MATH 465, LECTURE 4: TRANSVERSALITY

J. FRANCIS, NOTES BY I. BOBKOVA

In this lecture we will prove Thom's Transversality Theorem and apply it to complete the proof of the equivalence $\Omega_n^{\text{un}} \cong \pi_n MO$ begun in the previous lecture.

1. Transversality

An idea of "general position" seems to have existed very early in topology. This was made precise in the notion of a transverse intersection, which possibly originates in Thom's thesis in the early 1950s.

Definition 1.1. Let $f\colon P \to E$ be a smooth map of manifolds and $i\colon M \to E$ a smooth submanifold of E. f is transverse to i at a point x of M if, for any $p \in f^{-1}\{x\}$, the induced map

$$df + di : T_pP \oplus T_xM \longrightarrow T_xE$$

is surjective. If f is transverse to i at every point $x \in M$, then f is transverse to i, notated $f \pitchfork i$.

Remark 1.2. If $f^{-1}\{x\}$ is the empty set, then f is automatically transverse to i at x.

The notion of transversality generalizes of that a regular value of a map $f : P \to E$. That is, we have the following:

Example 1.3. Let the submanifold M consist of a single point $M = * \xrightarrow{x} E$. In this case, f is transverse to x if and only if x is a regular value of f.

As observed by Pontryagin in the 30s, the inverse image of a regular value always has the structure of a smooth manifold; this feature is part of what gives the notion of a regular value its importance. This generalizes.

Proposition 1.4. *If $f \pitchfork i$, then $f^{-1}M \hookrightarrow P$ is a smooth submanifold.*

Proof. Apply the inverse function theorem. □

Regular values occur in abundance, as follows from the the well-known theorem of Brown, Sard and Morse.

Theorem 1.5 (Brown–Sard–Morse)**.** *For any smooth map of manifolds $f\colon P \to E$, the regular values of f form a dense subspace of E.*

Thom's Transversality Theorem, the key geometric input making the work of [5] go, is a generalization of this result.

Theorem 1.6 (Thom Transversality)**.** *Let P be a smooth manifold, and let $i : M \hookrightarrow E$ be a smooth submanifold. The subspace $\mathrm{Map}^{\mathrm{sm}}_{\pitchfork i}(P,E) \subset \mathrm{Map}^{\mathrm{sm}}(P,E)$, consisting of those maps $f : P \to E$ for which f is transverse i, is dense.*

Remark 1.7. Additionally, every map $P \to E$ can be approximated within arbitrarily small ε by a smooth; i.e., $\mathrm{Map}^{\mathrm{sm}}(P,E)$ is a dense subspace of $\mathrm{Map}(P,E)$. Thus, by composing, we obtain that transverse to M maps, $\mathrm{Map}^{\mathrm{sm}}_{\pitchfork i}(P,E)$, form a dense subspace of all maps, $\mathrm{Map}(P,E)$.

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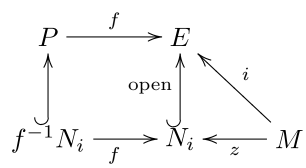
We will in fact prove a modification of this theorem, namely, the following statement: For any smooth map $f\colon P \to E$ there exists a smooth embedding $s\colon M \to E$ arbitrarily close to i, and for which f is transverse to s. This is easily seen to be equivalent.

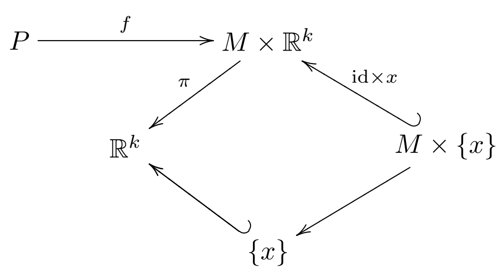
Proof. First, we will demonstrate that the transversality theorem for a general manifold E is a consequence of the particular case in which E has the structure of a vector bundle over M. A tubular neighborhood N_i of the embedding i is an open submanifold of E, so the inverse image $f^{-1}N_i$ therefore defines a smooth open submanifold of P:



Now, let us assume the the transversality theorem for $P' = f^{-1}N_i$ and $E' = N_i$. With this assumption, we can find an embedding $s : M \to N_i$ arbitrarily close to the zero section z and for which f and s are transverse in N_i. Composing the map s with the embedding of N_i into E, we thus obtain a map $\tilde{s} : M \to E$ that is transverse to $f : P \to E$.

Thus, it suffices to prove the transversality theorem under the assumption that E is a vector bundle over M. We will first consider the case where the vector bundle is trivial, which we will then make use of in the case of a general vector bundle.

