MATH 465, LECTURE 2: COBORDISM

J. FRANCIS, NOTES BY O. GWILLIAM

Our eventual goal is to understand the classification of manifolds in families, in which the fibers are all Cat homeomorphic, for Cat = Top, PL, or Diff. As a warm-up to this, we'll study families in which the fibers can change. In particular, we'll study cobordisms. Roughly speaking, a generic family $\pi : W \to [0,1]$ of n-manifolds parametrized by the closed interval is a cobordism between $\pi^{-1}(0)$ and $\pi^{-1}(1)$.

Definition 0.1. A *cobordism* of smooth n-manifolds M and M' is an (n+1)-manifold W with a diffeomorphism $M \amalg M' \xrightarrow{\cong} \partial W$.

Remark 0.2. Of course, we can decorate or modify this definition for different settings. For instance, if we work with topological rather than smooth manifolds, we require the map on ∂W to be a homeomorphism. Or, if we wish to work with oriented manifolds, we require M, M', and W to be oriented such that we have a orientation-respecting diffeomorphism $\partial W \to M \coprod \bar{M}'$, where the bar means "reversed orientation."

Our goal for the moment is to classify manifolds up to cobordism (roughly speaking, we want to consider the set/space of manifolds modulo the equivalence relation given by cobordism). Ideally, we would like a "moduli space $\mathcal{M}$ of all manifolds up to cobordism." Here are a few properties we should demand of $\mathcal{M}$, in order for it to earn this moniker:

(1) A point $* \to \mathcal{M}$ gives a 0-manifold.
(2) A path $[0,1] \xrightarrow{f} \mathcal{M}$ defines a cobordism of 0-manifolds between $f(0)$ and $f(1)$.

Notice that the empty 0-manifold $\emptyset^0$ equips $\mathcal{M}$ with a basepoint. Hence a loop $f \in \Omega\mathcal{M}$ specifies a closed 1-manifold, since it gives a cobordism from $\emptyset^0$ to itself. Analogously, a based map from S^2 into $\mathcal{M}$, namely an element $f \in \Omega^2\mathcal{M}$, specifies a closed 2-manifold.

Continuing, we want a relation

$$\Omega^n\mathcal{M} \simeq \{\text{closed compact } n\text{-manifolds}\}$$

in such a way that

$$\pi_0\Omega^n\mathcal{M} \simeq \{\text{closed compact } n\text{-manifolds}\}/\sim \text{ given by cobordism.}$$

In other words, we want a space whose homotopy type encodes manifolds up to cobordism. Our description above is only a heuristic, but we will make a space like this.

Historically, the first approach to this space was via the Pontryagin-Thom construction.

Pontryagin wanted to understand maps between spheres. He observed that if you have a smooth map $f : S^{n+k} \to S^k$, then the inverse image $f^{-1}(x)$ of a regular value $x \in S^k$ is an n-manifold and, moreover, that a path between two regular values gives a cobordism of n-manifolds. Hence maps between spheres encode something about cobordism, and the process of taking the inverse image of a regular value interrelates function theory and geometry.

What he showed precisely was that

$$\pi_{n+k}S^k \xrightarrow{\cong} \Omega_n^{\mathrm{fr}}(\mathbb{R}^{n+k}),$$

where the right hand symbol means "the set of compact, closed n-manifolds embedded in $\mathbb{R}^{n+k}$ with framings on their normal bundles, up to framed embedded cobordism." This is a really remarkable result, relating geometry (in the guise of manifolds) to the core concerns of homotopy theory.

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This result is suggestive that we might regard space $\Omega^{n+k}S^k$ as some sort of moduli space of n-manifolds with normally-framed embeddings into $\mathbb{R}^{n+k}$. Taking this hint, we will use a similar idea to construct $\mathcal{M}$. We will first recall the standard notion of the Thom space.

Definition 0.3. For a vector bundle $\pi : V \to B$, the *Thom space* $\mathrm{Th}(V)$ is the quotient $\mathrm{Disk}(V)/\mathrm{Sph}(V)$, where we pick a metric on the bundle for which $\mathrm{Disk}(V)$ is the unit disk bundle and $\mathrm{Sph}(V)$ is the unit sphere bundle.

