COURSE 465, SPRING 2010: TOPOLOGY OF MANIFOLDS

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This is a course on the classification of manifolds and their symmetries. We will cover some of the great theorems of manifold topology from the past 50 years, using methods of geometric topology and homotopy theory.

- Cobordism theory. Thom spectra MB, with structure group G. Equivalence of the unoriented cobordism ring and $\pi_* MO$. Cobordism classification of manifolds via the computation of $\pi_* MO$.

- Surgery. Smale's h-cobordism theorem and proof of the higher dimensional topological Poincaré conjecture. Simple homotopy and Whitehead torsion. The s-cobordism theorem of Barden, Mazur, and Stallings. Algebraic L-theory and the surgery exact sequence.

- Exotic spheres. Milnor's plumbing construction of exotic 7-spheres. Kervaire-Milnor computation of groups of exotic spheres. The J-homorphism, Bott periodicity, and the Adams conjecture.

Additional topics will be drawn from the following list.

- Smoothing theory. Classifying spaces for surgery: comparison of topological, PL, and smooth manifold structures after Mazur-Hirsch and Kirby-Siebenmann. The Hauptvermutung.

- Low-dimensional manifolds. Moise's theorem on existence and uniqueness of smooth structures on topological 3-manifolds. Hatcher's theorem on the homotopy equivalence $O(3) \to \mathrm{Diff}(S^3)$. Uniqueness of smoothings for PL 4-manifolds.

- Cobordism categories of manifolds. Galatius-Madsen-Tillmann-Weiss's computation of the classifying spaces of cobordism categories. Topological quantum field theories and higher categories. Hopkins-Lurie's generators & relations description of low-dimensional cobordism categories. Lurie's proof of the Baez-Dolan hypothesis.

Prerequisites: Algebraic topology at the level of 460, including a basic knowledge of manifolds, vector bundles, and characteristic classes. Familiarity with simplicial sets helpful, but not required.

Participants will have the option of giving a talk on a topic of paper of their choosing. Possible topics are

Topics for participants' talks: Chapman's proof of the topological invariance of Whitehead torsion. Hirsch-Smale immersion theory and the h-principle. The spherical space form problem, classifying free uniform actions on Euclidean spaces. Sullivan's study of PL manifolds, étale homotopy of algebraic varieties, and Galois symmetry. Action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on structure sets of smooth manifolds. Exotic smooth structures on $\mathbb{CP}^n$, lens spaces, or torii. Topological invariance of Pontryagin classes. Waldhausen's K-theory of spaces and the parametrized s-cobordism theorem. Reidemeister torsion. Rational cohomology of diffeomorphism groups and higher Reidemeister torsion invariants of smooth families. The Kervaire invariant. Representability of homology classes by manifolds. Freedman's classification of simply-connected topological 4-manifolds. Wall's finiteness obstruction. Siebenmann's end obstruction. Smale's theorem on the homotopy equivalence $\mathrm{Diff}(S^2) \simeq O(3)$.

Hatcher's proof of Smale's conjecture that $\mathrm{Diff}(S^3) \simeq O(4)$; homotopy equivalence of isometry groups and diffeomorphism groups of hyperbolic 3-manifolds.

References

[1] W. Browder. Surgery on Simply Connected Manifolds. www.maths.ed.ac.uk/∼aar/surgery/browder.pdf

[2] M. Cohen. A Course in Simple Homotopy Theory. www.math.uchicago.edu/∼shmuel/tom-readings

[3] S. Galatius, I. Madsen, U. Tillmann, M. Weiss. The Homotopy Type of the Cobordism Category. http://arxiv.org/abs/math/0605249

[4] M. Kervaire and J. Milnor. Groups of Homotopy Spheres, I., Annals of Mathematics 77 (1963) 504537. http://www.maths.ed.ac.uk/∼aar/papers/kervmiln.pdf

[5] R. Kirby and L. Siebenmann. Foundational Essays on Topological Manifolds, Smoothings, and Triangulations. http://www.maths.ed.ac.uk/∼aar/haupt/ks76.pdf

[6] J. Lurie. On the Classication of Topological Field Theories. http://math.mit.edu/∼lurie/papers/cobordism.pdf

[7] D. Sullivan. Geometric Topology: Localization, Periodicity, and Galois Symmetry. MIT notes, 1970. www.maths.ed.ac.uk/∼aar/books/gtop.pdf

[8] C. T. C. Wall. Surgery on Compact Manifolds. www.maths.ed.ac.uk/∼aar/books/scm.pdf