcal{C}) \to \mathcal{C}$. I.e., we have the following structure of a commutative diagram:

$$
\begin{tikzcd}
& \mathrm{Mod}_S(\mathcal{C}) \arrow[dl, "\Omega_\infty"'] \arrow[dr, "G"] & \\
\mathrm{Costab}(\mathrm{Mod}_S\,\mathcal{C}) \arrow[rr, "\widetilde{G}"] & & \mathcal{C}.
\end{tikzcd}
$$

To prove the functor $\widetilde{\Omega}_\infty$ is an equivalence, as noted above, it suffices to show that Ω_∞ satisfies the monadic Barr-Beck hypotheses. We are assured that the forgetful functor $G : \mathrm{Mod}_S(\mathcal{C}) \to \mathcal{C}$ is conservative and preserves G-split geometric realizations. The factorization above immediately implies that Ω_∞ is thus conservative and preserves G-split geometric realizations. Now we need only note that any Ω_∞-split simplicial object is in particular G-split, and therefore Ω_∞ in particular preserves Ω_∞-split geometric realizations. Ω_∞ thus satisfies the monadic Barr-Beck criteria, implying the result. $\square$

We have the following consequence of this result.

Corollary 5.7. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category which is stable and such that the monoidal structure distributes over colimits, and let $\mathcal{O}$ be an operad. Then natural functor $\widetilde{\Omega}_\infty : \mathcal{O}\text{-alg}(\mathcal{C}) \to \mathrm{Mod}_T(\mathrm{Costab}(\mathcal{O}\text{-alg}\,\mathcal{C}))$ is an equivalence.*

PROOF. Under these hypotheses on $\mathcal{C}$, there is an equivalence $\mathcal{O}\text{-alg}(\mathcal{C}) \simeq \mathrm{Mod}_S(\mathcal{C})$, where S is the monad on $\mathcal{C}$ defined by the free $\mathcal{O}$-algebra functor. Thus, we may apply the previous proposition. $\square$

The proposition above is completely self-opposite, i.e., we obtain the following by passing to opposite ∞-categories.

PROPOSITION 5.8. *Let B be a comonad on a stable ∞-category $\mathcal{C}$, and set $\mathcal{X}$ to be the ∞-category of B-comodules in $\mathcal{C}$. Then the naturally defined functor $\widetilde{\Sigma}^\infty : \mathcal{X} \to \mathrm{Comod}_C(\mathrm{Stab}\,\mathcal{X})$ is an equivalence.*

PROOF. The argument is identical to that given for the proposition above. $\square$

We finally suggest the following rough interpretation of some of the discussion above. The adjunction $\mathrm{Mod}_T(\mathrm{Costab}\,\mathcal{X}) \leftrightarrows \mathrm{Comod}_C(\mathrm{Stab}\,\mathcal{X})$ is a natural duality between algebraic and coalgebraic approximations to an unstable homotopy theory $\mathcal{X}$. For instance, if $\mathcal{X}$ arises as some type of theory of algebras, then the approximation as $\mathcal{X} \to \mathrm{Mod}_T(\mathrm{Costab}\,\mathcal{X})$ should automatically be an equivalence, and the approximation as $\mathrm{Comod}_C(\mathrm{Stab}\,\mathcal{X})$ will be some nontrivial coalgebraic approximation that may or may not be an equivalence. Likewise, if $\mathcal{X}$ arises as a theory of coalebras, the functor $\widetilde{\Sigma}^\infty$ will automatically be an equivalence, while the functor $\widetilde{\Omega}_\infty$ will implement some nontrivial duality – $\widetilde{\Omega}_\infty$ for coalgebras is some variant of the derived functor of primitives valued in comodules, just as $\widetilde{\Sigma}^\infty$ for algebras is a variant of the derived functor of indecomposables and valued in modules.

If however, $\mathcal{X}$ arises in some completely different manner, neither $\widetilde{\Sigma}^\infty$ or $\widetilde{\Omega}_\infty$ are guaranteed to be good approximations. For instance, if $\mathcal{X}$ is the ∞-category of groupoids then both functors are trivial, and more generally this will be true whenever the objects of an ∞-category are both n-connected and m-coconnected.

For an ∞-category of algebras $\mathcal{O}\text{-alg}(\mathcal{C})$, although the functor $\widetilde{\Omega}_\infty : \mathcal{O}\text{-alg}(\mathcal{C}) \to \mathrm{Mod}_T(\mathrm{Costab}\,\mathcal{O}\text{-alg}(\mathcal{C}))$ is guaranteed to be an equivalence, it not the case that the costabilization functor $\Omega_\infty : \mathcal{O}\text{-alg}(\mathcal{C}) \to \mathrm{Costab}(\mathcal{O}\text{-alg}(\mathcal{C}))$ is uninformative. Roughly, one can think that the costabilization $\Omega_\infty(A)$ contains not just the data of the underlying object of A in $\mathcal{C}$, but also the action of an algebra of stable operations on $\mathcal{O}$-algebras. That is, we have a natural commutative diagram

$$\begin{tikzcd}
\mathrm{Costab}(\mathcal{O}\text{-alg}) \arrow[dr, "\widetilde{G}"'] & \mathcal{O}\text{-alg} \arrow[l] \arrow[d, "G"] \\
 & \mathcal{C}
\end{tikzcd}$$

Set $R := \mathrm{Hom}(G, G)$. R can be thought as an algebra of operations acting on the underlying object of any $\mathcal{O}$-algebra, i.e., R is a type of $\mathcal{O}$-Dyer-Lashof algebra. Then there is a natural map $\mathrm{Costab}(\mathcal{O}\text{-alg}) \to \mathrm{Mod}_R(\mathcal{C})$ that factorizes $\widetilde{G}$, and this map should be close to an equivalence.

6. Algebra Structures on the Cotangent Complex

In this section, we will specialize the previously discussed duality between algebra and coalgebra approximations of an ∞-category $\mathcal{X}$ to the setting where $\mathcal{X}$ arises as $\mathcal{O}$-algebras in a stable ∞-category $\mathcal{C}$.

This class of examples also has a more classical motivation. In the case of a discrete commutative ring A, the module of Kähler differentials Ω_A has the structure of a Lie A-coalgebroid. After dualizing Ω_A, this gives the perhaps more familiar statement that vector fields on A forms a sheaf of Lie algebras on $\mathrm{Spec}\,A$ or, more specifically, a Lie A-algebroid. If A is provided an augmentation $A \to k$, the cotangent space of A has a Lie coalgebra structure, and likewise the tangent space of A, $T_{k|A}$, has the structure of a Lie algebra.

We will study derived analogues of the above statement in the general setting of algebras over an operad.

6.1. Bar Constructions. Let us start with following basic construction: let T be an augmented algebra in the monoidal ∞-category $\mathrm{Fun}'(\mathcal{C},\mathcal{C})$, i.e., an augmented monad. Denote the augmentation $\eta : T \to \mathrm{id}$, where id is the identity functor of $\mathcal{C}$. We assume that the underlying functor of T commutes with geometric realizations, and therefore the restriction functor $\mathrm{Res}_\eta : \mathcal{C} \to \mathrm{Mod}_T(\mathcal{C})$ has a left adjoint Ind_η. The composition $\mathrm{Ind}_\eta \circ \mathrm{Res}_\eta$ has the structure of a comonad on $\mathcal{C}$, with counit $\mu : C \to \mathrm{id}$, leading to the following canonical diagram:

$$\begin{tikzcd}
\mathrm{Mod}_T(\mathcal{C}) \arrow[rr, bend left, "\widetilde{\mathrm{Ind}}_\eta"] \arrow[dr, bend right, "\mathrm{Ind}_\eta"'] & & \mathrm{Comod}_C(\mathcal{C}) \arrow[dl, bend right, "\mathrm{Res}_\mu"'] \\
& \mathcal{C}. \arrow[ul, "\mathrm{Res}_\eta"'] \arrow[ur, "\mathrm{Coind}_\mu"'] &
\end{tikzcd}$$

The above construction is functorial in the augmented monad T, and thus we obtain a functor from augmented monads to coaugmented comonads

$$\mathrm{Monads}'^{\mathrm{aug}}(\mathcal{C}) \longrightarrow \mathrm{Comonads}'^{\mathrm{aug}}(\mathcal{C})$$

assigning to an augmented monad $\eta : T \to \mathrm{id}$ the comonad C associated to adjunction $(\mathrm{Ind}_\eta, \mathrm{Res}_\eta)$ on $\mathcal{C}$. From our previous discussion, the value of the induction functor $\mathrm{Ind}_\eta(X)$ is calculated by the geometric realization of the simplicial bar construction $\mathrm{id} \circ_T X \simeq |\mathrm{Bar}(\mathrm{id}; T; X)|$. Since colimits in $\mathrm{Fun}'(\mathcal{C},\mathcal{C})$ are calculated objectwise in the target, there is a natural equivalence $\mathrm{id} \circ_T X \simeq (\mathrm{id} \circ_T \mathrm{id})(X)$. Thus, the functor defined above is given by the usual bar construction of an augmented algebra, which we can rewrite

$$\mathrm{Alg}^{\mathrm{aug}}(\mathrm{Fun}'(\mathcal{C},\mathcal{C})) \xrightarrow{\mathrm{Bar}} \mathrm{Coalg}^{\mathrm{aug}}(\mathrm{Fun}'(\mathcal{C},\mathcal{C})).$$

Remark 6.1. Again, it is necessary to require the functor underlying the monad T to commute with geometric realizations. Forgoing this assumption may entail the following interrelated mishaps: there may be no comonad structure on the bar construction $\mathrm{id} \circ_T \mathrm{id}$; the ∞-category $\mathrm{Mod}_T(\mathcal{C})$ might not have colimits; and the induction functor Ind_η may fail to exist.

