1. Recapitulation

Recall we constructed manifolds $M_{ij}$ by fibering $S^3$ over $S^4$. The regular 7-sphere is obtained by the Hopf fibration. $M_{ij}$ is classified by a map $f_{ij}: S^4 \rightarrow BSO(4)$.

Recall we computed $\pi_4 BSO(4) \cong \pi_4 B(Sp(1) \times Sp(1)) \cong \mathbb{Z} \times \mathbb{Z}$, indexed by $(i, j)$. The connecting map in the long exact sequence of homotopy groups is an isomorphism if $i - j = 1$ and so $M_{ij}$ is 6-connected. In particular, $M_{ij}$ is homotopy equivalent to $S^7$.

2. Secondary Invariants

Therefore, we need to obtain an invariant which can distinguish homotopic 7-manifolds. Our current selection of invariants is insufficient for this purpose. For instance, Pontryagin classes only exist in $4k$-dimensions. Stiefel-Whitney numbers are also zero for our 7-manifolds. However, analogous to our construction of the Whitehead group, we can construct a secondary invariant. Heuristically, these exist typically when our primary invariant is zero: The secondary invariant describes in what manner the primary invariant is trivial.

The cobordism class is our primary invariant. By Thom’s theory we will show that $\Omega^7_{SO} = 0$, namely that every oriented 7-manifold is bounded by an oriented 8-manifold. Therefore, we may choose some 8-manifold $B^8$ with $\partial B^8 \cong M^7$ and compute an associated characteristic number of the 8-manifold. We will check to what degree this number is well defined.

**Proposition 2.1.** $\Omega^7_{SO} = 0$

**Proof.**

We now define the secondary invariant which is the main focus of this lecture.

**Proposition 2.2.** Let $M^7$ be an oriented manifold with $H_3(M) = H_4(M) = 0$. Choose an oriented 8-manifold $B^8$ with boundary $\partial B^8 \cong M^7$. Then, the following is an invariant of $M$:

\[
\lambda(M) = \langle 2p_1^2(B), [B, M] \rangle - \text{Sig}(B) \mod 7.
\]

Here, the signature of the manifold with boundary $B$ is understood to mean the signature of the nondegenerate bilinear form on $H^4(B) \cong H^4(B, M)$ – this isomorphism makes use of the fact that the boundary $M$ has middle cohomology equal zero.

We will need Hirzebruch’s Signature Theorem, which gives the signature of a 4k-dimensional manifold in terms of the Pontryagin numbers (with coefficients in terms of the L-genus of the formal power series $\frac{\sqrt{z}}{\tanh(\sqrt{z})}$). Signature is a cobordism invariant ([?], [?]). In particular, we will need the signature of an 8-manifold.

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However, to motivate the construction, we will compute the signature of a 4-manifold $N^4$.

$$\text{Sig}(N^4) = \langle \frac{p_1}{3}, [N] \rangle$$

We check the scaling factor (here a third) since signature is determined by the first Pontryagin number. $\Omega^3 \otimes \mathbb{Q} = \mathbb{Q}$. We only need to check on one 4-manifold: we will use $CP^2$. Recall that in general the $i$-th Pontryagin class of a real vector bundle $E \to N$ is given in terms of the even Chern classes of the complexification of $E$:

$$p_i(E) = (-1)^i c_{2i}(E \otimes \mathbb{C}) \in H^{4i}(N; \mathbb{Z})$$

For $CP^2$, we compute the total Pontryagin character, $p(v) = \bar{c}(v \otimes \mathbb{C})$:

$$c(CP^2) = (1 + x)^3 \text{ mod } x^3 \Rightarrow \bar{p}(CP^2) = (1 + x)^3(1 - x)^3 \text{ mod } x^3 = 1 - 3x^2 \Rightarrow p_1(CP^2) = 3x^2$$

Finally, we use that the signature of $CP^2$ is $1$ to obtain the $1/3$ factor.

We are interested in the case of 8-manifolds. The signature of an arbitrary 8-manifold can be checked using two specific test cases, e.g., $CP^4$ and $CP^2 \times CP^2$. We obtain (exercise):

$$\text{Sig}(N^8) = \left\langle \frac{7p_2 - p_1^2}{45}, [N] \right\rangle$$

We are now in a position to complete the proof of our proposition.

**Proof.** Choose two 8-manifolds $B$ and $B'$ which both bound $M$. $\partial B \cong \partial B' \cong M$. Consider $N = B \cup_M B'$ as an oriented 8-manifold.

**Remark 2.3.** This proposition is closely related to the construction of “levels” of quantizations in quantum field theory, e.g., as in the Weiss-Zumino-Witten model for loop-groups.

