Recall the theorem from the previous lecture:

**Theorem 0.1.** If $M$ is a Riemannian manifold and $x, y \in M$ are points on $M$ satisfying certain conditions, then the map

$$\Omega_{x,y}^\text{min} M \to \Omega_{x,y} M$$

is $(k - 1)$-connected, where $k$ is the smallest index of a non-minimal geodesic between $x$ and $y$.

In this lecture we will use the theory of non-minimal geodesics on a Riemannian manifold to prove Bott’s result about periodicity of homotopy groups of infinite unitary and orthogonal groups.

1. **Minimal geodesics**

Recall from the last time that we have the exponential map

$$T_I U(n) \xrightarrow{\exp} U(n)$$

$$A \xrightarrow{\exp} \Sigma A^k_{I^T}.$$

from the Lie algebra of $U(n)$ (which consists of skew Hermitian matrices) to $U(n)$, which sends $A$ to $\Sigma A^k_{I^T}$. This map is invariant with respect to the adjoint action.

**Question.** For which matrices $A \in T_I U(n)$ is $\exp A = -I$? (If we can find this space, it would be the space of minimal geodesics.)

First notice that any matrix $A$ can be diagonalized by conjugation by some element $g \in U(n)$ and this conjugation does not change the value of exponential map, since $\exp(gAG^{-1}) = g \cdot \exp(A) \cdot g^{-1} = g(-I)g^{-1} = -I$. So we can assume that $A$ is diagonal, hence it is a matrix of the form $A = \text{diag}(ia_1, ia_2, \ldots, ia_n)$, where $a_i$ are real numbers (because we know that $A$ is skew Hermitian). Then $\exp(A) = \text{diag}(e^{ia_1}, \ldots, e^{ia_n})$ hence $e^{ia_j} = 1$ and $a_j = \pi k_j$, where $k_j$ is an odd integer. The length of geodesics will be given by $\pi \sqrt{k_j^2}$ and it is minimized when $k_j = \pm 1$. The eigenspace of this matrix is a direct sum of negative eigenspace (given by $k_j = -1$) and positive eigenspace(orthogonal complement of the negative). The positive eigenspace can form any subspace of $\mathbb{C}^n$. Hence

$$\Omega_{I,-I}^\text{min} U(n) \cong \prod_{0 \leq k \leq n} Gr_k(\mathbb{C}^n).$$

Recall also that $\text{Lie}(SU(n)) \subset \text{Lie}(U(n))$ and let $n = 2m$. Then

$$\Omega_{I,-I}^\text{min} SU(2m) \cong Gr_m(\mathbb{C}^{2m})$$

by the same argument as before, taking into account that matrices in $\text{Lie}(SU(n))$ have zero trace (so negative and positive eigenspaces have equal dimensions).
2. Bott periodicity

We will use the following lemma which will be given without proof.

**Lemma 2.1.** The smallest index of a non-minimal geodesic in $SU(2m)$ is $2m + 2$.

**Corollary 2.2.** The space of minimal geodesics, which we just identified with the Grassmannian is $(2m+1)$-connected.

$$Gr_m(C^{2m}) \cong \Omega^1 SU(2m) \to \Omega SU(2m)$$

$Z \times \Omega SU \cong \Omega U$ and it follows from the corollary that

$$Z \times Gr_m(C^{2m}) \to \Omega U(2m)$$

is $(2m + 1)$-connected. On the other hand

$$\lim \rightarrow Gr_m(C^{2m}) = BU.$$ 

So we proved that $Z \times BU$ is homotopy equivalent to $\Omega U$ and we have

$$\Omega^2 U \cong \Omega(Z \times BU) \cong \Omega BU \cong U$$

We know that $\pi_0 U = 0$ and $\pi_1 U = Z$, so $\pi_i U = \begin{cases} 0, & i \text{ even} \\ Z, & i \text{ odd} \end{cases}$.

We can also apply these methods to the orthogonal group to get:

$$\Omega O = O/U$$
$$\Omega^2 O = U/Sp$$
$$\Omega^3 O = Z \times BSp$$
$$\Omega^4 O = Sp$$
$$\Omega^5 O = Sp/U$$
$$\Omega^6 O = U/O$$
$$\Omega^7 O = Z \times BO$$
$$\Omega^8 O = O$$

$\pi_0 Sp = \pi_0(Sp/U) = \pi_0(U/O) = \pi_0(U/Sp) = 0$, because they are connected.

$\pi_0 O = \pi_0(O/U) = Z/2$, because they have two components.

$\pi_0(Z \times BSp) = \pi_0(Z \times BO) = Z$ and the result follows:

$$\pi_k O = \pi_0 \Omega^k O = \begin{cases} 0, & k = 2, 4, 5 \text{ or } 6 \\ Z, & k = 3 \text{ or } 7 \\ Z/2, & k = 1 \text{ or } 8 \end{cases}$$