Now let us assume that $\mathcal{C}$ is a monoidal ∞-category, and that the monoidal structure $\otimes$ on $\mathcal{C}$ distributes over filtered colimits. We thus obtain a functor $\mathcal{C} \to \mathrm{Fun}'(\mathcal{C},\mathcal{C})$ sending an object X to the functor $X \otimes (-)$. This functor is monoidal, and thus results in a functor $\mathrm{Alg}^{\mathrm{aug}}(\mathcal{C}) \to \mathrm{Monads}'^{\mathrm{aug}}(\mathcal{C})$ sending an algebra A to the monad with underlying functor $T = A \otimes (-)$. Applying the construction above results in a comonad $C \simeq \mathrm{id} \circ_T \mathrm{id}$ whose underlying functor is equivalent to $C(X) \simeq (1 \otimes_A 1) \otimes X$. This results in a coaugmented comonad structure on the functor $1 \otimes_A 1 \otimes (-)$, and hence a coaugmented coalgebra structure on $1 \otimes_A 1$.

Example 6.2. Let $\mathcal{C}$ be the monoidal ∞-category of symmetric sequences in a symmetric monoidal ∞-category $\mathcal{A}$. Augmented algebras in $\mathcal{C}$ are thus the same as augmented operads in $\mathcal{A}$, and coaugmented coalgebras are likewise the same as augmented cooperads. The construction above thus gives a functor $\mathrm{Operads}^{\mathrm{aug}}(\mathcal{C}) \to \mathrm{Cooperads}^{\mathrm{aug}}(\mathcal{C})$ computed by the bar construction.

Now let $\mathcal{C}$ be a symmetric monoidal ∞-category, not just monoidal as as above, and assume further that its monoidal structure distributes over colimits. Then there is a sequence of monoidal functors $\mathcal{C} \to \mathcal{C}^\Sigma \to \mathrm{Fun}'(\mathcal{C},\mathcal{C})$, where we map an object X of $\mathcal{C}$ to the symmetric sequence which is X concentrated in degree one. Since these are monoidal functors, they induce functors from the corresponding ∞-categories of algebras. These are compatible with the bar construction in each case because of the following lemma.

Lemma 6.3. *If $\mathcal{C}$ is a monoidal ∞-category which is cocomplete and whose monoidal structure distributes over geometric realizations, then the functors $\mathcal{C}^\Sigma \to \mathrm{Fun}'(\mathcal{C},\mathcal{C}) \to \mathrm{Fun}(\mathcal{C},\mathcal{C})$ commute with geometric realizations.*

Proof. Colimits in an ∞-category of functors are computed in the target. Thus, the inclusion $\mathrm{Fun}'(\mathcal{C},\mathcal{C}) \to \mathrm{Fun}(\mathcal{C},\mathcal{C})$ preserves colimits, and it suffices to just consider the functor $\mathcal{C}^{\Sigma} \to \mathrm{Fun}(\mathcal{C},\mathcal{C})$. Let $X_{\bullet}$ be a simplicial object in symmetric sequences $\mathcal{C}^{\Sigma}$ and $iX_{\bullet}$ the associated simplicial object in endofunctors of $\mathcal{C}$. There is a natural map $|iX_{\bullet}| \to i|X_{\bullet}|$. To check the map is an equivalence, we check their values on an arbitrary test object C in $\mathcal{C}$. The map above can then be expanded as

$$|\coprod_{n\geq 0} X_{\bullet}(n) \otimes_{\Sigma_n} C^{\otimes n}| \longrightarrow \coprod_{n\geq 0} |X_{\bullet}(n)| \otimes_{\Sigma_n} C^{\otimes n}$$

which is an equivalence because the monoidal structure was assumed to distribute over geometric realizations (and because coproducts, geometric realizations, and Σ_n-orbits are all colimits and hence commute for formal reasons). $\square$

Since the bar construction is computed as a geometric realization, and the functors $\mathcal{C} \to \mathcal{C}^{\Sigma} \to \mathrm{Fun}'(\mathcal{C},\mathcal{C})$ are monoidal and compatible with geometric realizations, we obtain the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Monads}'^{\mathrm{aug}}(\mathcal{C}) \arrow[r, "\mathrm{Bar}"] & \mathrm{Comonads}'^{\mathrm{aug}}(\mathcal{C}) \\
\mathrm{Operads}^{\mathrm{aug}}(\mathcal{C}) \arrow[r, "\mathrm{Bar}"] \arrow[u] & \mathrm{Cooperads}'^{\mathrm{aug}}(\mathcal{C}) \arrow[u] \\
\mathrm{Alg}^{\mathrm{aug}}(\mathcal{C}) \arrow[r, "\mathrm{Bar}"] \arrow[u] & \mathrm{Coalg}^{\mathrm{aug}}(\mathcal{C}) \arrow[u]
\end{tikzcd}$$

The induction functor Ind_{η} often has a more computable form. Let $\mathcal{C}$ be a stable monoidal ∞-category whose monoidal structure $\otimes$ is compatible with colimits. Denote the unit of the monoidal structure by 1, as usual. The ∞-category of augmented algebras $\mathrm{Alg}^{\mathrm{aug}}(\mathcal{C})$ is the equivalent to $\mathrm{Mod}_T(\mathcal{C})$, where $T(X)$ is the free augmented algebra on X, $T(X) = \coprod_{n\geq 0} X^{\otimes n}$. T is an augmented monad with $\eta : T \to \mathrm{id}$.

Proposition 6.4. *Let $\mathcal{C}$ be as above, and let $I : \mathrm{Alg}^{\mathrm{aug}}(\mathcal{C}) \to \mathcal{C}$ denote the functor that assigns to an augmented algebra A of $\mathcal{C}$ the kernel of the counit map $1 \otimes_A 1 \to 1$. Then there is a natural equivalence $\mathrm{Ind}_{\eta}(A)[1] \simeq I(A)$.*

Proof. Note that under the hypothesis above, the induction functor Ind_{η} is implementing the stabilization of the ∞-category of augmented algebras in $\mathcal{C}$, i.e., $\mathrm{Ind}_{\eta} \simeq \Sigma^{\infty}$. Since the functor $I : \mathrm{Alg}^{\mathrm{aug}}(\mathcal{C}) \to \mathcal{C}$ is colimit preserving, by the universal property of stabilization there is thus a natural transformation $\mathrm{Ind}_{\eta} \to I$. To verify that a natural transformation between colimit preserving functors is an equivalence, it suffices to check on a collection of objects that generate the ∞-category under colimits. Thus, we can check on free algebras, which we now do.

Consider $A \simeq T(X)$ the free augmented algebra on an object X of $\mathcal{C}$. There is a cofiber sequence $\mathrm{Ker}(\epsilon) \to A \xrightarrow{\epsilon} 1$, and the kernel of the augmentation is equivalent to the following coproduct calculated in $\mathcal{C}$, $\mathrm{Ker}(\epsilon) \simeq \coprod_{n\geq 1} X^{\otimes n}$. Since the monoidal structure is compatible with infinite coproducts, we can deduce formally that $\mathrm{Ker}(\epsilon) \simeq X \otimes A$. I.e., there is a cofiber sequence $X \otimes A \to A \to 1$. Applying the exact functor $(-)\otimes_A 1$ obtains a new cofiber sequence $(X\otimes A)\otimes_A 1 \to A \otimes_A 1 \to 1 \otimes_A 1$, which can be simplified to $X \to 1 \to 1 \otimes_A 1$, or $1 \to 1\otimes_A 1 \to X[1]$. The counit map $1 \otimes_A 1 \to 1$ splits this cofiber sequence, and we thus derive the natural equivalence $1 \otimes_A 1 \simeq 1 \oplus X[1]$, i.e., that $X[1] \simeq I(A)$. Since $\mathrm{Ind}_{\eta}(A) \simeq X$, this completes the proof. $\square$

Remark 6.5. This result can be interpreted as saying that a shift of the cotangent complex of an augmented algebra has the structure of a non-unital coalgebra.

In some sense, the essential ingredient in the proof above was the recursive form of the free algebra functor $T(X)$, that $T(X) \simeq 1 \oplus X \otimes T(X)$. The free algebra functor may have such a recursive form, even without the requirement that the monoidal structure distributes over colimits. The following result is due to Rezk.

Proposition 6.6. [**Re**] *Let $\mathcal{C}$ be a monoidal ∞-category which is cocomplete and such that the functor $\mathcal{C} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}$ preserves colimits in the left variable and filtered colimits in each variable. Then there is a natural equivalence $T(X) \simeq 1 \amalg X \otimes T(X)$ in $\mathcal{C}$.*

A consequence is that the previous proposition holds in greater generality.