By the long exact sequence of a cofibration, and our initial assumption of vanishing middle cohomology of $M$, $H_3(M) = H_4(M) = 0$, we observe that the middle cohomology of $B \cup_M B'$ is obtained as a direct sum of that of $B$ and $B'$.

$$H^4(B \cup_M B') = H^4(B) \oplus H^4(B')$$

Using this direct sum splitting, we obtain that

$$\text{Sig}(B \cup_M B') = \text{Sig}(B) + \text{Sig}(B') = \text{Sig}(B) - \text{Sig}(B')$$

The tangent bundle of $B \cup_M B'$ is classified by a map $T : B \cup_M B' \to BSO(8)$ which restricts to the inclusions of $B$ and $B'$ respectively. Again since $H_4(M) = 0$,

$$p_1(B \cup_M B') = p_1(B) + p_1(B') = p_1(B) - p_1(B')$$

$$\Rightarrow p_1^2(B \cup_M B') = p_1^2(B) + p_1^2(B')$$

Let $\lambda(X) = \langle 2p_1^2(X), [X] \rangle - \text{Sig}(X)$ be our putative invariant. We show that $\lambda(B) = \lambda(B')$ is an integer multiple of $7$.

Note the sign change due to the orientation of $B'$ as opposed to $B \cup_M B'$ (recall we denote $N = B \cup_M B'$):

$$\langle 2p_1^2(B), [B] \rangle - \langle 2p_1^2(B'), [B'] \rangle = \langle 2p_1^2(N), [N] \rangle$$

Using that $\text{Sig}(B) - \text{Sig}(B') = \text{Sig}(N)$, and dropping the pairing with the fundamental class $[N]$ in our notation (all Pontryagin classes appearing from now until the end of the proof will be those of $N$), we obtain that

$$\lambda(B) - \lambda(B') = 2p_1^2 - \text{Sig}(N)$$

Using the signature theorem, $\text{Sig}(N) = \frac{7p_2 - p_1^2}{45}$ (note $7$ is coprime to $45$), we are done. □

**Exercise 2.4.** Do the same procedure for 3-manifolds (every 3-manifold bounds a 4-manifold). Do you obtain an interesting invariant?
3. \(\lambda\) Invariant in action

Recall we constructed the following map of fibrations:

\[
\begin{array}{ccc}
S^3 & \longrightarrow & M^7_{ij} \\
\downarrow & & \downarrow f_{ij} \\
S^3 & \longrightarrow & BSO(3)
\end{array}
\]
\[
\begin{array}{ccc}
S^4 & \longrightarrow & S^4 \\
\downarrow & & \downarrow f_{ij} \\
BSO(3) & \longrightarrow & BSO(4)
\end{array}
\]

We view the classifying map \(f_{ij} \in \pi_4(BSO(4)) \cong \pi_3(SO(4)) \cong \pi_3(Sp(1) \times Sp(1))\) and thus as an element \((i, j) \in \mathbb{Z} \times \mathbb{Z}\).

Let \(f_{ij}\) be such that \(i - j = 1\), and so \(M^7\) is a homology 7-sphere. Let \(k = i + j\) be the free variable and denote \(M^7_k = M^7_{ij}\).

We have constructed \(M^7\) as a (3-)sphere bundle over a base manifold \((S^4)\). Therefore, writing \(M^7_k = Sph(\xi_k)\), we naturally obtain \(M^7_k\) as the boundary of the 8-manifold \(B = Disk(\xi_k)\). In particular Sig\((B) = 1\).

**Lemma 3.1** (1). \(p_1(\xi_{i,j}) = \pm 2(i + j)\). Here \(\iota\) denotes the standard generator of \(H^4(S^4)\).

We will use this lemma to prove:

**Lemma 3.2** (2). \(\lambda(M^7_k) = k^2 - 1 \mod 7\)

Now, as long as \(k^2\) is not congruent to 1 mod 7, \(M^7_k\) cannot be diffeomorphic to the usual 7-sphere.

*Proof.* We prove the first lemma. Consider reversing the orientation of our fiber \(S^3\). The first Pontryagin class is invariant under this change. Observe that:

\[
\xi(-1f_{i,j}) = \xi(f_{-j,-i})
\]

Therefore, our formula must be symmetric in \(i\) and \(j\). So, for some constant \(c \in \mathbb{Z}\), \(p_1 = c(i + j)\iota\)

We just need to check in one non-zero example what this constant is. For \(k = 1\) we have the usual Hopf fibration, and so \(Disk(\xi_1) \cong \mathbb{H}P^2 - D^8\). As a power series,

\[
p(\mathbb{H}P^n) = \frac{(1 + x)^{2n+2}}{1 + 3x} \Rightarrow p_1(\mathbb{H}P^2) = 2x \Rightarrow c = \pm 2
\]

\(\square\)