THE CLASSIFICATION OF IRREDUCIBLE ADMISSIBLE MOD p REPRESENTATIONS OF A p-ADIC GL_n.

FLORIAN HERZIG

Abstract. Let F be a finite extension of $\mathbb{Q}_p$. Using the Satake transform, we define what it means for an irreducible admissible smooth representation of an F-split p-adic reductive group over $\overline{\mathbb{F}}_p$ to be supersingular. We then give the classification of irreducible admissible smooth $\mathrm{GL}_n(F)$-representations over $\overline{\mathbb{F}}_p$ in terms of supersingular representations. As a consequence we deduce that *supersingular* is the same as *supercuspidal*. These results generalise the work of Barthel–Livné for $n = 2$. For general split reductive groups we obtain similar results under stronger hypotheses.

1. Introduction

Let F be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$, uniformiser ϖ, and residue field k of order q. Let $G_{/\mathcal{O}}$ be a split connected reductive group and fix a maximal split torus $T_{/\mathcal{O}}$. Let $\Phi \subset X^*(T)$ denote the set of roots and choose a system of positive roots Φ^+. Denote by $B_{/\mathcal{O}}$ denote the associated Borel subgroup and by $U_{/\mathcal{O}}$ its unipotent radical. Let W be the Weyl group and let $K = G(\mathcal{O})$, a hyperspecial maximal compact subgroup of $G(F)$.

1.1. **Arrangement of the paper.** In Section 4 we define what it means for an irreducible admissible representation to be supersingular. Our definition uses the mod p Satake transform [Her] and generalises the definition for GL_2 of Barthel–Livné [BL95], [BL94].

The main results of this paper are contained in the last two sections. For $G = \mathrm{GL}_n$ we prove the following. We determine the Jordan–Hölder factors of parabolic inductions of supersingular representations (Theorems 8.3 and 8.4). We prove that any irreducible admissible representation occurs in such an induced representation (Thm. 9.10) and we establish a uniqueness result (Thm. 9.12). As a consequence we prove that *supersingular* is the same as *supercuspidal* (Thm. 9.15) and we determine the Jordan–Hölder factors of parabolic inductions of irreducible admissible representations (Thm. 8.5). For general groups G, we obtain results under stronger hypotheses (see Theorems 8.6 and 9.19).

The main tools we used may be of independent interest. In Section 3 we establish an isomorphism between the quotient by a maximal ideal of the

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Hecke algebra of a certain compactly induced representation (from K to G) with a parabolically induced representation, under certain conditions. This generalises results of Barthel–Livné for GL_2 [BL95], [BL94]. In Section 6 we give a criterion when quotients of different compactly induced representations are isomorphic. This allows us to "change of weight" in smooth representations under certain conditions. In Section 7 we discuss generalised Steinberg representations: their irreducibility (completing results of Grosse-Klönne [GK]), their weights and their Hecke eigenvalues.

1.2. **Notation.** The letters P, Q usually denote standard parabolic subgroups, i.e., parabolic subgroups containing the Borel B. A Levi decomposition $P = M \ltimes N$ of a standard parabolic is implicitly assumed to be standard, i.e., M is the unique Levi subgroup of P that contains T. If $P = MN$ is a parabolic we denote by $\overline{P} = M\overline{N}$ the opposite parabolic (it depends on the choice of Levi M).

We denote by Z the *connected* centre of G. Similarly Z_M denotes the connected centre of a Levi subgroup M. We denote by $X_*(T)_-$ the set of antidominant coweights of T. The set of simple roots in Φ^+ is denoted by Δ. If $P = MN$ is a standard parabolic, we similarly define Δ_M (with respect to $\Phi_M^+ = \Phi_M \cap \Phi^+$).

We denote by $\mathrm{red} : K = G(\mathcal{O}) \to G(k)$ the reduction map and by $K(1)$ its kernel, which is a pro-p group. For a standard parabolic subgroup P, we let $\mathcal{P} := \mathrm{red}^{-1}(P(k))$ be the corresponding parahoric subgroup. We will also write I for the standard Iwahori subgroup $\mathcal{B}$ and define $I(1) := \mathrm{red}^{-1}(U(k))$ (the pro-p Sylow subgroup of I).

If H is a group and σ and H-representation, we denote by $\mathrm{soc}_H\,\sigma$ the H-socle, i.e., the largest semisimple subrepresentation.

We usually write G, P, M, ... when we really mean $G(F)$, $P(F)$, $M(F)$, ... This should cause no confusion.

All representations in this paper, unless otherwise stated, live on $\bar{k}$-vector spaces.

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Contents

2. Hecke actions and the Satake transform

2.1. **Background.**

2.1.1. *Serre weights.*

Definition 2.1. A *Serre weight* is an isomorphism class of irreducible representations V of the finite group $G(k)$ over $\bar{k}$.

The set of q-restricted weights is defined to be:

$$X_q(T) = \{\lambda \in X(T) : 0 \leq \langle \lambda, \alpha^\vee \rangle < q \quad \forall \alpha \in \Delta\}.$$

We also define:

$$X^0(T) = \{\lambda \in X(T) : \langle \lambda, \alpha^\vee \rangle = 0 \quad \forall \alpha \in \Phi\}.$$

For $\nu \in X^*(T)$ dominant, let $F(\nu)$ denote the irreducible $G_{/\bar{k}}$-module of highest weight ν. Via the inclusion $G(k) \to G(\bar{k})$, we can consider $F(\nu)$ as $G(k)$-representation.