Proposition 6.7. *Let $\mathcal{C}$ be a stable monoidal ∞-category whose monoidal structure preserves small colimits in the left variable and filtered colimits in each variable. Then there is a natural equivalence $\mathrm{Ind}_\eta(A)[1] \simeq I(A)$, where Ind_η and I are defined as above.*

Proof. The argument above applies verbatim. However, some extra care is required, because without the assumption that the monoidal structure distributes over coproducts, the equivalence between augmented and non-unital no longer holds. $\square$

This extra generality allows the inclusion of several very interesting examples into the above algebra/coalgebra schema.

Example 6.8. For instance, let $\mathcal{C}$ be the monoidal ∞-category $\mathrm{Fun}'(\mathcal{A}, \mathcal{A})$ of sifted colimit preserving endofunctors of a stable ∞-category $\mathcal{A}$. The composition product $\circ$ distributes over all small colimits on the left, and $\circ$ distributes over sifted colimits in each variable. Then augmented algebras in $\mathcal{C}$ (i.e., augmented monads in $\mathcal{A}$) are equivalent to modules over a certain augmented monad T in $\mathcal{C}_1$.

$$\begin{tikzcd}
\mathrm{Alg}^{\mathrm{aug}}(\mathrm{Fun}'(\mathcal{A},\mathcal{A})) \arrow[rr, "\mathrm{Bar}"] \arrow[d, equal] & & \mathrm{Coalg}^{\mathrm{aug}}(\mathrm{Fun}'(\mathcal{A},\mathcal{A})) \arrow[d, equal, "\text{shift } \pm 1"] \\
\mathrm{Mod}_T(\mathrm{Fun}'(\mathcal{A},\mathcal{A})) \arrow[rr, "\widetilde{\mathrm{Ind}}_\eta"] \arrow[dr, "\mathrm{Ind}_\eta"'] & & \mathrm{Comod}_C \\
& \mathrm{Fun}'(\mathcal{A},\mathcal{A})^{\mathrm{id}/}_{/\mathrm{id}} &
\end{tikzcd}$$

And a similar thing, once properly formulated, holds for the ∞-category of operads.

6.2. Local Structure on the Cotangent Complex. The upshot of much of the discussion above is that certain standard dualities between algebras and coalgebras – such as the bar construction from algebras to coalgebras – are implementing in particular cases the previous general algebra-coalgebra duality obtained by costabilization versus stabilization. This recommends the general procedure for application to slightly more exotic algebra structures, such as algebras over a more general operad.

We now address the structure on the cotangent complexes of augmented, or nonunital, $\mathcal{O}$-algebras. The idea is that the Koszul duality of algebras over operads may be interpreted as assigning the tangent complex to a formal moduli functor, and can thus be viewed in terms of stabilization. From our study of stabilization, we know that the stabilization $\mathrm{Stab}(\mathcal{O}\text{-alg}^{\mathrm{nu}})$ is equivalent to $\mathcal{C}$. From the previous section, we know that the value $\Sigma^\infty A \simeq \mathsf{L}(A)$ carries the structure of a non-unital $1 \circ_{\mathcal{O}} 1$-coalgebra.

6.3. Structure on the Global Cotangent Complex. Let A be a unital $\mathcal{O}$-algebra. We then have the following picture from our previous discussion, specifying that the absolute cotangent complex of A is a comodule over comonad $C = \mathsf{L}_A \circ (A \oplus -)$.

$$
\begin{tikzcd}
\mathcal{O}\text{-alg}_{/A} \arrow[rr, bend left, "\widetilde{\mathsf{L}_A}"] \arrow[dr, bend right, "\mathsf{L}_A"'] & & \mathrm{Comod}_C(\mathrm{Mod}_A^{\mathcal{O}}) \arrow[dl, bend right] \\
& \mathrm{Mod}_A^{\mathcal{O}} \arrow[ul, "A\oplus -"'] \arrow[ur] &
\end{tikzcd}
$$

In the case where the operad $\mathcal{O}$ is the commutative, or $\mathcal{E}_\infty$, operad. The situation above recovers derived versions of the familiar notions of Lie algebroids and coalgebroids. Let $\mathcal{E}_\infty$ denote the commutative operad in pointed spaces, obtained from the commutative operad in spaces by adjoining for each n a disjoint basepoint to the nth space of the operad.

Definition 6.9. The derived Lie cooperad coLie is the simplicial bar construction of the pointed commutative operad $\mathcal{E}_\infty$, i.e., $\mathrm{coLie} = 1 \circ_{\mathcal{E}_\infty} 1$.

Remark 6.10. Although it is more standard to deal with Lie algebras only in the stable context, such as chain complexes, the notion of coLie coalgebras is sensible in any pointed ∞-category without the hypothesis of stability.

Let A be a commutative algebra in stable symmetric monoidal ∞-category $\mathcal{C}^\otimes$, and let C denote the comonad on A-modules given by $C(M) = \mathsf{L}_A(A \oplus M)$. Then the ∞-category of comodules over the comonad C is describes an ∞-category of coLie-A-coalgebroids:

$$
\begin{tikzcd}
\mathcal{E}_\infty\text{-alg}_{/A} \arrow[rrr, bend left, "\tilde{\mathsf{L}}_A"] \arrow[dr, bend right, "A\otimes -"'] \arrow[ddrr, bend right, "\mathsf{L}_A"'] & & & \mathrm{Comod}_C(\mathrm{Mod}_A) \arrow[r, "\simeq"] \arrow[d, bend right] & \{\mathrm{coLie} - A - \mathrm{coalgebroids}\} \\
& \mathcal{E}_\infty\text{-alg}_A^{\mathrm{aug}} \arrow[ul] \arrow[rr, bend left, "\tilde{\mathsf{L}}_{A|A}"] \arrow[dr, bend right, "\mathsf{L}_{A|A}"'] & & \mathrm{Comod}_{C'}(\mathrm{Mod}_A) \arrow[r, "\simeq"] \arrow[u] \arrow[dl, bend right] & \{\mathrm{coLie} - A - \mathrm{coalgebras}\} \\
& & \mathrm{Mod}_A \arrow[ul, "A\oplus -"'] \arrow[ur] & &
\end{tikzcd}
$$

We will describe in a future treatment of these ideas how, in a similar sense, the global cotangent complex of a general $\mathcal{O}$-algebra A has the structure of a $1 \circ_{\mathcal{O}} 1$-A-coalgebroid, and the global tangent complex of A has the structure of an $(1 \circ_{\mathcal{O}} 1)^\vee$-$A$-algebroid.

CHAPTER 4

Algebraic Geometry over $\mathcal{E}_n$-Rings

1. Gluing $\mathcal{E}_n$-Rings

In this section, we develop an approach to algebraic geometry over $\mathcal{E}_n$-algebras based on Grothendieck's notion of the functor of points, similar to the approaches of [**L0**] or [**TV2**]. To make sense of an algebro-geometric object X defined over a certain category of rings $\mathcal{R}$, we should specify the data of the object and the property of the geometry: to give the data of object, one must assign R-points of X, for every $R \in \mathcal{R}$, which one may think of the R-valued solutions of the equations defining X; second, one specifies in what sense solutions glue together. This second condition may be interpreted as giving a Grothendieck topology on the (opposite) ∞-category $\mathcal{R}$, which is equivalent to usual notion of a Grothendieck topology on the homotopy category of $\mathcal{R}^{\mathrm{op}}$.

Thus, our first order of business will be to define several topologies on the ∞-category of $\mathcal{E}_n$-rings.

1.1. Grothendieck Topologies on $\mathcal{E}_n$-Rings. The theory of derived algebraic geometry over connective rings more closely resembles classical algebraic geometry, as opposed to when our $\mathcal{E}_n$-rings are allowed to have nonzero homotopy groups in negative dimensions.

We recall the definition of a flat module for an A_∞-ring given in [**L2**].

Definition 1.1. Let A be an A_∞-ring, and let M be a left A-module. M is flat if:

- $\pi_0 M$ is a flat $\pi_0 A$-module in the classical sense (i.e., $(-)\otimes_{\pi_0 A}\pi_0 M$ is an exact functor);
- The natural maps $\pi_n A \otimes_{\pi_0 A} \pi_0 M \to \pi_n M$ are isomorphisms.

We will further say the module M is faithfully flat if it is flat and the functor $(-)\otimes_A M$ is conservative.

Using the above notion of flatness, we now present a notion of an étale map. In the case $n = \infty$, this is precisely the definition given in [**L4**] and very similar to the notion presented in [**TV1**].

Definition 1.2. An $\mathcal{E}_n$-ring map $f : A \to B$ is étale if:

- B is flat as a left A-module;
- The map $\pi_0 f : \pi_0 A \to \pi_0 B$ is an étale map of discrete commutative rings.

Using this definition of an étale map, we can give a notion of a Zariski open.

Definition 1.3. An $\mathcal{E}_n$-ring map $f : A \to B$ is a Zariski open immersion if it is étale and the map $\pi_0 f$ is a Zariski open immersion of discrete commutative rings.

We are now in a position to define the topologies of interest on $\mathcal{E}_n$-rings. As described in [**TV1**] or [**L**], a topology on a homotopy theory of rings $\mathcal{R}$ will be equivalent to giving a Grothendieck topology on $\mathrm{h}\mathcal{R}^{\mathrm{op}}$, the opposite of the homotopy category of $\mathcal{R}$.

