THE H-PRINCIPLE, LECTURE 20: THE H-PRINCIPLE FOR MICROFLEXIBLE SHEAVES

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1. A generalization of McDuff's theorem

We have now proved that if either X is a connected topological space, or M is a compact open manifold, there is a weak homotopy equivalence between $\mathrm{Conf}_X(M)$ and the sections $\Gamma(T_M^\infty \wedge_M X)$. As with many topological results, there is a relative version of this theorem.

Theorem 1.1. *Suppose that $M_0 \subset M$ and that M is contained in the interior of some manifold W. Assume that X is connected or that the pair (M, M_0) is connected. Then there is a weak homotopy equivalence*

$$\mathrm{Conf}_X(M, M_0) \to \Gamma(W - M_0, W - M; T_W^\infty \wedge_W X).$$

This theorem specializes to our previous results in the following way. Suppose that M is compact and open. Let $W = M \cup \partial M \times [0,1]$ and $M_0 = \partial M$. The right hand side above becomes

$$\Gamma(W - \partial M, W - M; T_W^\infty \wedge_W X).$$

These sections are precisely those that vanish over the half open collar of ∂M, which is the same as $\Gamma(M, \partial M; T_M^\infty \wedge_M X)$.

Now consider the case when M is non-compact and without boundary. For simplicity, assume that M is the interior of $\overline{M}$, for some $\overline{M}$ compact with boundary. We apply the theorem with $W = \overline{M}$ and $M_0 = \emptyset$. Re-writing the equivalence we obtain

$$\mathrm{Conf}_X(\overline{M}, \emptyset) \to \Gamma(\overline{M}, \partial M; T_{\overline{M}}^\infty \wedge_{\overline{M}} X).$$

The right hand side is the compactly supported sections, $\Gamma_c(T_M^\infty \wedge_M X)$, which is again McDuff's theorem.

We will not give a proof of this result, since it is essentially the same as the proof of the non-relative statement. It uses induction on a handle decomposition of W. One checks that every time we add a handle $U \to U + \varphi$, the induced map on the configuration spaces is a quasi-fibration.

Remark 1.2. The map $\mathrm{Conf}_X(M) \to \Gamma$ is interesting even if it is not a homotopy equivalence. It is closely related to the map $B\Sigma_\infty \to \Omega^\infty \Sigma^\infty S^0 = \mathrm{colim}_n\, \Omega^n S^n$. Barratt, Priddy and Quillen proved that this map is a homology equivalence, but not a homotopy equivalence.

2. Flexibility revisited

Recall that Mfld_n is the category of compact n-manifolds with morphisms the smooth embeddings. We saw that certain sheaves on Mfld_n can be approximated as sections of bundles. Recall the following definition.

Definition 2.1. A sheaf $\mathcal{F}$ is *flexible* if the map $\mathcal{F}(V) \to \mathcal{F}(U)$ is a Serre fibration for every embedding of compact manifolds $U \hookrightarrow V$. $\mathrm{Shv}^{\mathrm{flex}}(\mathrm{Mfld}_n)$ is the subcategory of $\mathrm{Shv}(\mathrm{Mfld}_n)$ consisting of flexible sheaves.

The following underlies the importance of these properties, and is certainly implicit in Gromov's thinking.

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Theorem 2.2. *There is a natural homotopy equivalences of topological categories*

$$\mathrm{Spaces}^{\mathrm{fib}}_{/BO_n} \xrightarrow{\ \Gamma\ } \mathrm{Shv}^{\mathrm{flex}}(\mathrm{Mfld}_n)$$

$$\mathrm{Shv}^{\mathrm{flex}}(\mathrm{Mfld}_n) \xrightarrow[\mathrm{ev}(D^n)]{} \mathrm{Spaces}^{\mathrm{fib}}_{/BO_n}$$

where $\mathrm{Spaces}^{\mathrm{fib}}_{/BO_n}$ *is the topological category of spaces* B *over* BO_n *for which the map* $B \to BO_n$ *is a Serre fibration. The value of the sheaf* Γ_B *on a manifold* M *is the space of sections* $\Gamma_M(\tau^{-1}B)$*, where* $\tau : M \to BO_n$ *is the map classifying the tangent bundle of* M*.The homotopy inverse functor* $\mathrm{Shv}^{\mathrm{flex}}(\mathrm{Mfld}_n) \to \mathrm{Spaces}^{\mathrm{fib}}_{/BO_n}$ *assigns to a sheaf* $\mathcal{F}$ *the homotopy quotient* $EO_n \times_{O_n} \mathcal{F}(D^n) \simeq \mathcal{F}(D^n)_{\mathrm{h}O_n}$*.*

Proof. Induction on a handle decomposition of M.