Proposition 2.2. *Suppose that the derived subgroup of G is simply connected. Then all Serre weights are of the form $F(\nu)$, where $\nu \in X_q(T)$. Moreover for ν, $\nu' \in X_q(T)$, we have $F(\nu) \cong F(\nu')$ as $G(k)$-representations if and only if $\nu - \nu' \in (q-1)X^0(T)$.*

This goes back to Steinberg if G is semisimple; in general, see Prop. 1.3 in the appendix to [Her09]. If the derived subgroup of G fails to be simply connected, Serre weights can be described using a z-extension of G (as in the proof of [Her, Lemma 2.5]).

We will sometimes denote by 1 the trivial Serre weight. If $P = MN$ is a standard parabolic, we denote by $X^0_M(T)$ and $F^M(\nu)$ the analogues of $X^0(T)$ and $F(\nu)$ for the Levi M.

For the following, very useful Lemma, see [Her, Lemma 2.5].

Lemma 2.3. *Suppose that V is a Serre weight and that $P = MN$ is a standard parabolic. Then $V^{N(k)}$ and $V_{\overline{N}(k)}$ are Serre weights for M and the natural, $M(k)$-linear map $V^{N(k)} \to V_{\overline{N}(k)}$ is an isomorphism. In particular, $V^{U(k)} \cong V_{\overline{U}(k)}$ is one-dimensional.*

Suppose the derived subgroup of G is simply connected. If $V \cong F(\nu)$ for some $\nu \in X_q(T)$, then $V^{N(k)} \cong F^M(\nu)$. Moreover, as a subspace of $F(\nu)$, $V^{N(k)}$ is the sum of all weight spaces $F(\nu)_{\nu'}$ with $\nu - \nu' \in \mathbb{Z}\Phi_M$.

Definition 2.4. A Serre weight V is said to be *M-regular* if $\mathrm{Stab}_W(V^{U(k)}) \subset W_M$.

Here $\mathrm{Stab}_W(V^{U(k)})$ denotes the set of $w \in W$ that preserve the one-dimensional, $T(k)$-stable subspace $V^{U(k)} \subset V$. Note that if $V = F(\nu)$, then $\mathrm{Stab}_W(V^{U(k)}) = \mathrm{Stab}_W(\nu)$.

Lemma 2.5. *The map $V \mapsto V^{N(k)}$ from M-regular Serre weights for G to Serre weights for M is a bijection.*

Proof. Suppose first that the derived subgroup of G is simply connected. Then the same is true for M. Let $\overline{V}$ be a Serre weight for M. Then $\overline{V} \cong F^M(\nu)$ for some $\nu \in X(T)$ such that $0 \le \langle \nu, \alpha^\vee \rangle \le q-1$ for all $\alpha \in \Delta_M$. We need to find $\nu' \in X(T)$ such that (a) $0 \le \langle \nu', \alpha^\vee \rangle \le q-1$ for all $\alpha \in \Delta_M$, (b) $0 < \langle \nu', \alpha^\vee \rangle \le q-1$ for all $\alpha \in \Delta - \Delta_M$, and (c) $\nu - \nu' \in (q-1)X_M^0(T)$. The first two conditions express that ν' is q-restricted and that the Serre weight $V \cong F(\nu')$ is M-regular. Condition (c) expresses that $V^{N(k)} \cong \overline{V}$. Clearly there is such a ν' and it is uniquely determined up to $(q-1)X^0(T)$. This completes the proof.

In the general case, pick a z-extension $1 \to R \to \widetilde{G} \to G \to 1$, just as in [Her, Lemma 2.5]. We know there is a unique Serre weight V for $\widetilde{G}$ such that $V^{\widetilde{N}(k)} \cong \overline{V}$ as $\widetilde{M}(k)$-representation. Since $R(k)$ acts trivially on $\overline{V}$ and since it acts centrally on V, we see that V descends to a $G(k)$-representation. The uniqueness of V is even easier. □

2.1.2. *Smooth representations.* We recall that a smooth G-representation π is said to be *admissible* if π^H is finite-dimensional for all open subgroups H of G. It is sufficient to verify this condition for one open pro-p subgroup H. An irreducible admissible G-representation π has a central character, which we denote by $\omega_\pi : Z \to \bar{k}^\times$.

Any irreducible smooth K-representation factors through $G(k)$ (as $K(1)$ is pro-p); it is thus a Serre weight. If π is admissible, $\mathrm{soc}_K \pi \subset \pi^{K(1)}$ is finite-dimensional and non-zero, so π contains a Serre weight V. (It is clear that even any smooth G-representation contains a Serre weight.) We will also say that V is a Serre weight of π.

For a closed subgroup $H \subset G$ and a smooth H-representation σ we denote by $\mathrm{Ind}_H^G \sigma$ (resp., $\text{c-Ind}_H^G \sigma$) the representation that is induced (resp., compactly induced) from σ. If H is open, then $\text{c-Ind}_H^G \sigma \cong \bar{k}[G] \otimes_{\bar{k}[H]} \sigma$, so c-Ind_H^G is a left adjoint to the forgetful functor. In this case, we will also denote by $[g, x] \in \text{c-Ind}_H^G \sigma$ the element that is supported on Hg^{-1} and sends g^{-1} to $x \in \sigma$. If P is a parabolic subgroup, then Ind_P^G is *exact*. (This is because the map $G \to G/P$ has continuous sections; see [Eme, Prop. 4.1.5].)