Definition 1.4. The étale topology $\mathcal{T}_{\acute{e}t}$ on $\mathcal{E}_n$-rings has admissible morphisms given by étale maps $A \to A'$ and covering families consisting of collections $\{f_\alpha : A \to A_\alpha : f_\alpha \text{ étale},\ \alpha \in I\}$ for which there exists a finite subset $I' \in I$ such that the map

$$A \to \prod_{\alpha \in I'} A_\alpha$$

is faithfully flat.

The étale topology will our primary focus. We also give a definition of the Zariski topology.

DEFINITION 1.5. The Zariski topology $\mathcal{T}_{Zar}$ on $\mathcal{E}_n$-rings has admissible morphisms $f : A \to A'$ that are étale and such that $\pi_0 f$ is a Zariski open immersion. The covering families are collections of Zariski admissible maps subject to the same conditions as for the étale topology.

1.2. Localization of $\mathcal{E}_n$-Rings. An important facet of the classical theory of commutative rings is the ability to localize, i.e., to invert an element x of a ring A to obtain a ring $A[x^{-1}]$ that models the original ring A away from x. In this section, we will study such localization procedures for $\mathcal{E}_n$-rings. The availability of such features will be important in the study of the algebraic geometry of $\mathcal{E}_n$-rings, and we will make immediate use of it in the subsequent section to construct the Zariski spectrum of an $\mathcal{E}_n$-ring.

We begin by considering what functorial property a localization should possess. For the definition below, let A be an $\mathcal{E}_n$-ring, x an element of $\pi_0 A$, and denote by X the functor that A corepresents, i.e., $X(B) = \mathrm{Map}_{\mathcal{E}_n}(A, B)$.

DEFINITION 1.6. For A, x, and X as above, then $X - x$ is a functor from $\mathcal{E}_n$-rings to Spaces, together with a natural transformation of functors $X - x \to X$, such that for any B the map $(X - x)(B) \to X(B) = \mathrm{Map}_{\mathcal{E}_n}(A, B)$ is a homotopy equivalence between the left hand side and the subspace of the right hand side consisting of those components for which the image of x is invertible in $\pi_0 B$.

In other words, this identifies what functor the localization of a ring A at an element x should corepresent. Our reason for choosing the suggestive notation for $X - x$ will be clear later in discussion of the related geometry. In the following definition, let A, x, and $X - x$ be as above.

DEFINITION 1.7. The localization of A at x is an $\mathcal{E}_n$-ring $A[x^{-1}]$ together with a map $A \to A[x^{-1}]$, such that $A[x^{-1}]$ corepresents the functor $X - x$ defined above.

If the $\mathcal{E}_n$-ring $A[x^{-1}]$ exists, then it is essentially unique by the ∞-categorical Yoneda lemma.

PROPOSITION 1.8. *For A an $\mathcal{E}_n$-ring with $x \in \pi_0 A$, the localization $A[x^{-1}]$ exists.*

PROOF. We will make the following construction, which we will afterward show satisfies the correct universal property of the localization. Consider the $\mathcal{E}_n$-ring $A \amalg F(\mathbb{S})$, the coproduct in $\mathcal{E}_n$-rings of A with the free $\mathcal{E}_n$-ring on the sphere spectrum. There exist a natural map of of graded abelian groups $\pi_* A \otimes \pi_* \mathbb{S} \to \pi_*(A \amalg F(\mathbb{S}))$. Thus, the choice of the element x of $\pi_0 A$ and $1 \in \pi_0 \mathbb{S}$ thus gives an element of $\pi_0(A \amalg F(\mathbb{S}))$, which may be represented by a map of spectra $\mathbb{S} \to A \amalg F(\mathbb{S})$. This map is adjoint to a map of $\mathcal{E}_n$-rings $F(\mathbb{S}) \to A \amalg F(\mathbb{S})$. Likewise the identity map $\mathbb{S} \to \mathbb{S}$ of spectra is adjoint to a map of $\mathcal{E}_n$-rings $F(\mathbb{S}) \to \mathbb{S}$. We now define "$A[x^{-1}]$" to lie in the pushout square

$$\begin{tikzcd}
F(\mathbb{S}) \arrow[r] \arrow[d] & A \amalg F(\mathbb{S}) \arrow[d] \\
\mathbb{S} \arrow[r] & \text{“}A[x^{-1}]\text{”}.
\end{tikzcd}$$

We now check that "$A[x^{-1}]$" satisfies the correct mapping property. By definition, for any $\mathcal{E}_n$-ring B we have the following homotopy pullback square of spaces

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{E}_n}(\text{“}A[x^{-1}]\text{”}, B) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{E}_n}(A \amalg F(\mathbb{S}), B) \arrow[d] \\
\mathrm{Map}_{\mathcal{E}_n}(\mathbb{S}, B) \arrow[r] & \mathrm{Map}_{\mathcal{E}_n}(F(\mathbb{S}), B).
\end{tikzcd}$$

We now analyze this diagram. Note first that $\mathrm{Map}_{\mathcal{E}_n}(\mathbb{S}, B)$ is homotopy equivalent to a point, since $\mathbb{S}$ is the initial object of $\mathcal{E}_n$-rings. Second, we have that $\mathrm{Map}_{\mathcal{E}_n}(F(\mathbb{S}), B) \simeq \mathrm{Map}(\mathbb{S}, B)$. We now rewrite the above diagram

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{E}_n}(\text{“}A[x^{-1}]\text{”}, B) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{E}_n}(A, B) \times \mathrm{Map}(\mathbb{S}, B) \arrow[d] \\
* \arrow[r] & \mathrm{Map}(\mathbb{S}, B).
\end{tikzcd}$$

The fiber of the identity map $* \to \mathrm{Map}(\mathbb{S}, B)$ exactly consists of the those maps $A \to B$ and $\mathbb{S} \to B$ such that the product of the right map with x is the identity in B. This thus presents the fiber $\mathrm{Map}_{\mathcal{E}_n}(\text{“}A[x^{-1}]\text{”}, B)$ as the space of maps from A to B such that the image of x is invertible. Hence, we obtain the equivalence “$A[x^{-1}]$” $\simeq A[x^{-1}]$, and therefore $A[x^{-1}]$ exists. $\square$

It is worthwhile to note the following property of localizations, which will be relevant to our later discussion of the cotangent complex of a scheme over $\mathcal{E}_n$-rings.

Lemma 1.9. *Let B be a localization of A, i.e., there is an $\mathcal{E}_n$-ring map $A \to B$ and a map of spectra $V \to A$ such that B is universal among $\mathcal{E}_n$-rings such that the image of V is invertible. Then the relative cotangent complex $L_{B|A}$ is contractible.*

Proof. From our discussion of stabilization and the cotangent complex previously, the relative cotangent complex $L_{B|A}$ can be computed by the stabilization of ∞-category of $\mathcal{E}_n$-A-algebras over B. That is, there is an equivalence $B \oplus L_{B|A} \simeq \Omega^\infty_B \Sigma^\infty_B(B \amalg_A B)$. The pair of functors $\Sigma^\infty_B : \mathcal{E}_n\text{-alg}_B \leftrightarrows \mathrm{Mod}_B^{\mathcal{E}_n} : \Omega^\infty_B$ is an adjunction of pointed ∞-categories, and thus both Σ^∞_B and Ω^∞_B preserve the final objects, providing an equivalence $\Omega^\infty_B \Sigma^\infty_B(B) \simeq B$. $B \amalg_A B$ is the colimit in $\mathcal{E}_n$-rings of the diagram $B \leftarrow A \to B$. Since B is a localization of A, we can see that $\mathcal{E}_n$-ring maps out of $B \amalg_A B$ satisfy the same universal property that B satisfies, and thus there is an equivalence $B \simeq B \amalg_A B$. This then computes that $\Omega^\infty_B \Sigma^\infty_B(B \amalg_A B)$ is equivalent to B, implying that $L_{B|A}$ is contractible.

$\square$

1.3. Derived Schemes and Stacks. In pursuing algebraic geometry over some particular type of rings, it common to simply define the affine objects as the full subcategory of presheaves given by representable objects. In the case of algebraic geometry over $\mathcal{E}_n$-rings, as with usual commutative algebra, it possible to geometrically describe this full subcategory of affine objects, as a sheaf of $\mathcal{E}_n$-algebras on a topological space.

We now present the notion of the Zariski spectrum of an $\mathcal{E}_n$-ring A. Recall, for a (discrete) commutative ring A, the Zariski spectrum of A consists of a topological space $\mathrm{Spec}\, A$ together with a sheaf of commutative rings $\mathcal{O}_{\mathrm{Spec}\, A}$. The space $\mathrm{Spec}\, A$ has as underlying set prime ideals $\mathfrak{p} \subset A$, and its topology is generated by specifying open sets for every $x \in A$ defined by $U_x = \{\mathfrak{p} : x \notin \mathfrak{p}\}$. The sheaf of rings $\mathcal{O}_{\mathrm{Spec}\, A}$ is determined by its values on these generating open sets, on which it is defined to take values $\mathcal{O}_{\mathrm{Spec}\, A}(U_x) := A[x^{-1}]$.