First case: E a trivial vector bundle.
Let E be a trivial k-dimensional vector bundle over M, $E \cong M \times \mathbb{R}^k$, and let $f : P \to E$ be any smooth map, as before. Given a point $x : * \to \mathbb{R}^k$, consider the following commuting diagram:



Observe that x is a regular value of composite map $\pi \circ f$ if and only if f is transverse to $\mathrm{id} \times x$. To see this, first assume that the derivative map $d(\pi \circ f)|_p$ is a surjection onto the tangent space $T_x\mathbb{R}^k$, for p a point in the inverse image $f^{-1}(M \times \{x\})$. Then $d(\pi_* f)|_p \oplus d(\mathrm{id} \times x)$ is a surjection onto $T_{f(p)}M \times \mathbb{R}^k$, since this tangent space $T_{f(p)}$ can be split as a direct sum $T_x\mathbb{R}^k \oplus T_pP$, where each of these summands is surjected upon by one of the two derivative maps. The converse, that the transversality of f and $\mathrm{id} \times x$ implies that x is a regular value of $\pi \circ f$, obtains by the reverse bit of linear algebra. The Brown–Sard–Morse theorem now implies that the collection of $x \in \mathbb{R}^k$ that regular values of $\pi \circ f$ forms a dense subspace of $\mathbb{R}^k$. Thus, a value of x for which $f \pitchfork \mathrm{id} \times \{x\}$, is dense in $\mathbb{R}^k$. We may therefore select a regular value x arbitrarily close to the origin $0 \in \mathbb{R}^k$, and f will be transverse to $\mathrm{id} \times x$. This proves the transversality theorem in the case of E a trivial bundle.

Second case: E a general vector bundle.
Let E be any vector bundle over M, and let $f : P \to E$ be a smooth map, as before. We can choose $E^\perp$ such that the direct sum of vector bundles $E \oplus E^\perp$ is a trivial bundle. Choosing a trivialization $E \oplus E^\perp \cong M \times \mathbb{R}^k$, our situation is summarized in the following diagram:

$$\begin{array}{ccccc} f^{-1}(E \oplus E^{\perp}) & \xrightarrow{\tilde{f}} & E \oplus E^{\perp} & \overset{\sim}{=\!=} & M \times \mathbb{R}^k \\ \downarrow & & \downarrow{\scriptstyle \pi_E} & & \uparrow{\scriptstyle \mathrm{id}\times x} \\ P & \xrightarrow{f} & E & & M \end{array}$$

By forming the pullback $f^{-1}(E \oplus E^{\perp})$ (which is manifold, since it fibers smoothly over P), we put ourselves in the situation of the first case: For the smooth map $\tilde{f}\colon f^{-1}(E \oplus E^{\perp}) \to M \times \mathbb{R}^k$, valued in a trivial vector bundle, there exists a a point $x : * \to \mathbb{R}^k$ such that the map $\mathrm{id} \times x\colon M \to M \times \mathbb{R}^k$ is transverse to $\tilde{f}$.

We now define the embedding $s : M \to E$ to be the composite $\pi_E \circ (\mathrm{id} \times x)$. We now show that f is indeed transverse to s, and this will complete the proof of the transversality theorem.

The transversality $\tilde{f} \pitchfork \mathrm{id} \times \{x\}$ implies that for any $e \in E \oplus E^{\perp}$ in the image of $\tilde{f}$ and $\mathrm{id} \times \{x\}$ and $\tilde{e} \in \widetilde{f^{-1}}(e)$ the following diagram commutes

$$\begin{array}{ccc} T_{\tilde{e}} f^{-1}(E \oplus E^{\perp}) \oplus T_e M & \twoheadrightarrow & T_e(E \oplus E^{\perp}) \\ \downarrow & \searrow & \downarrow \\ T_{\pi(\tilde{e})} E \oplus T_e M & \dashrightarrow & T_{\pi_E(e)} E \end{array}$$

The surjectivity of the dotted arrow in the above diagram is forced by the surjectivity of all the other maps in this diagram, so we conclude the transversality of $f \pitchfork \pi_E \circ (\mathrm{id} \times x)$. I.e., f is transverse to s. □

We now make immediate use of the transversality theorem to finish Thom's proof of the equivalence $\Omega_n^{\mathrm{un}} \cong \pi_n MO$.