Suppose that H is a compact open subgroup and that V is a finite-dimensional smooth H-representation. We define the *Hecke algebra* of V to be $\mathcal{H}_H(V) := \mathrm{End}_G(\text{c-Ind}_H^G V)$. Using the above adjunction, we can and usually will think of it as $\bar{k}$-algebra of compactly supported functions

$f : G \to \mathrm{End}_{\bar{k}} V$ satisfying $f(h_1 g h_2) = h_1 f(g) h_2$ for all h_1, $h_2 \in H$, $g \in G$, where the multiplication is given by convolution. Note that if π is a smooth G-representation, $\mathcal{H}_H(V)$ naturally acts on $\mathrm{Hom}_H(V, \pi) \cong \mathrm{Hom}_G(\text{c-Ind}_H^G V, \pi)$. (This is a right action.)

Suppose now that V is a Serre weight. we will usually write $\mathcal{H}_G(V)$ instead of $\mathcal{H}_K(V)$. We will see in a moment that $\mathcal{H}_G(V)$ is commutative. If $V = 1$ then $\mathcal{H}_G(V)$ is the usual unramified Hecke algebra.

2.2. **The mod p Satake transform.** We begin by recalling some results of [Her]. Let T^- denote the submonoid of T,

$$T^- = \{t \in T : \mathrm{ord}_F(\alpha(t)) \le 0 \quad \forall \alpha \in \Delta\},$$

and let $\mathcal{H}_T^-(V^{U(k)})$ denote the subalgebra of $\mathcal{H}_T(V^{U(k)})$ consisting of those $\varphi : T \to \bar{k}$ that are supported on T^-.

Theorem 2.6. *Suppose that V is a Serre weight. Then*

$$\mathcal{S}_G : \mathcal{H}_G(V) \to \mathcal{H}_T(V^{U(k)})$$
$$f \mapsto \left(t \mapsto \sum_{u \in U/U(\mathcal{O})} f(tu)\Big|_{V^{U(k)}} \right)$$

is an injective $\bar{k}$-algebra homomorphism with image $\mathcal{H}_T^-(V^{U(k)})$.

In particular, $\mathcal{H}_G(V) \cong \bar{k}[X_*(T)_-]$ is commutative and noetherian (Gordon's lemma shows that $X_*(T)_-$ is finitely generated). We recall that $G = \coprod K\lambda(\varpi)K$, where λ ranges over $X_*(T)_-$ (refined Cartan decomposition). Moreover, $\mathcal{H}_G(V)$ has a basis given by T_λ ($\lambda \in X_*(T)_-$), where T_λ has support $K\lambda(\varpi)K$ and sends $\lambda(\varpi)$ to the endomorphism $V \twoheadrightarrow V_{N_\lambda(k)} \xleftarrow{\sim} V^{N_{-\lambda}(k)} \hookrightarrow V$ (see §2.4 for the definition of $P_\lambda = M_\lambda N_\lambda$). We also denote by $\tau_\lambda \in \mathcal{H}_T(V^{U(k)})$ the element supported on $\lambda(\varpi)T(\mathcal{O})$ that sends $\lambda(\varpi)$ to 1. We have:

$$\mathcal{S}_G(T_\lambda) = \sum_{\substack{\mu \in X_*(T)_- \\ \mu \ge_{\mathbb{R}} \lambda}} a_\lambda(\mu)\tau_\mu, \quad \text{with } a_\lambda(\mu) \in \bar{k} \text{ and } a_\lambda(\lambda) = 1$$

Here, $\mu \ge_{\mathbb{R}} \lambda$ means that $\mu - \lambda$ is a non-negative real linear combination of the simple coroots.

For $z \in Z$ we also define $T_z \in \mathcal{H}_G(V)$ such that $\mathrm{supp}(T_z) = Kz$ and $T_z(z) = \mathrm{id}_V$. The following formulae for z, z_1, $z_2 \in Z$, $z_0 \in Z(\mathcal{O})$ will be useful later.

$$T_{z_1 z_2} = T_{z_1} T_{z_2},\ T_1 = 1, \tag{2.7}$$

$$T_{z_0 z} = \omega_V(z_0)^{-1} T_z, \tag{2.8}$$

$$T_z = z^{-1} \quad \text{on c-Ind}_K^G V, \tag{2.9}$$

where $\omega_V : Z(k) \to \bar{k}^\times$ is the central character of V.

2.3. **Variants of the Satake transform.** Let $P = MN$ be a standard parabolic subgroup. There is a "partial" Satake homomorphism ${}'\mathcal{S}_G^M : \mathcal{H}_G(V) \to \mathcal{H}_M(V_{\overline{N}(k)})$ (since P is standard, it is determined by M). It is defined by

$$\begin{aligned} {}'\mathcal{S}_G^M : \mathcal{H}_G(V) &\to \mathcal{H}_M(V_{\overline{N}(k)}) \\ f &\mapsto \left(m \mapsto p_{\overline{N}} \sum_{\overline{N}(\mathcal{O})\backslash \overline{N}} f(\overline{n}m) \right), \end{aligned}$$

where $p_{\overline{N}}$ denotes the projection $V \twoheadrightarrow V_{\overline{N}(k)}$. We also have $\mathcal{S}_G^M : \mathcal{H}_G(V) \to \mathcal{H}_M(V^{N(k)})$, defined in the same way as $\mathcal{S}_G$.

We have a simple compatibility between ${}'\mathcal{S}_G^M$ and $\mathcal{S}_G^M$. There is an algebra isomorphism $\mathcal{H}_G(V) \xrightarrow{\sim} \mathcal{H}_G(V^*)$ which sends φ to φ' with $\varphi'(g) = \varphi(g^{-1})^*$. (Recall that $\mathcal{H}_G(V)$ is commutative.) Similarly we have an isomorphism $\mathcal{H}_M(V^{\overline{N}(k)}) \xrightarrow{\sim} \mathcal{H}_M((V^*)_{\overline{N}(k)})$. In the following lemma we exceptionally compute $\mathcal{S}_G^M$ with respect to $\overline{P} = M\overline{N}$ (a standard parabolic for Φ^-).