We will make a similar definition for the Zariski spectrum of an $\mathcal{E}_n$-ring A, which will consist of a topological space $\mathrm{Spec}\, A$ equipped with a sheaf of $\mathcal{E}_n$-rings $\mathcal{O}_{\mathrm{Spec}\, A}$. Further, the topological space $\mathrm{Spec}\, A$ will be equivalent the usual Zariski spectrum of the commutative ring $\pi_0 A$. That this the correct approach to making the spectrum of a derived ring is far from obvious, but we will see that constructs an ∞-category equivalent to the opposite ∞-category of $\mathcal{E}_n$-rings, and that it correctly models how we might wish to glue $\mathcal{E}_n$-rings together.

This allows the following construction of Zariski spectra for $\mathcal{E}_n$-rings.

Definition 1.10. The Zariski spectrum $\mathrm{Spec}\, A$ of an $\mathcal{E}_n$-ring A consists of a topological space equipped with a sheaf of $\mathcal{E}_n$-rings, $(\mathrm{Spec}(\pi_0 A), \mathcal{O})$. The underlying topological space is the usual Zariski spectrum of $\pi_0 A$, and the sheaf of $\mathcal{E}_n$-rings $\mathcal{O}$ on $\mathrm{Spec}(\pi_0 A)$ is determined by the values on the generating opens $\mathcal{O}(U_x) := A[x^{-1}]$, for each $x \in \pi_0 A$.

Let X and Y be the Zariski spectra of $\mathcal{E}_n$-rings A and B. We define the space of maps from X to Y to be

$$\mathrm{Map}(X,Y) := \coprod_{\pi_0 B \xrightarrow{f} \pi_0 A} \mathrm{Map}^0_{\mathcal{E}_n}(\mathcal{O}_Y, f_* \mathcal{O}_X).$$

Let $\mathrm{Aff}(\mathcal{E}_n)$ denote the ∞-category of Zariski spectra of $\mathcal{E}_n$-rings, with objects and maps defined as above.

Above, we gave a descpription of an affine scheme associated to an $\mathcal{E}_n$-ring as a topological space with a sheaf of $\mathcal{E}_n$-rings. We now adopt a more abstract approach as in [**L0**], adapted to the $\mathcal{E}_n$ setting. Here, a scheme will be a Grothendieck topos equipped with a sheaf of $\mathcal{E}_n$-rings. This approach is well suited to constructing a large supply of stacks over $\mathcal{E}_n$-rings that nonetheless have recognizably geometric behavior. Much of this treatment could be conducted much more generally, as Toën and Vezzosi do in [**TV2**] and Lurie does in [**L5**].

Recall from [**L**] the notion of an ∞-topos: an ∞-topos $\mathcal{X}$ is a ∞-category equivalent to a localization of an ∞-category of presheaves of spaces on a small ∞-category $\mathscr{P}(\mathcal{A})$ such that the localization functor $L : \mathscr{P}(\mathcal{A}) \to \mathcal{X}$ preserves finite limits and filtered colimits. Intuitively, there should exist some Grothendieck topology on $\mathscr{P}(\mathcal{A})$ with respect to which the objects of $\mathcal{X}$ are sheaves. These topoi will serve as the underlying "space" in our ringed space approach to geometrically describing moduli functors for $\mathcal{E}_n$-rings.

Definition 1.11. An $\mathcal{E}_n$-ringed ∞-topos $(\mathcal{X}, \mathcal{O})$ consists of an ∞-topos $\mathcal{X}$ together with a sheaf $\mathcal{O}$ valued in $\mathcal{E}_n$-rings, i.e., a limit preserving functor $\mathcal{O} : \mathcal{X}^{\mathrm{op}} \to \mathcal{E}_n$ -rings.

Let $\mathcal{T}$ be a topology on $\mathcal{E}_n$ -rings. We now describe the notion of the $\mathcal{T}$-spectrum of an $\mathcal{E}_n$-ring A in terms of a universal property of $\mathcal{T}$-locality. For $\mathcal{T}$ the Zariski topology, this gives a notion corresponding to our previous explicit construction of Zariski spectra. For $\mathcal{T}$ the étale topology, this gives appropriate affine building blocks for more general stacks analogous to Deligne-Mumford stacks.

Definition 1.12. Let $\mathcal{X}$ be an ∞-topos with an $\mathcal{E}_n$-valued sheaf $\mathcal{O}$, and let $\mathcal{T}$ be a topology on $\mathcal{E}_n$ -rings. Then $\mathcal{O}$ is $\mathcal{T}$-local if for any $U \in \mathcal{X}$ and a $\mathcal{T}$-admissible cover $\{f_\alpha : \mathcal{O}(U) \to R_\alpha\}$, the collection $\{\mathrm{Sol}(f_\alpha) \to U\}$ forms a cover of U in $\mathcal{X}$, where $\mathrm{Sol}(f_\alpha)$ is the object of $\mathcal{X}_{/U}$ such that $\mathrm{Hom}_{\mathcal{O}(U)/}(R_\alpha, \mathcal{O}(U')) \simeq \mathrm{Hom}_{/U}(U', \mathrm{Sol}(f_\alpha))$.

The first example of a $\mathcal{E}_n$-ringed topos which is local with respect to a topology on $\mathcal{E}_n$-rings is given by the Zariski spectrum of an $\mathcal{E}_n$-ring considered above. It is an observation of Grothendieck that this notion of locality is a defining property of the spectrum of a commutative ring. Rewinding, suppose we wanted to describe a commutative ring A a space with a sheaf of rings. The most obvious way to achieve this is simply to let the space be a single point and the sheaf of rings just be the constant sheaf with value A. However, this ringed space is not local with respect to the Zariski topology in the sense of the above definition, so one could ask for one that is. This leads to one characterization of the classical spectrum of a commutative ring. Of course, one need not have restricted oneself to the Zariski topology, so we have the definition below.

Definition 1.13. For $A \in \mathcal{E}_n$ -rings and $\mathcal{T}$ a topology on $\mathcal{E}_n$ -rings, the $\mathcal{E}_n$-ringed ∞-topos $\mathrm{Spec}_{\mathcal{T}} A$ is defined as the universal $\mathcal{T}$-locally $\mathcal{E}_n$-ringed ∞-topos with a map to $(*, A)$.

Recall that classically, a scheme is defined as a ringed space that is locally affine with respect to the Zariski topology. Given a topology on $\mathcal{E}_n$, we may now give a notion of affine schemes with respect to this topology.

Definition 1.14. A $\mathcal{T}$-locally affine scheme (i.e., $\mathcal{T}$-scheme) over $\mathcal{E}_n$ -rings consists of an $\mathcal{E}_n$-ringed ∞-topos $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ such that $\mathcal{T}$-locally $\mathcal{X}$ is equivalent to $\mathrm{Spec}_{\mathcal{T}} A$, for $A \in \mathcal{E}_n$ -rings.

Several topologies are available in the case of $\mathcal{E}_n$-rings. For the Zariski topology $\mathcal{T}_{Zar}$, the resulting notion of an affine scheme is closest to usual classical schemes. However, a more flexible notion results if we use the étale topology $\mathcal{T}_{ét}$ and thus allow more gluings. The notion of a $\mathcal{T}_{ét}$-scheme is very close to the classical notion of a Deligne-Mumford stack, which is typically defined to be a ringed topos which is étale locally affine that further satisfies the condition that the diagonal map is separated.

Derived schemes that are étale locally affine obtain many of the good properties of $\mathcal{E}_n$-rings which are étale local. For instance, in future work we will show that the relative cotangent complex of an étale map is contractible, so one may expect statements about the deformation theory of $\mathcal{E}_n$-rings to directly transfer to $\mathcal{T}_{ét}$-schemes. We will see an example of this in our later discussion of the cotangent complex. One might not expect such automatic transfers for a more general class of stacks which are locally affine with respect to a finer topology, since the relative cotangent complex of a flat or smooth map will not typically vanish.

We now turn our attention to defining more general classes of stacks over $\mathcal{E}_n$-rings. First, any $\mathcal{E}_n$-ringed ∞-topos $\mathcal{X}$ defines a moduli functors on $\mathcal{E}_n$-rings. For instance, if $\mathcal{X}$ is local with respect to the étale topology $\mathcal{T}_{ét}$, then it defines a moduli functor that is a sheaf with respect to the étale topology on $\mathcal{E}_n\text{-rings}^{\mathrm{op}}$. To make this precise, we need the notion of an étale local map.

Definition 1.15. Set $f : \mathcal{X} \to \mathcal{Y}$, $f^*\mathcal{O}_{\mathcal{Y}} \to \mathcal{O}_{\mathcal{X}}$, a map of $\mathcal{E}_n$-ringed ∞-topoi. Further, let U be an object of $\mathcal{Y}$, let φ be any étale $\mathcal{E}_n$-ring map $\mathcal{O}_{\mathcal{Y}}(U) \to A$, and denote by $\varphi' : \mathcal{O}_{\mathcal{X}}(f^*U) \to \mathcal{O}_{\mathcal{X}}(f^*U) \coprod_{\mathcal{O}_{\mathcal{Y}}(U)} A$ the pushout in $\mathcal{E}_n$-rings of φ along the map $\mathcal{O}_{\mathcal{Y}}(U) \to \mathcal{O}_{\mathcal{X}}(U)$. Then f is étale local if then the natural map $f^*\mathrm{Sol}(\varphi) \to \mathrm{Sol}(\varphi')$ is an equivalence, for any U and φ.

