Throughout this lecture our handlebody $W^n$ of dimension $n$ will be obtained from $\partial_0 W \times [0,1]$ by attaching $q$-handles $\phi^q_i$ for $i$ in the index set $I_q$, $q = 0, \ldots, n$. That is

$$W = \partial_0 W \times [0,1] + \sum_{q=0}^{n} \sum_{i \in I_q} \phi^q_i.$$

Note that attaching the handles in this order does not impose any restrictions by the reordering result of lecture 7. We also have a natural filtration

$$\partial_0 W \times [0,1] =: W_{-1} \subset W_0 \subset \cdots \subset W_n = W$$

where for $k \geq 0$, $W_k$ is obtained from $W_{k-1}$ by attaching handles of index $k$. That is

$$W_k := W_{k-1} + \sum_{i \in I_k} \phi^k_i.$$

Our goal is to prove the following lemma which gives a homological condition under which the hypotheses of the cancellation lemma are satisfied.

**Lemma 0.1.** Let $W^n$ be as above with $n \geq 6$. Let $2 \leq q \leq n - 3$, and let $f : S^q \to \partial_1 W_q$ be an embedding such that $[f] = \pm [\phi^q_1]$ in $C^\text{cell}_q(W)$. Then $f$ is isotopic to a map that intersects the transverse sphere of $\phi^q_1$ transversely in a single point and does not intersect the transverse sphere sphere of any other $\phi^q_i$.

The remainder of this lecture will reduce this lemma to the so-called Whitney trick (to be proven next time) by better understanding the $C^\text{cell}_q(W)$. Recall that by definition $C^\text{cell}_q(W) = H_q(W_q, W_{q-1})$. Hence by collapsing $W_{q-1}$ we have

$$C^\text{cell}_q(W) \cong \bigoplus_{i \in I_q} H_q(D^q \times D^{n-q}, S^{q-1} \times D^{n-q}).$$

Each term of the sum is itself isomorphic to $H_q(D^q, S^{q-1})$ and so it is one dimensional. Let $[\phi^q_i]$ denote a generator so that the collection $\{[\phi^q_i] \}_{i \in I_q}$ is a basis for $C^\text{cell}_q(W)$.

In the statement of the lemma we view $f : S^q \to \partial_1 W_q$ as an element of $C^\text{cell}_q(W)$. How can we do this? Well, we have maps

$$\pi_q(W_q) \to H_q(W_q) \to H_q(W_q, W_{q-1}) = C^\text{cell}_q(W)$$

where the first map is the Hurewicz homomorphism (i.e. $[f] \mapsto f_*([S^n])$) and the second map is from the definition of relative homology.

The idea of the proof is to write $[f]$ in terms of the $[\phi^q_i]$. Consider the maps

$$r : S^n, \varnothing \xrightarrow{f} (W_q, W_{q-1}) \xrightarrow{p} (\bigcup_{q} D^q \times D^{n-q}, \bigcup_{q} S^{q-1} \times D^{n-q})$$

where the first map is the Hurewicz homomorphism (i.e. $[f] \mapsto f_*([S^n])$).
where \( p \) is the collapse map and \( p_i \) projects onto the \( i \)th factor.

Since \( p \) induces an isomorphism on homotopy and \( p_i \) induces projection onto the subgroup generated by \( \phi_i^q \), \( p_i \circ f \) determines an element of \( \pi_q(D^q, S^{q-1}) \cong \mathbb{Z} \), that is an integer which we denote \( [p_i \circ f] \). Then

\[
[f] = \sum_{i \in I_q} [p_i \circ f] \cdot [\phi_i^q].
\]

To compute the coefficients \( [p_i \circ f] \) geometrically we look at the inverse image of a regular value. Namely let \( 0 \in (D^q, S^{q-1}) \) be a regular value. Then

\[
[p_i \circ f] = \sum_{x \in p_i^{-1}(0)} \epsilon_x
\]

where \( \epsilon_x = \pm 1 \) is determined by the oriented intersection of \( f(S^q) \) and \( \phi_i^q \) in \( W \). This is defined generally as follows.

Let \( V \) be any manifold and \( f : M \to V \), \( i : M' \to V \) smooth maps of manifolds with \( \dim M + \dim M' = \dim V \) and \( f \pitchfork i \) and choices of orientation of \( TM \) and \( N_i M' \). Then for \( x \in M \cap M' \) we have the following diagram

\[
\begin{align*}
T_x M & \xrightarrow{df} T_x V \cong T_x M' \oplus N_i M'|_x \\
& \cong N_i M'|_x
\end{align*}
\]

where the diagonal map is an isomorphism by transversality. That map is a map between oriented vector spaces. We define \( \epsilon_x = \pm 1 \) depending on whether the map is orientation preserving or reversing.

We apply this to \( V = \partial_1 W_q \), \( M = S^q \), \( M' = \{0\} \times S^{n-q-1} \). The relevant bundles are orientable and we choose orientations. Then we have

\[
[f] = \sum_{i \in I_q} \sum_{x \in f(S^q) \cap \phi_i^q(\{0\} \times S^{n-q-1})} \epsilon_x [\phi_i^q]
\]

Note that changing the choice of orientation will only only change the coefficient of each basis vector \( [\phi_i^q] \) up to a sign.

We now return to the proof of the homology lemma. We start by isotoping \( f \) to a map that is transverse to each of the transverse spheres of the \( q \)-handles (apply lecture 4, theorem 1.6 to \( P = S^q \), \( E = \partial_1 W_q \), \( M = \sqcup I_q S^{n-q-1} \)). Then by hypothesis \( [f] = [\phi_i^q] \) where we can now write \( [f] \) using equation 1. Thus \( f(S^q) \cap \phi_i^q(\{0\} \times S^{n-q-1}) \) is non-empty.

The conclusion of the lemma is exactly that we can isotope \( f \) such that the sum in equation 1 has only one term. So if this is already the case, we are done. Otherwise there is cancellation in the sum and we will apply the Whitney trick to isotope \( f \) to eliminate the superfluous intersection points in pairs.

References