Lemma 2.10. *We have the following commutative diagram.*

$$\begin{array}{ccc} \mathcal{H}_G(V) & \xrightarrow{\mathcal{S}_G^M} & \mathcal{H}_M(V^{\overline{N}(k)}) \\ \cong \downarrow & & \downarrow \cong \\ \mathcal{H}_G(V^*) & \xrightarrow{{}'\mathcal{S}_G^M} & \mathcal{H}_M((V^*)_{\overline{N}(k)}) \end{array}$$

In particular ${}'\mathcal{S}_G := {}'\mathcal{S}_G^T$ *is injective. Its image consists of those elements of* $\mathcal{H}_T((V^*)_{\overline{U}(k)})$ *that are supported on* T^- *and is isomorphic to* $\bar{k}[X_*(T)_-]$.

Proof. This is straightforward, noting that $V^{\overline{N}(k)} \to V$ is dual to $V^* \to (V^*)_{\overline{N}(k)}$. □

We will see in Cor. 2.18 that $\mathcal{S}_G^M$ and ${}'\mathcal{S}_G^M$ are actually the same, under the natural identification of $\mathcal{H}_M(V^{N(k)})$ and $\mathcal{H}_M(V_{\overline{N}(k)})$. This does not seem to be obvious from the definition.

Proposition 2.11. *The map* ${}'\mathcal{S}_G^M$ *is an injective algebra homomorphism. It is a localisation map of integral domains. Also* ${}'\mathcal{S}_G = {}'\mathcal{S}_M \circ {}'\mathcal{S}_G^M$.

The same of course also holds for $\mathcal{S}_G^M$.

Proof. It is easy to see that the map is well defined and an algebra homomorphism (see also [Her]). A direct calculation shows that ${}'\mathcal{S}_G = {}'\mathcal{S}_M \circ {}'\mathcal{S}_G^M$. Since ${}'\mathcal{S}_G$ is injective, so is ${}'\mathcal{S}_G^M$. Finally note that $\bar{k}[X_*(T)_-] \hookrightarrow \bar{k}[X_*^M(T)_-]$ is a localisation map of integral domains, where $X_*^M(T)_-$ are the antidominant coweights for T in M. (It suffices to invert any $\lambda \in X_*(T)_-$ such that for $\alpha \in \Delta$, $\langle \lambda, \alpha \rangle = 0$ if and only if $\alpha \in \Delta_M$.) □

Suppose now that V is a Serre weight for G and σ a smooth M-representation. By the Iwasawa decomposition we have $(\mathrm{Ind}_{\overline{P}}^{G}\sigma)|_K \cong \mathrm{Ind}_{\overline{P}(\mathcal{O})}^{K}\sigma$, so the natural map

$$\begin{aligned} \mathrm{Hom}_K(V, \mathrm{Ind}_{\overline{P}}^{G}\sigma) &\to \mathrm{Hom}_{M(\mathcal{O})}(V_{\overline{N}(k)}, \sigma) \\ f &\mapsto \overline{f} = \big(\overline{v} \mapsto f(v)(1)\big) \end{aligned} \tag{2.12}$$

is a bijection (here v is any lift of $\overline{v}$). Note that $\mathcal{H}_G(V)$ naturally acts on the left-hand side, and on the right-hand side via ${}'\mathcal{S}_G^M$.

Lemma 2.13. *The natural map in* (2.12) *is* $\mathcal{H}_G(V)$*-equivariant.*

Proof. For $\varphi \in \mathcal{H}_G(V)$ let $\varphi_M = {}'\mathcal{S}_G^M(\varphi)$. We compute

$$\begin{aligned} (\overline{f} * \varphi_M)(\overline{v}) &= \sum_{M(\mathcal{O})\backslash M} m^{-1} f\Bigg(\sum_{\overline{N}(\mathcal{O})\backslash \overline{N}} \varphi(\overline{n}m)v\Bigg)(1) \\ &= \sum_{M(\mathcal{O})\backslash M}\ \sum_{\overline{N}(\mathcal{O})\backslash \overline{N}} f(\varphi(\overline{n}m)v)(m^{-1}\overline{n}^{-1}) \end{aligned}$$

and

$$(\overline{f * \varphi})(\overline{v}) = (f * \varphi)(v)(1) = \sum_{K\backslash G} f(\varphi(g)v)(g^{-1}).$$

By the Iwasawa decomposition $G = K\overline{P}$ these two expressions are equal. □

2.4. **Various lemmas.** Suppose that $\lambda \in X_*(T)$. Let $P_\lambda = M_\lambda N_\lambda$ denote the parabolic subgroup of G defined by λ. For the following proposition, see [Her, Prop. 3.8].

Proposition 2.14. *Let* $t = \lambda(\varpi)$*. Then* $\mathrm{red}(K \cap K^t) = P_\lambda(k)$*.*

Define $Z_M^- := Z_M \cap T^-$ and $Z_M^+ := Z_M \cap (T^-)^{-1}$. We also need

$$Z_M^{--} := \{z \in Z_M : \mathrm{ord}_F(\alpha(z)) < 0 \quad \forall \alpha \in \Delta - \Delta_M\}.$$

Lemma 2.15. *Suppose $P = MN$ is a standard parabolic subgroup. We let $\overline{\mathcal{P}}^- = \overline{N} \cap \overline{\mathcal{P}}$, $\overline{\mathcal{P}}^0 = M \cap \overline{\mathcal{P}}$, $\overline{\mathcal{P}}^+ = N \cap \overline{\mathcal{P}}$. Then $\overline{\mathcal{P}} = \overline{\mathcal{P}}^-\overline{\mathcal{P}}^0\overline{\mathcal{P}}^+$ (in any order). T^- contracts $\overline{\mathcal{P}}^-$ and expands $\overline{\mathcal{P}}^+$.*

If $h_0 \in Z_M^{--}$ then for any open subset $\Omega \subset N$, we have $(\overline{\mathcal{P}}^+)^{h_0^n} \subset \Omega$ for $n \gg 0$.