Thus, given any $\mathcal{E}_n$-ringed ∞-topos $\mathcal{X}$ which is étale local, we can define a functor $X : \mathcal{E}_n\text{-rings} \to \mathrm{Spaces}$ with values $X(A) := \mathrm{Map}_{ét}(\operatorname{Spec} A, \mathcal{X})$, the space of étale local maps of $\mathcal{E}_n$-ringed ∞-topoi from the étale spectrum of A into $\mathcal{X}$. X is a sheaf on $\mathcal{E}_n\text{-rings}^{\mathrm{op}}$ with respect to the étale topology.

Proposition 1.16. *The functor that to an étale scheme in $\mathcal{E}_n$-rings assigns the associated moduli functor, $\mathcal{T}_{ét}\text{-Schemes} \to \mathrm{Fun}(\mathcal{E}_n\text{-rings}, \mathrm{Spaces})$, is fully faithful.*

Proof. Maps in both ∞-categories are local in the source, and thus it suffices to check homs for which the source is affine. This case, however, is then a consequence of the ∞-categorical Yoneda lemma. □

To consider more general analogues of Artin stacks in $\mathcal{E}_n$-rings, it is just as tractable to focus on the moduli functor approach, rather than that of the ringed space approach, as in common in classical algebraic geometry. The following definitions are essentially those of Toën and Vezzosi in [**TV2**] and Lurie in [**L0**] placed in context of $\mathcal{E}_n$-rings.

First, to work with more general stacks it is standard to relax the condition that a stack have an étale cover by an affine, to that of having a smooth cover. We thus require a definition of smoothness in the $\mathcal{E}_n$-ring setting.

Definition 1.17. A map $f : X \to Y$ is smooth if f can be factored étale locally on X as étale map followed by a projection. I.e., f is smooth if there exists an étale covering $\{X_\alpha \to X\}$ such that each $f|_{X_\alpha}$ be factorized as a composition $f|_{X_\alpha} : X_\alpha \xrightarrow{i} Y \times W \xrightarrow{p} Y$, where i is étale and p is the projection.

Definition 1.18. A moduli functor $X : \mathcal{E}_n\text{-rings} \to \mathrm{Spaces}$ is an n-geometric stack if it is a sheaf for the étale topology, there exists a smooth cover $U \to X$ where U is a coproduct of affines, and the fiber product $\operatorname{Spec} A \times_X U$ is an $(n-1)$-geometric stack for any affine $\operatorname{Spec} A \to X$. X is 0-geometric stack if it is affine.

Remark 1.19. The case of 1-geometric stacks in the above definition is particularly interesting. These are stacks whose diagonal is affine, i.e., whose diagonal map $X \to X \times X$ is a relative affine. Examples include the classifying stack BG for any affine algebraic group G in $\mathcal{E}_n$-rings.

We can obtain a different notion of an n-stack by changing the class of 0-stacks with which the inductive definition starts.

Definition 1.20. An n-geometric stack X is an Artin n-stack if for any discrete $\mathcal{E}_n$-ring R the space $X(R)$ is n-coconnective.

In other words, a moduli functor X is an Artin n-stack if and only if X is an n-geometric stack and $\pi_i X(R) = 0$ for any $i > n$ and R discrete.

Remark 1.21. This class of Artin n-stacks should be characterized by conditions as in Artin's representability theorem, or Lurie's version for simplicial commutative rings [**L0**]. In a future elaboration of this work, we intend show that if a moduli functor X is an étale sheaf, has a cotangent complex, preserves filtered colimits, is nilcomplete, infinitesimally cohesive and takes n-coconnective values on discrete $\mathcal{E}_n$-rings, then X is represented by an n-stack.

2. Quasicoherent Sheaves

The two notions of modules for $\mathcal{E}_n$-algebras likewise lead to two notions of sheaves on a stack in $\mathcal{E}_n$-algebras. That is, let $\mathcal{M}$ be a notion of modules for $\mathcal{O}$-algebras, by which we will mean that $\mathcal{M}$ has at least the structure of a covariant functor $\mathcal{M} : \mathcal{O}\text{-alg} \to \mathrm{Cat}_\infty$, assigning an ∞-category to every algebra and an induction functor $f_! \mathcal{M}(A) \to \mathcal{M}(B)$ for every map $f : A \to B$. We can thereby prolong $\mathcal{M}$ to assign values to moduli functors of spaces $\mathscr{P}(\mathcal{E}_n\text{-alg}^{\mathrm{op}})$. We give the following definition, replacing $\mathcal{E}_n$-alg with an arbitrary presentable ∞-category $\mathcal{C}$.

Definition 2.1. Let $\mathcal{M} : \mathcal{X} \to \mathrm{Cat}_\infty$ be a covariant ∞-category valued functor. $\mathcal{M}$ is prolonged to the ∞-category of space-valued functors of $\mathcal{X}$ by assigning to X the ∞-category

$$\mathcal{M}(X) := \lim_{A \in \mathcal{X}^{\mathrm{op}}_{/X}} \mathcal{M}(A).$$

This defines a functor $\mathcal{M} : \mathscr{P}(\mathcal{X}^{\mathrm{op}}) \to \mathrm{Cat}_\infty$.

The above admits the following, perhaps more conceptual, reformulation. That is, an object $M \in \mathcal{M}(X)$ is a natural transformation of functors $M : \pi_\infty \circ X \to \mathcal{M}$, where the ∞-groupoid functor π_∞ represents the fully faithful embedding of spaces into ∞-categories (as ∞-categories with invertible 1-morphisms).

$$\begin{tikzcd} & \mathrm{Spaces} \arrow[dr, "\pi_\infty"] & \\ \mathcal{X} \arrow[ur, "X"] \arrow[rr, "\mathcal{M}"'] & & \mathrm{Cat}_\infty \end{tikzcd}$$

In other words, we can describe the value of $\mathcal{M}$ on X as the hom $\mathcal{M}(X) \simeq \mathrm{Hom}(X, \mathcal{M})$ taken in the functor category of $\mathcal{X}$-diagrams, $(\mathrm{Cat}_\infty)^{\mathcal{X}}$.

Remark 2.2. Since we have not made any smallness requirements on $\mathcal{X}$, this prolonged version version of $\mathcal{M}$ may take very large values for an arbitrary moduli functor. That is, the homs in $\mathcal{M}(X)$ are not guaranteed to be small. However, in practice this will not be an issue, as all the presheaves that we will consider will satisfy an appropriate smallness condition.

The considerations above apply to the particular case where $\mathcal{X}$ is $\mathcal{O}$-alg, or more particularly to $\mathcal{E}_n$-rings, which is our main focus. However, we have several notions of modules for an $\mathcal{E}_n$-ring A, and this leads to several notions of sheaves for a moduli functor. First, we present the notion of sheaf corresponding to left modules.

Definition 2.3. For X a moduli functor on $\mathcal{E}_n$-rings, $X \in \mathrm{Fun}(\mathcal{E}_n\text{-rings}, \mathrm{Spaces})$, the ∞-category of quasicoherent sheaves on X is defined as the above construction applied for the functor $\mathcal{M} = \mathrm{Mod}$, where Mod is the covariant functor of left modules assigning Mod_A to an $\mathcal{E}_n$-ring A and assigning the induction functor $B \otimes_A -$ to an $\mathcal{E}_n$-ring map $A \to B$. In other words, we have

$$\mathrm{QC}_X := \lim_{A \in \mathrm{Aff}/X} \mathrm{Mod}_A \simeq \mathrm{Hom}(X, \mathrm{Mod}).$$

We also have a notion of $\mathcal{E}_n$-A-modules, leading to the following notion:

DEFINITION 2.4. For X a moduli functor on $\mathcal{E}_n$-rings, the ∞-category of $\mathcal{E}_i$-quasicoherent sheaves on X, $1 \leq i \leq n$, is the above construction applied for the functor $\mathcal{M} = \mathrm{Mod}^{\mathcal{E}_i}$, where $\mathrm{Mod}^{\mathcal{E}_i}$ the covariant functor of $\mathcal{E}_i$-modules, defined by assigning $\mathrm{Mod}_A^{\mathcal{E}_i}$ to A and the induction functor $f_! \simeq U_B \otimes_{U_A} -$ to an $\mathcal{E}_n$-ring map $f : A \to B$. More precisely, we have

$$\mathrm{QC}_X^{\mathcal{E}_i} := \lim_{A \in \mathrm{Aff}/X} \mathrm{Mod}_A^{\mathcal{E}_i} \simeq \mathrm{Hom}(X, \mathrm{Mod}^{\mathcal{E}_i})$$

The interactions of these two types of sheaves provides the theory of derived algebraic geometry over $\mathcal{E}_n$-rings much of the flavor that distinguishes it from the case of $\mathcal{E}_n\infty$-rings.