□

In other words, flexible sheaves on manifolds adhere to the h-principle.

Remark 2.3. There is a similar statement for a fixed manifold M, that "$\mathrm{Shv}^{\mathrm{flex}}(M)$" is equivalent to $\mathrm{Spaces}^{\mathrm{fib}}_{/M}$. However, it is a little cumbersome to define the diffeomorphism actions to define $\mathrm{Shv}^{\mathrm{flex}}(M)$. Perhaps the best way is to take the category of manifolds over M, with isotopies of embeddings as morphisms. However, composition in this category is no longer strictly associative, but governed by the $\mathcal{E}_1$ operad. So we will skirt this issue entirely, at least in this class.

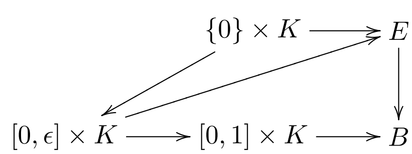
3. The h-principle for microflexible sheaves

Recall from the proof of the Hirsch-Smale theorem that while it was very difficult to show that the restriction map $\mathrm{Imm}(D^n, N) \to \mathrm{Imm}(S^{k-1} \times [0,1] \times D^{n-k}, N)$ is a Serre fibration (the technical crux of the proof), it was quite easy to show an weaker result, that the map is a Serre *microfibration.* That is:

Definition 3.1. A map $E \to B$ is a Serre microfibration if for any CW complex K and any commutative diagram

$$\begin{array}{ccc} \{0\} \times K & \longrightarrow & E \\ \downarrow & & \downarrow \\ [0,1] \times K & \longrightarrow & B \end{array}$$

there then exists a sufficiently small positive number ϵ and for which there exists a lift:



Example 3.2. If $U \subset V$ is an open subspace, then the inclusion map $U \hookrightarrow V$ is a Serre microfibration. However, it is clearly not a Serre fibration unless it is a homeomorphism, $U \cong V$.

Example 3.3. The restriction map $\mathrm{Imm}(D^n, N^n) \to \mathrm{Imm}(S^{n-1}, N^n)$ is a Serre microfibration, but it is not a Serre fibration.

Definition 3.4. A sheaf $\mathcal{F}$ of topological spaces on M is microflexible if the restriction map $\mathcal{F}(K') \to \mathcal{F}(K)$ is a Serre microfibration for every closed inclusion of compact subspaces $K \to K'$. It is flexible if the maps $\mathcal{F}(K') \to \mathcal{F}(K)$ are all Serre fibrations.

The condition of being microflexible is weaker, but much easier to check. Some of the importance of the notion of flexibility was reflected in the following result, see Lecture 4.

Proposition 3.5. *If a map of flexible sheaves $\mathcal{F} \to \mathcal{F}'$ on M is a weak homotopy equivalence $\mathcal{F}(U) \to \mathcal{F}'(U)$ for every $U \subset M$ for which U is contractible, then $\mathcal{F}(M) \to \mathcal{F}'(M)$ is a weak homotopy equivalence.*

Theorem 3.6 (Gromov)**.** *Diffeomorphism invariant microflexible sheaves on open manifolds are equivalent to flexible sheaves, and consequently adhere to the h-principle. Equivalently, the inclusion*

$$\mathrm{Shv}^{\mathrm{flex}}(\mathrm{Mfld}_n^{\mathrm{open}}) \to \mathrm{Shv}^{\mathrm{mflex}}(\mathrm{Mfld}_n^{\mathrm{open}})$$

is a homotopy equivalence.

By popular demand, we will not prove this result.

References

[1] Gromov, Mikhael. Partial differential relations. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 9. Springer-Verlag, Berlin, 1986. x+363 pp.