Proof. Use Bruhat–Tits. □

Lemma 2.16. *Suppose that $P = MN$ and $Q = LN'$ are standard parabolics. Suppose that V is both M and L-regular. Then $p_{\overline{N'}}(kV^{N(k)}) \neq 0$ for $k \in K$ implies $k \in \overline{\mathcal{Q}}\mathcal{P}$.*

If $\mathrm{Stab}_W(V^{U(k)})$ equals W_M (resp., W_L), we may drop the assumption that V is L-regular (resp., M-regular).

Proof. It is easy to reduce to the case that the derived subgroup of G is simply connected, just as in [Her, Lemma 2.5]. (Note that the statements we want to prove just concern the finite group $G(k)$.) In this case $V = F(\nu)$ for some $\nu \in X_1(T)$, moreover $V^{N(k)} = F(\nu)^N$.

We claim that for all weights μ of $F(\nu)^N$ and all $\alpha \in \Phi^+ - \Phi^+_M$ we have $\langle \mu, \alpha^\vee \rangle > 0$. We know that $F(\nu)^N$ is the irreducible M-representation of highest weight ν, so its weights lie in the convex hull of $w\nu$ ($w \in W_M$). Note that any W_M preserves $\Phi^+ - \Phi^+_M$. (Reduce to the case of a simple reflection s_β with $\beta \in \Phi_M$. It preserves Φ_M and $\Phi^+ - \{\beta\}$.) Therefore $\langle w\nu, \alpha^\vee \rangle = \langle \nu, w^{-1}\alpha^\vee \rangle \geq 0$ for all $w \in W_M$. If equality holds, then $s_{w^{-1}\alpha}(\nu) = \nu$. Thus $s_{w^{-1}\alpha} \in W_M$ (as V is M-regular), which implies that $\alpha \in \Phi_M$, a contradiction. The claim follows.

By the rational Bruhat decomposition, $G(k) = \coprod_{W_L \backslash W / W_M} \overline{Q}(k)\dot{\sigma}P(k)$. It will thus suffice to show that $p_{\overline{N'}}(\dot{\sigma}V^{N(k)}) \neq 0$ for $\sigma \in W$ implies $\sigma \in W_L W_M$. Since the natural map $F(\nu)^{N'} \to F(\nu)_{\overline{N'}}$ is an isomorphism (Lemma 2.3), we see that there is a weight μ of $F(\nu)^N$ such that $\sigma\mu$ is a weight of $F(\nu)^{N'}$. By the previous paragraph,

$$\langle \mu, \alpha^\vee \rangle > 0 \quad \forall \alpha \in \Phi^+ - \Phi^+_M, \quad \langle \mu, \beta^\vee \rangle > 0 \quad \forall \beta \in \Phi^+ - \Phi^+_L.$$

It follows that $\sigma(\Phi^+ - \Phi^+_M) \subset \Phi^+ \cup \Phi^-_L$.

We claim that there is a $w \in W_M$ such that $\sigma w(\Phi^+) \subset \Phi^+ \cup \Phi^-_L$. Suppose there is a simple root α of M such that $\sigma(\alpha) < 0$. Then s_α preserves $\Phi^+ - \Phi^+_M$, so $\sigma s_\alpha(\Phi^+ - \Phi^+_M) \subset \Phi^+ \cup \Phi^-_L$ while σs_α maps one fewer simple root of M to a negative root. By induction we find a $w \in W_M$ such that $\sigma w(\Phi^+ - \Phi^+_M) \subset \Phi^+ \cup \Phi^-_L$ and σw maps all simple roots of M to positive roots. This implies the claim.

Equivalently, $w^{-1}\sigma^{-1}(\Phi^- - \Phi^-_L) \subset \Phi^-$. The same argument as in the previous paragraph shows that there is a $w' \in W_L$ such that $w^{-1}\sigma^{-1}w'(\Phi^-) \subset \Phi^-$. This shows that $\sigma = w'w^{-1} \in W_L W_M$, which completes the proof.

To justify the final statement, suppose that $\mathrm{Stab}_W(V^{U(k)}) = W_M$. Then $V^{N(k)} = V^{U(k)}$ is one-dimensional. (One can see this directly using the Bruhat decomposition; alternatively note that $\nu \in X^0_M(T)$.) Thus if $p_{\overline{N'}}(\dot{\sigma}V^{N(k)}) \neq 0$ then $\sigma\nu$ is an extremal weight of $F(\nu)^{N'}$. The latter representation is irreducible for L and of highest weight ν. Therefore $\sigma\nu \in W_L\nu$, so $\sigma \in W_L \,\mathrm{Stab}_W(\nu) = W_L W_M$. The argument in case $\mathrm{Stab}_W(V^{U(k)}) = W_L$ is similar, or follows by duality, □

Corollary 2.17. *Suppose that $P = MN$ contains B and that $\lambda \in X_*(T)_-$. Suppose that V is M-regular, and suppose that either* $\mathrm{Stab}_W(V^{U(k)}) = W_M$ *or that V is also M_λ-regular. Let $t = \lambda(\varpi)$.*

(i) *If $T_\lambda(g)|_{V^{N(k)}} \neq 0$ then $g \in Kt\mathcal{P}$. If $p_{\overline{N}} \circ T_\lambda(g) \neq 0$ then $g \in \overline{\mathcal{P}}tK$.*

(ii) *We have $\mathcal{S}^M_G(T_\lambda) = T^M_\lambda$ and $'\mathcal{S}^M_G(T_\lambda) = T^M_\lambda$.*

In particular if $h \in Z^-_M$ then $'\mathcal{S}^M_G(T_h) = T^M_h$. Moreover $T^M_h(h)$ is the identity on $V_{\overline{N}(k)}$.