There exists a natural functor $G : \mathrm{Mod}_A^{\mathcal{E}_n} \to \mathrm{Mod}_A$, intuitively defined by forgetting the space of actions of A on a module M except in a single direction. This functor has a left adjoint F, which can be computed by the bar construction $U_A \otimes_A -$, where $U_A^{\mathcal{E}_n}$ denotes the $\mathcal{E}_n$-enveloping algebra of A, as previously discussed. For instance, the ∞-category $\mathrm{Mod}_A^{\mathcal{E}_1}$ is equivalent to A-bimodules, or $A \otimes A^{\mathrm{op}}$-modules, there is a forgetful functor $\mathrm{Mod}_{A\otimes A^{\mathrm{op}}} \to \mathrm{Mod}_A$ given by remembering the left A action with left adjoint $(A \otimes A^{\mathrm{op}}) \otimes_A -$. The sequence of operad maps $\mathcal{E}_1 \to \mathcal{E}_2 \to \ldots \to \mathcal{E}_n$ have an associated sequence of forgetful functors on modules, each with a left adjoint computed by $U_A^{\mathcal{E}_i+1} \otimes_{U_A^{\mathcal{E}_i}} -$. This obtains a sequence of induction functors

$$\mathrm{Mod}_A \longrightarrow \mathrm{Mod}_A^{\mathcal{E}_1} \longrightarrow \mathrm{Mod}_A^{\mathcal{E}_2} \longrightarrow \cdots \longrightarrow \mathrm{Mod}_A^{\mathcal{E}_n}\ ,$$

and for more general X defined over $\mathcal{E}_n$-algebras, we thus obtain a sequence of functors

$$\mathrm{QC}_X \longrightarrow \mathrm{QC}_X^{\mathcal{E}_1} \longrightarrow \mathrm{QC}_X^{\mathcal{E}_2} \longrightarrow \cdots \longrightarrow \mathrm{QC}_X^{\mathcal{E}_n}\ .$$

This composite functor $F : \mathrm{QC}_X \to \mathrm{QC}_X^{\mathcal{E}_n}$ will be of particular interest to us.

Recall that for any map of $\mathcal{E}_n$-rings $f : A \to B$, these induction and restriction functors sit together in the following diagram of adjunction.

$$\begin{tikzcd}
\mathrm{Mod}_A^{\mathcal{E}_n} \arrow[r, bend left, "f_!"] \arrow[d, shift left, "G"] & \mathrm{Mod}_B^{\mathcal{E}_n} \arrow[l, "\mathrm{Res}_f"] \arrow[d, shift left, "G"] \\
\mathrm{Mod}_A \arrow[u, bend left, "F"] \arrow[r, bend left, "B\otimes_A -"] & \mathrm{Mod}_B \arrow[l, "\mathrm{Res}_f"] \arrow[u, bend left, "F"]
\end{tikzcd}$$

where there is further there is an equivalence $\mathrm{Mod}_A^{\mathcal{E}_n} \simeq \mathrm{Mod}_{GF}(\mathrm{Mod}_A) \simeq \mathrm{Mod}_{U_A}$.

The situation is similar for quasicoherent sheaves on stacks over $\mathcal{E}_n$-rings. There is a natural adjunction of ∞-categories $F : \mathrm{QC}_X \leftrightarrows \mathrm{QC}_X^{\mathcal{E}_n} : G$, where $T = GF$ is a monad on QC_X associated to the functor G above. We can set $U_X := FG(\mathcal{O}_X)$, where $\mathcal{O}_X$ is the structure sheaf of X regarded as an $\mathcal{E}_n$-quasicoherent sheaf on X, so that U_X is an $\mathcal{E}_n$-quasicoherent sheaf that can be regarded as a left quasicoherent sheaf of algebras on X. The adjunction above gives a functor $\mathrm{QC}_X^{\mathcal{E}_n} \to m_{U_X}(\mathrm{QC}_X)$. Unlike in the affine case, this functor is not guaranteed to be an equivalence, though it often is in many interesting examples.

Given a map of stacks $f : X \to Y$, we thereby obtain a corresponding diagram of adjunctions

$$\begin{tikzcd}
\mathrm{QC}_Y^{\mathcal{E}_n} \arrow[r, bend left, "f^*"] \arrow[d, shift left, "G"] & \mathrm{QC}_X^{\mathcal{E}_n} \arrow[l, "f_*"] \arrow[d, shift left, "G"] \\
\mathrm{QC}_Y \arrow[u, bend left, "F"] \arrow[r, bend left, "f^\star"] & \mathrm{QC}_X\ . \arrow[l, "f_\star"] \arrow[u, bend left, "F"]
\end{tikzcd}$$

Remark 2.5. These two notions of quasicoherent sheaves play quite different roles. In some sense, the ∞-category QC_X, rather than $\mathrm{QC}_X^{\mathcal{E}_n}$, is a better categorical approximation to the structure of X. A map $f : X \to Y$ give rise to $\mathcal{E}_{n-1}$-monoidal functor $f^\star : \mathrm{QC}_Y \to \mathrm{QC}_X$, and under good conditions there should be an equivalence between such functors and such maps, which is a form of Tannakian duality. No such relation is apparent with $\mathrm{QC}_X^{\mathcal{E}_n}$. However, the ∞-category $\mathrm{QC}_X^{\mathcal{E}_n}$ is more geometric. The cotangent complex is naturally an element of $\mathrm{QC}_X^{\mathcal{E}_n}$, not QC_X. Further, there is a relative spectrum functor Spec_X that takes an $\mathcal{E}_n$-algebra A in $\mathrm{QC}_X^{\mathcal{E}_n}$ and produces a stack $\mathrm{Spec}_X A \to X$ that is a relative affine over X. We will study these features in more detail in a future elaboration of this work, where we will see in particular that $\mathrm{QC}_X^{\mathcal{E}_n}$ is the $\mathcal{E}_{n-1}$-Drinfeld center of QC_X when X is a 1-geometric stack. This last statement is closely related to work with Ben-Zvi and Nadler in [**BFN**].

3. The Cotangent Complex

Before stating the relative version of the cotangent construction, it is convenient to first reformulate the previous absolute definition. Let X be a moduli functor on $\mathcal{O}$-algebras, C an $\mathcal{O}$-algebra, and let $x \in X(C)$. For M an $\mathcal{O}$-C-module, we will denote by $\mathrm{fiber}_x(M)$ the fiber over the point x of the natural map $X(C \oplus M) \to X(C)$. That is, we have a Cartesian square of spaces below:

$$\begin{tikzcd} \mathrm{fiber}_x(M) \arrow[r] \arrow[d] & X(C \oplus M) \arrow[d] \\ \{x\} \arrow[r] & X(C) \end{tikzcd}$$

Definition 3.1. The cotangent complex of X, L_X, is the $\mathcal{O}$-quasicoherent sheaf on X defined by having a natural equivalence

$$\mathrm{Hom}_{\mathrm{Mod}_C^{\mathcal{O}}}(L_X(x), M) \simeq \mathrm{fiber}_x(M).$$

If the quasicoherent sheaf L_X exists, it is as usual unique up to a natural equivalence. In the study of classical Artin stacks, the existence of a cotangent complex is a key factor distinguishing between general moduli problems and those that may be described more geometrically.

Let $f : X \to Y$ now be a map of moduli functors, and let us now define a relative notion of the cotangent complex. Given a point x in $X(C)$ and M define the relative fiber, $\mathrm{fiber}_x(M; f)$, to lie in a Cartesian square below:

$$\begin{tikzcd} \mathrm{fiber}_x(M; f) \arrow[r] \arrow[d] & X(C \oplus M) \arrow[d] \\ \{x\} \arrow[r] & X(C) \times_{Y(C)} Y(C \oplus M). \end{tikzcd}$$

We may now ask for some $\mathcal{O}$-quasicoherent sheaf to corepresent the above functor of M.

Definition 3.2. The relative cotangent complex $L_{X|Y}$ is an $\mathcal{O}$-quasicoherent sheaf on X with the structure of a natural equivalence for any C and M, of

$$\mathrm{Hom}_{\mathrm{Mod}_C^{\mathcal{O}}}(L_{X|Y}(x), M) \simeq \mathrm{fiber}_x(M; f).$$

This construction specializes to one above when Y is $\mathrm{Spec}\, k$. It is worth rewording the definition to make the connection with the previous definition of the relative cotangent complex $L_{B|A}$ of a map of $\mathcal{O}$-algebras $A \to B$. Let $X = \mathrm{Spec}\, B$ and $Y = \mathrm{Spec}\, A$, where by $\mathrm{Spec}\, R$ we are just denoting the moduli functors corepresented by the $\mathcal{O}$-algebra R, i.e., $\mathrm{Spec}\, R := \mathrm{Map}_{\mathcal{O}\text{-alg}}(R, -)$. The Cartesian square above can be rewritten as

$$\begin{tikzcd}
\mathrm{fiber}_x(M;f) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{O}\text{-alg}}(B, C\oplus M) \arrow[d] \\
\{x\} \arrow[r] & \mathrm{Map}_{\mathcal{O}\text{-alg}}(B,C)\times_{\mathrm{Map}_{\mathcal{O}\text{-alg}}(A,C)} \mathrm{Map}_{\mathcal{O}\text{-alg}}(A, C\oplus M).
\end{tikzcd}$$

where x the point of $X(C)$ is now a map $x : B \to C$. To pick out the $\mathcal{O}$-B-module that has this property, i.e., that corepresents this functor, it suffices to specialize to the case where x is the identity map $B \to B$. In this case, the diagram simplifies to

$$\begin{tikzcd}
\mathrm{fiber}_{\mathrm{id}_B}(M;f) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{O}\text{-alg}}(B, B\oplus M) \arrow[d] \\
\{\mathrm{id}_B\} \arrow[r] & \mathrm{Map}_{\mathcal{O}\text{-alg}}(B,B)\times_{\mathrm{Map}_{\mathcal{O}\text{-alg}}(A,B)} \mathrm{Map}_{\mathcal{O}\text{-alg}}(A, B\oplus M).
\end{tikzcd}$$

from which the fiber can be identified as $\mathrm{fiber}_{\mathrm{id}_B}(M;f) \simeq \mathrm{Map}_{\mathcal{O}\text{-alg}^{A/}_{/B}}(B, B\oplus M)$, exactly the value of the cotangent complex defined previously.