2. Completion of the proof of $\Omega_n \cong \pi_n MO$

Recall from the last lecture the construction of well-defined homomorphism $\Theta : \Omega_n^{\mathrm{un}} \to \pi_n MO$ defined via the Pontryagin-Thom collapse map of the tubular neighborhood of an n-manifold M embedded into Euclidean space.

Theorem 2.1. Θ *is an isomorphism.*

Proof. Let us consider a class $[f] \in \pi_n MO$ and choose a representative

$$f\colon (S^{n+k}, *) \to (\mathrm{Th}(\gamma_s^k), *)$$

for k sufficiently large. By smooth approximation, we may select f so that its restriction f' to the inverse image of complement of a small neighborhood of the basepoint of $\mathrm{Th}(\gamma_s^k)$,

$$f' : S^{n+k} - f^{-1}(*) \longrightarrow \mathrm{Th}(\gamma_s^k) - \{*\} \cong \mathrm{Disk}^{\circ}(\gamma_s^k)$$

is a smooth map of manifolds. (For convenience, I assume that this neighborhood is just the point, itself.) Our goal is to define an n-dimensional manifold M which corresponds to this class $[f]$, so that $\Theta([M]) = [f]$. This will, of course, imply the surjectivity of our map Θ.

We may apply Thom's Transversality Theorem and choose an embedding s of $\mathrm{Gr}_k(\mathbb{R}^s) \hookrightarrow \mathrm{Th}(\gamma_s^k) - \{*\}$ near the zero section, such that s is transverse to f'. Define the desired n-manifold M as a pullback of the following diagram:

$$\begin{array}{ccc} f'^{-1}(\mathrm{Gr}_k(\mathbb{R}^s)) & \hookrightarrow & S^{n-k} - f^{-1}(*) \\ \downarrow & & \downarrow \\ \mathrm{Gr}_k(\mathbb{R}^s) & \hookrightarrow & \mathrm{Th}(\gamma_s^k) - \{*\} \end{array}$$

I.e., M is the transverse intersection of S^{n+k} and $\mathrm{Gr}_k(\mathbb{R}^s)$ inside $\mathrm{Th}(\gamma_s^k)$.

Note that M comes with an embedding into S^{n+k}, and the basepoint of S^{n+k} is, by construction, not in the image of this embedding. By identifying $S^{n+k} - \{*\} \cong \mathbb{R}^{n+k}$, we obtain an embedding of M into the Euclidean space $\mathbb{R}^{n+k}$. Applying the Pontryagin-Thom collapse to the normal bundle of this embedding, as in the previous lecture, we obtain a pointed map $\Theta(M) : S^{n+k} \to MO(n+k)$.

Let us now construct a homotopy between the map $\Theta(M)$ and our original map f. f can be chosen to so that its restriction to M exactly classifies the normal bundle of M, and its restriction to the tubular neighborhood of M in S^{n+k} agrees with the Pontryagin-Thom collapse map. By contracting whatever the map f does outside of M's tubular neighborhood to zero, we obtain a homotopy between f and $\Theta(M)$, and thus we have shown the surjectivity of Θ.

Injectivity of Θ: Since Θ is a homomorphism, to demonstrate the injectivity of Θ it suffices to assume that for an n-manifold M for which $\Theta[M] = 0$, that it is therefore the case that $M \simeq \partial W^{n+1}$. I.e., M is a boundary of an $(n+1)$-manifold. Since we are assuming the map $\Theta(M)$ is null-homotopic, let us choose a regular homotopy

$$[0,1] \times S^{n+k} \to \mathrm{Th}(\gamma_s^k)$$

from the map $\Theta(M)$ to the constant map $\{1\} \times S^{n+k} \to * \to \mathrm{Th}(\gamma_s^k)$. Note that the above transverse inverse image construction applied to the constant map produces an empty n-manifold. Thus, applying this the transversality construction to the map

$$[0,1) \times S^{n+k} \hookrightarrow [0,1] \times S^{n+k} \to \mathrm{Th}(\gamma_s^k)$$

produces an $(n+1)$-dimensional manifold W^{n+1} with boundary M.

□

Remark 2.2. The above proof can repeated essentially verbatim to prove the more general isomorphism $\Omega_*^B \cong \pi_* MB$. Here B consists of a sequence $\ldots B_n \to B_{n+1}$ with compatible fibrations $\alpha_n : B_n \to BO(n)$, Ω_n^B consists of cobordism classes of n-manifolds with structure B on their normal bundles, and MB is the (Thom) spectrum with $MB(n) = \mathrm{Th}(\alpha_n^* \gamma^n)$. The proof is the same, one need only verify at each step that the B structure can be carried along through each of constructions.

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