This generalises [Her, Proposition 1.4].

Proof. (i) Writing $g = k'tk$ with k', $k \in K$, we see that $T_\lambda(t)k|_{V^{N(k)}} \neq 0$, so $p_{N_\lambda}(kV^{N(k)}) \neq 0$. Lemma 2.16 shows that $k \in \mathcal{P}_\lambda \cdot \mathcal{P}$. By Prop. 2.14, $\mathrm{red}(\mathcal{P}_\lambda) = \mathrm{red}(K \cap K^t)$, so $k \in (K \cap K^t)\mathcal{P}$ and $g = k'tk \in Kt\mathcal{P}$.

The other part is similar, or follows by duality.

(ii) Take any $m \in M$ and suppose that $T_\lambda(mn)|_{V^{N(k)}} \neq 0$ (some $n \in N$) is a non-zero term contributing to $(\mathcal{S}_G^M T_h)(m)$. By part (i) and by Lemma 2.15 (twice), we have

$$mn \in Kt\mathcal{P} \cap P = KtP(\mathcal{O}) \cap P = P(\mathcal{O})tP(\mathcal{O}) = M(\mathcal{O})tM(\mathcal{O})N(\mathcal{O}).$$

Thus $m \in M(\mathcal{O})tM(\mathcal{O})$ and $n \in N(\mathcal{O})$. This shows that $\mathcal{S}_G^M(T_\lambda)$ is supported on $M(\mathcal{O})tM(\mathcal{O})$. Moreover $(\mathcal{S}_G^M T_\lambda)(t) = T_\lambda(t)|_{V^{N(k)}}$ is a linear projection, so $\mathcal{S}_G^M(T_\lambda) = T_\lambda^M$. The other statement follows similarly or by using duality.

The final claim follows since h is central in M. ◻

Corollary 2.18. *Let $P = MN$ be any standard parabolic. Under the natural identification of $\mathcal{H}_M(V^{N(k)})$ and $\mathcal{H}_M(V_{\overline{N}(k)})$ we have $\mathcal{S}_G^M = {}'\mathcal{S}_G^M$.*

Proof. We first consider the case when M is the standard Levi with $\mathrm{Stab}_W(V^{U(k)}) = W_M$. Under the natural identification, we have by Cor. 2.17 that $\mathcal{S}_G^M(T_\lambda) = T_\lambda^M = {}'\mathcal{S}_G^M(T_\lambda)$, for any $\lambda \in X_*(T)_-$. Note that $V^{N(k)}$ is one-dimensional (for example, by the Bruhat decomposition), so $\mathcal{S}_M = {}'\mathcal{S}_M$ for this Serre weight. By transitivity, $\mathcal{S}_G = {}'\mathcal{S}_G$.

If M is arbitrary we use that ${}'\mathcal{S}_G = {}'\mathcal{S}_M \circ {}'\mathcal{S}_G^M$ and $\mathcal{S}_G = \mathcal{S}_M \circ \mathcal{S}_G^M$. By the above we already know that $\mathcal{S}_G = {}'\mathcal{S}_G$ and $\mathcal{S}_M = {}'\mathcal{S}_M$. The claim follows by using the injectivity of these maps. ◻

Lemma 2.19. *Suppose $\overline{P} = M\overline{N}$. Then $G = \overline{\mathcal{P}}TK$.*

Suppose the subset $X \subset G$ has finite image in $\overline{\mathcal{P}}\backslash G$. Then there exists $h \in Z_M^-$ such that $hX \subset \overline{\mathcal{P}}T^-K$.

See also [SS91, Lemma 12].

Proof. For the first claim, suppose $g \in G$. By the Cartan decomposition, there is a $\lambda \in X_*(T)_-$ such that $g \in K\lambda(\varpi)K$. By Prop. 2.14, we have

$$\overline{\mathcal{P}}\backslash K\lambda(\varpi)K/K \cong \overline{P}(k)\backslash G(k)/\overline{P}_{-\lambda}(k).$$

The rational Bruhat decomposition shows that $g \in \overline{\mathcal{P}}\dot{w}\lambda(\varpi)K = \overline{\mathcal{P}}({}^w\lambda(\varpi))K$, for some $w \in W$. Alternatively one could use the Bruhat–Tits decomposition $G = \overline{\mathcal{B}}N(T)\overline{\mathcal{B}}$.