Note that varying the metric preserves the homeomorphism-type of the Thom space. To give a feel for what the Thom space is, observe that for a compact base B, $\mathrm{Th}(V) \cong V^+$, the one-point compactification of the bundle. In general, the Thom space is naturally pointed by the image of the sphere bundle, just as V^+ is pointed by the point at infinity.

Exercise 0.4. Show that $\mathrm{Th}(V \oplus V) \cong \mathrm{Th}(V) \wedge \mathrm{Th}(W)$. In particular, show that $\mathrm{Th}(V \oplus \underline{\mathbb{R}}^n) \cong \Sigma^n \mathrm{Th}(V)$. Hence, the Thom space plays nicely with our favorite operations on spaces.

We now apply this construction to the most natural bundles: let $\gamma^n \to BO(n) = \mathrm{Gr}_n(\mathbb{R}^\infty)$ denote the universal n-plane bundle.

Definition 0.5. $MO(n) := \mathrm{Th}(\gamma^n)$.

Consider the following sequence of inclusions

$$\cdots \to BO(n-1) \xrightarrow{g_{n-1}} BO(n) \xrightarrow{g_n} BO(n+1) \to \cdots .$$

Observe that $g_n^* \gamma^{n+1} \cong \gamma^n \oplus \underline{\mathbb{R}}^1$. Hence, by the exercise,

$$\mathrm{Th}(g_n^* \gamma^{n+1}) \cong \Sigma \mathrm{Th}(\gamma^n) = \Sigma MO(n).$$

Notice that a pullback of bundles $g^*V \to V$ induces a map between the Thom spaces $\mathrm{Th}(g^*V) \to \mathrm{Th}(V)$. Putting these constructions together, we obtain the Thom spectrum.

Definition 0.6. Let MO denote the spectrum given by the sequence of spaces $\{MO(n)\}$ with the maps

$$\mathrm{Th}(g_n^* \gamma^{n+1}) = \Sigma MO(n) \to MO(n+1) = \mathrm{Th}(\gamma^{n+1}).$$

This is a connective spectrum and we can extract the associated infinite loop space to get the moduli space $\mathcal{M}$ we want.

Definition 0.7. $\mathcal{M} := \mathrm{colim}_{n\to\infty} \Omega^n MO(n) = \Omega^\infty MO$.

This space will have all the properties we wanted. To see that it exists, you simply need to apply the adjunction between looping Ω and suspension Σ to our definition of the Thom spectrum.

To justify our assertions that this infinite loop space *is* the moduli space we desire, we will prove the following theorem.

Theorem 0.8 (Thom)**.** *$\Omega_n^{\mathrm{un}} \simeq \pi_n MO$.*

Here Ω_n^{un} denotes the set of compact closed unoriented smooth n-manifolds up to cobordism. For a spectrum, the homotopy groups are defined as follows:

$$\pi_n MO := \varinjlim_{k\to\infty} \pi_{n+k} MO(k).$$

Remark 0.9. Rather than taking the above as the definition of $\mathcal{M}$, a more honest approach to constructing this moduli space of manifolds $\mathcal{M}$ would be to first make the (∞,∞)-category of cobordisms Cob_∞, and then take the classifying space. A result of either Galatius-Madsen-Tillmann-Weiss or Lurie states that $B\mathrm{Cob}_\infty \simeq \Omega^\infty MO$.

<u>Question</u>: Why does "smooth" show up on the left hand side? The other stuff never seemed to involve this hypothesis. <u>A</u>: The right had side involves vector bundles, and the tangent bundle of a smooth manifold has the structure of a vector bundle; this is not the case for topological or PL manifolds, which both have a notion of a "tangent bundle," but which is *not* a vector bundle.

Since we're interested in these cases also, it might have made sense to do this more general case first. However, the proof of Thom's theorem relies on transversality, which manifestly depends on smoothness. There is an analogous result, proved much later by Kirby and Siebenmann, for topological manifolds, but it requires a topological substitute for transversality. The spectrum for topological manifolds is denoted $M\mathrm{Top}$, for PL manifolds $M\mathrm{PL}$, and once you have this transversality result you can prove $\Omega_n^{\mathrm{Top}} \cong \pi_n M\mathrm{Top}$ and $\Omega_n^{\mathrm{PL}} \cong \pi_n M\mathrm{PL}$. We'll talk about this stuff eventually.