Proposition 3.3. *Let $X \xrightarrow{f} X' \to X''$ be maps of moduli functors of $\mathcal{O}$-algebras, and assume that the relative cotangent complexes exist. Then there is a natural cofiber sequence of $\mathcal{O}$-quasicoherent sheaves on X given by*

$$f^*L_{X'|X''} \to L_{X|X''} \to L_{X|X'}.$$

We now explore the means by which the cotangent complex detects properties of maps of moduli functors.

We now turn to the problem of describing the cotangent complex of a scheme in a similar way to how we identified the cotangent complex of an $\mathcal{E}_n$-algebras from the cofiber sequence $A \to U_A \to L_A[n]$. We will require the following lemma.

Recall the adjunction $F : \mathrm{QC}_X \leftrightarrows \mathrm{QC}_X^{\mathcal{E}_n} : G$, with $U_X = FG(\mathcal{O}_X)$. We can now globalize our previous description of the cotangent complex of an $\mathcal{E}_n$-algebra.

Theorem 3.4. *Let X be a scheme over $\mathcal{E}_n$-rings. Then there is a cofiber sequence $U_X \to \mathcal{O}_X \to L_X[n]$ in the ∞-category of $\mathcal{E}_n$-quasicoherent sheaves on X.*

Proof. By assumption, X can be described as an $\mathcal{E}_n$-ringed space with a cover by affine. Let $f : \coprod_\alpha \operatorname{Spec} A_\alpha \to X$ be such a cover. Applying the functor f_α^* to the cofiber sequence $U_X \to \mathcal{O}_X \to \mathrm{Coker}$ gives a map $f^*U_X \to f^*\mathcal{O}_X$ of $\mathcal{E}_n$-A-modules. There is an equivalence $f^*U_X \simeq U_A$, without any hypotheses on the map f. Using that f is an open embedding, we furthermore have the equivalence $f^*\mathcal{O}_X \simeq A$. (Note that this is false generally, as for a general map of affine $B \to B'$, $f_!B \not\simeq B'$, but this is true for a localization $B \to B[x^{-1}]$ since the relative cotangent complex of a localization vanishes.)

Thus, we obtain that the pullback of the cofiber sequence $U_X \to \mathcal{O}_X \to \mathrm{Coker}$ is equivalent to the cofiber sequence $U_A \to A \to L_A[n]$. Thus, there is an equivalence $f_\alpha^*\mathrm{Coker} \simeq L_A[n] \simeq f_\alpha^* L_X[n]$. Since, the map $\coprod_\alpha \operatorname{Spec} A_\alpha \to X$ was a cover, we obtain that the equivalence $\mathrm{Coker} \simeq L_X[n]$, and therefore there is a cofiber sequence $U_X \to \mathcal{O}_X \to L_X[n]$ of sheaves on X.

□

This result also globalizes the relation between derivations and Hochschild cohomology, which we will see after we formulate the natural definition of $\mathcal{E}_n$-Hochschild homology for a stack.

Definition 3.5. The $\mathcal{E}_n$-Hochschild cohomology of a moduli functor X of $\mathcal{E}_n$-rings with coefficients in an $\mathcal{E}_n$-quasicoherent sheaf M is

$$\mathrm{HH}^*_{\mathcal{E}_n}(X;M) = \mathrm{Hom}_{\mathrm{QC}_X^{\mathcal{E}_n}}(\mathcal{O}_X, M).$$

We thus have the following corollary of the above theorem.

Corollary 3.6. *Let X be a scheme over $\mathcal{E}_n$-rings, and let M be an $\mathcal{E}_n$-quasicoherent sheaf on X. Then there is a cofiber sequence of spectra*

$$\mathrm{Hom}_{\mathrm{QC}_X^{\mathcal{E}_n}}(L_X, M)[-n] \longrightarrow \mathrm{HH}^*_{\mathcal{E}_n}(X; M) \longrightarrow \Gamma M$$

where ΓM is the global sections of M. In the particular case of $M = \mathcal{O}_X$, this cofiber sequence can be rewritten as $\Gamma T_X[-n] \to \mathrm{HH}^*_{\mathcal{E}_n}(X) \to \Gamma\mathcal{O}_X$, where T_X is the tangent complex of X. In this case we obtain a relation between the tangent complex of X, the Hochschild cohomology of X, and the sheaf cohomology of X.

Remark 3.7. The previous result is true in greater generality. For instance, the result holds if X is a $\mathcal{T}_{\acute{e}t}$-scheme, i.e., if X is étale locally affine. However, the proof requires first showing that the relative cotangent complex of an étale map of $\mathcal{E}_n$-rings is contractible, which we postpone to to a future, more detailed, treatment of this subject.

Bibliography

[BM] Maria Basterra and Michael A. Mandell, Homology and cohomology of E_∞ ring spectra, Math. Z. 249 (2005), no. 4, 903944.

[BBD] Alexander Beilinson, Joseph Bernstein, and Pierre Deligne, Faisceaux pervers, Analysis and topology on singular spaces, I (Luminy, 1981), Astérisque, vol. 100, Soc. Math. France, Paris, 1982, pp. 5171.

[BFN] David Ben-Zvi, John Francis, and David Nadler, Integral transforms and Drinfeld centers in derived algebraic geometry, arXiv:math.AG/08050157.

[Bö] Carl-Friedrich Bödigheimer, Stable splittings of mapping spaces, Algebraic topology (Seattle, Wash., 1985), Lecture Notes in Math., vol. 1286, Springer, Berlin, 1987, pp. 174187.

[Fr] Benoit Fresse, Lie theory of formal groups over an operad, J. Algebra 202 (1998), no. 2, 455511.

[GK] Victor Ginzburg and Mikhail Kapranov, Koszul duality for operads, Duke Math. J. 76 (1994), no. 1, 203272.

[Go] Thomas Goodwillie, Calculus. III. Taylor series, Geom. Topol. 7 (2003), 645711 (elec- tronic).

[Hu] Po Hu, Higher string topology on general spaces, Proc. London Math. Soc. (3) 93 (2006), no. 2, 515544.

[Jo] AndréJoyal, Quasi-categories and Kan complexes, J. Pure Appl. Algebra 175(2002), no. 1-3, 207222, Special volume celebrating the 70th birthday of Professor Max Kelly.

[Ko] Maxim Kontsevich, Operads and motives in deformation quantization, Lett. Math. Phys. 48 (1999), no. 1, 3572, Mosh?e Flato (19371998).

[La] A. Lazarev, Homotopy theory of A_∞ ring spectra and applications to MU-modules, K- Theory 24 (2001), no. 3, 243281.

[L0] Jacob Lurie, Derived algebraic geometry, Ph.D. thesis, MIT PhD Thesis 2004.

[L] Jacob Lurie, Higher topos theory. arXiv:math.CT/0608040.

[L1] Jacob Lurie, Derived Algebraic Geometry 1: Stable ∞-categories. arXiv:math.CT/0608228.

[L2] Jacob Lurie, Derived Algebraic Geometry 2: Noncommutative Algebra. arXiv:math.CT/0702299.

[L3] Jacob Lurie, Derived Algebraic Geometry 3: Commutative algebra. arXiv:math.CT/0703204.

[L4] Jacob Lurie, Derived Algebraic Geometry 4: Deformation theory. arXiv:math.CT/0703204.

[L5] Jacob Lurie, Derived Algebraic Geometry 5: Structured spaces. arXiv:math.CT/0703204.

[Ma] J. P. May, The geometry of iterated loop spaces, Springer-Verlag, Berlin, 1972, Lectures Notes in Mathematics, Vol. 271.

[Mc] Dusa McDuff, Configuration spaces of positive and negative particles, Topology 14 (1975), 91107.

[Qu] aniel Quillen, Homotopical algebra, Lecture Notes in Mathematics, No. 43, Springer- Verlag, Berlin, 1967.

[Re] Charles Rezk, Spaces of algebra structures and cohomology of operads, Ph.D. thesis, MIT PhD Thesis 1996.

[TV1] Bertrand Toën and Gabriele Vezzosi, Homotopical algebraic geometry. I. Topos theory, Adv. Math. 193 (2005), no. 2, 257372.

[TV2] B. Toën and G. Vezzosi, Homotopical algebraic geometry II: geometric stacks and applications, arXiv:math.AG/0404373.