For the second: say $X \subset \bigcup \overline{\mathcal{P}}t_ik_i$ (finite union). Since $(G, \overline{\mathcal{B}}, N(T))$ is a generalised Tits system [Iwa66], we know that for all $n' \in N(T)$ there are $n_1, \ldots, n_r \in N(T)$ such that

$$n\overline{\mathcal{B}}n' \subset \bigcup \overline{\mathcal{B}}nn_j\overline{\mathcal{B}} \quad \forall n \in N(T)$$

(by induction from the BN axioms). Thus there are $n_{ij} \in N(T)$ such that $h\overline{\mathcal{B}}t_i \subset \bigcup_j \overline{\mathcal{B}}hn_{ij}\overline{\mathcal{B}}$ for all i and for all $h \in Z_M^-$. It follows that

$$h\overline{\mathcal{P}}t_ik_i = \overline{\mathcal{P}}^-\overline{\mathcal{P}}^0h\overline{\mathcal{P}}^+t_ik_i \subset \bigcup_j \overline{\mathcal{P}}hn_{ij}K,$$

as $\overline{\mathcal{P}}^+ \subset \overline{\mathcal{B}} \subset \overline{\mathcal{P}}$. Writing $n_{ij} = t_{ij}\dot{w}_{ij}$ with $t_{ij} \in T$ and $w_{ij} \in W$ we see that the right-hand side equals $\bigcup_j \overline{\mathcal{P}}ht_{ij}K$. Since $M(\mathcal{O}) \subset \overline{\mathcal{P}}$ we may replace t_{ij} with any W_M-conjugate without changing its double coset. In this way we can ensure that t_{ij} is antidominant as element of M. Then it is possible to find an $h \in Z_M^-$ such that $ht_{ij} \in T^-$ for all i, j. $\square$

2.5. **Compatibilities between Hecke actions.** Let $\overline{V}$ be a Serre weight for M. Consider the following subspaces of Hecke algebras:

$$\{\varphi : \mathrm{supp}(\varphi) \subset M(\mathcal{O})Z_M^-M(\mathcal{O})\} \subset \mathcal{H}_M(\overline{V}), \tag{2.20}$$

$$\{\varphi : \mathrm{supp}(\varphi) \subset \mathcal{P}Z_M^-\mathcal{P}\} \subset \mathcal{H}_{\mathcal{P}}(\overline{V}), \tag{2.21}$$

$$\{\varphi : \mathrm{supp}(\varphi) \subset \overline{\mathcal{P}}Z_M^-\overline{\mathcal{P}}\} \subset \mathcal{H}_{\overline{\mathcal{P}}}(\overline{V}). \tag{2.22}$$

The following lemma shows that each of them is a (commutative) subalgebra and that we can naturally identify them. We call them $\mathcal{H}(\overline{V})$. When writing $\varphi \in \mathcal{H}$ later on, it should be clear from the context what $\overline{V}$ is.

Lemma 2.23. *The subspaces* (2.20), (2.21), (2.22) *are subalgebras. The map* $\varphi \mapsto \varphi|_M$ *from* (2.21) *to* (2.20) (*resp., from* (2.22) *to* (2.20)) *is an algebra isomorphism.*

(compare with Vigneras [Vig04])

Proof. We concentrate on the first case. The second is similar or may be deduced by duality.

The map is clearly injective. To check surjectivity, note that the T_h^M for $h \in Z_M^-$ span (2.20). For $h \in Z_M^-$ define $E_h : G \to \mathrm{End}(\overline{V})$ in (2.21) with support $\mathcal{P}h\mathcal{P}$ and such that $E_h(h) = \mathrm{id}_{\overline{V}}$. It is well defined: if $p_1h = hp_2$ for $p_i \in \mathcal{P}$ then $p_1^0 = p_2^0 \in \mathcal{P}^0$ if we use Iwahori decomposition for $\mathcal{P} = \mathcal{P}^-\mathcal{P}^0\mathcal{P}^+$. So $p_1E_h(h) = E_h(h)p_2 \in \mathrm{End}(\overline{V})$. Clearly $E_h|_M = T_h^M$.

To check that it is a homomorphism, it suffices to show that $E_{h_1} * E_{h_2} = E_{h_1h_2}$ ($h_i \in Z_M^-$). We have $\mathcal{P}h_1\mathcal{P}h_2\mathcal{P} = \mathcal{P}(h_1\mathcal{P}^-\mathcal{P}^0)(\mathcal{P}^+h_2)\mathcal{P} = \mathcal{P}h_1h_2\mathcal{P}$ by Lemma 2.15.

$$(E_{h_1} * E_{h_2})(h_1h_2) = \sum_{\mathcal{P}/(\mathcal{P}\cap {}^{h_1}\mathcal{P})} E_{h_1}(ph_1)E_{h_2}(h_1^{-1}p^{-1}h_1h_2).$$

Since $\mathcal{P}\cap {}^{h_1}\mathcal{P} = ({}^{h_1}\mathcal{P}^-)\mathcal{P}^0\mathcal{P}^+$, we may replace the index set by $\mathcal{P}^-/{}^{h_1}\mathcal{P}^-$. To obtain a non-zero term we also need $(h_1^{-1}p^{-1}h_1)h_2 \in \mathcal{P}h_2\mathcal{P} = \mathcal{P}^-(\mathcal{P}^0h_2)\mathcal{P}$. As the product map $\overline{N} \times M \times N \to G$ is injective, $p \in {}^{h_1}\mathcal{P}^-$. Thus $(E_{h_1} * E_{h_2})(h_1h_2) = \mathrm{id}_{\overline{V}}$. $\square$

Suppose that V is an M-regular Serre weight. Consider the subspace:

$$\{\varphi : \operatorname{supp}(\varphi) \subset KZ_M^- K\} \subset \mathcal{H}_G(V). \tag{2.24}$$

The following lemma shows that it is a subalgebra and that we can identify it with $\mathcal{H}(\overline{V})$, where $\overline{V}$ denotes $V^{N(k)} \xrightarrow{\sim} V_{\overline{N}(k)}$. We therefore denote it by $\mathcal{H}(V)$.

Lemma 2.25. *Assume that V is M-regular and let $\overline{V}$ denote the representation $V^{N(k)} \xrightarrow{\sim} V_{\overline{N}(k)}$. The subspace* (2.24) *is a subalgebra. There is an algebra isomorphism $i^{\mathcal{P}}$ from* (2.24) *to* (2.21)*, which is characterised as follows. For any φ in* (2.24) *the map $i^{\mathcal{P}}(\varphi)$ is supported on $\mathcal{P}Z_M^-\mathcal{P}$ and for $g \in \mathcal{P}Z_M^-\mathcal{P}$, $i^{\mathcal{P}}(\varphi)(g) \in \operatorname{End}(V^{N(k)})$ is the restriction of $\varphi(g) \in \operatorname{End}(V)$ to $V^{N(k)}$.*

Similarly there is an algebra isomorphism $i_{\overline{\mathcal{P}}}$ from (2.24) *to* (2.22) *such that for any φ in* (2.24) *the map $i_{\overline{\mathcal{P}}}(\varphi)$ is supported on $\overline{\mathcal{P}}Z_M^-\overline{\mathcal{P}}$ and for $g \in \overline{\mathcal{P}}Z_M^-\overline{\mathcal{P}}$, $i_{\overline{\mathcal{P}}}(\varphi)(g) \in \operatorname{End}(V_{\overline{N}(k)})$ is induced by $\varphi(g) \in \operatorname{End}(V)$.*

These two identifications of (2.24) *with $\mathcal{H}(\overline{V})$ coincide. The identification of* (2.24) *with* (2.20) *is given by ${}'\mathcal{S}_G^M$.*

Proof. We verify that $\varphi(g)$ induces an endomorphism of $V_{\overline{N}(k)}$ for $g \in \mathcal{P}Z_M^-\mathcal{P}$ by checking it for $g = h \in Z_M^-$. Take $\overline{n} = \overline{N}(\mathcal{O})$ and note that $\varphi(h)\overline{n} = {}^h\overline{n}\,\varphi(h)$, where ${}^h\overline{n} \in \overline{N}(\mathcal{O})$ since h contracts $\overline{N}(\mathcal{O}) = \mathcal{P}^-$. Thus $i_{\overline{\mathcal{P}}}$ is well defined.

We now use Lemma 2.23 and its proof. Since $T_h(h)$ induces the identity map on $V_{\overline{N}(k)}$ (see the proof of Cor. 2.17), $i_{\overline{\mathcal{P}}}(T_h) = E_h$. So T_h in (2.24) is mapped to $E_h|_M = T_h^M$ in (2.20). By Cor. 2.17, this composite map is given by ${}'\mathcal{S}_G^M$, thus it is an injective algebra homomorphism. As the T_h^M span (2.20), it is an isomorphism.

The algebra isomorphism $i^{\mathcal{P}}$ is obtained by duality from $i_{\overline{\mathcal{P}}}$ (interchanging positive and negative roots and V with V^*).

The two identifications coincide since for $m \in M(\mathcal{O})Z_M^-$, the endomorphisms of $V^{N(k)}$ and $V_{\overline{N}(k)}$ induced by $\varphi(m)$ are identified via $V^{N(k)} \xrightarrow{\sim} V_{\overline{N}(k)}$. ☐

In the following proposition we will use natural maps $\text{c-Ind}_{\mathcal{P}}^G V^{N(k)} \to \text{c-Ind}_K^G V \to \text{c-Ind}_{\overline{\mathcal{P}}}^G V_{\overline{N}(k)}$. They are obtained by Frobenius reciprocity from $V^{N(k)} \to V \subset \text{c-Ind}_K^G V$, respectively $V \to \text{c-Ind}_{\overline{\mathcal{P}}}^K V_{\overline{N}(k)} \subset \text{c-Ind}_{\overline{\mathcal{P}}}^G V_{\overline{N}(k)}$. Alternatively, as Hecke operators they are supported on K and map the identity of K to the natural map $V^{N(k)} \to V$, respectively $V \to V_{\overline{N}(k)}$.

Proposition 2.26. *Assume that V is M-regular. We have the following diagram of Hecke operators.*

$$\begin{array}{ccccc}
\text{c-Ind}_{\mathcal{P}}^{G} V^{N(k)} & \twoheadrightarrow & \text{c-Ind}_{K}^{G} V & \hookrightarrow & \text{c-Ind}_{\mathcal{P}}^{G} V_{\overline{N}(k)} \\
\downarrow{\scriptstyle i^{\mathcal{P}}(\varphi)} & \swarrow & \downarrow{\scriptstyle \varphi} & \swarrow & \downarrow{\scriptstyle i_{\overline{\mathcal{P}}}(\varphi)} \\
\text{c-Ind}_{\mathcal{P}}^{G} V^{N(k)} & \twoheadrightarrow & \text{c-Ind}_{K}^{G} V & \hookrightarrow & \text{c-Ind}_{\mathcal{P}}^{G} V_{\overline{N}(k)}
\end{array}$$

For all $\varphi \in \mathcal{H} \subset \mathcal{H}_G(V)$ the two squares commutes. If moreover $\operatorname{supp}(\varphi) \subset KZ_M^{--}K$ *then there are diagonal arrows making the whole diagram commute.*

Proof. We first deal with the left half of the diagram. Without loss of generality, φ is supported on a single double coset KhK with $h \in Z_M^-$. By Frobenius reciprocity it suffices to check the two maps around the left square agree on $V^{N(k)} \subset \text{c-Ind}_{\mathcal{P}}^G V^{N(k)}$. A vector $v \in V^{N(k)}$ is mapped to $[1, v] \in \text{c-Ind}_K^G V$ which in turn is mapped to $\sum_{K\backslash KhK}[g^{-1}, \varphi(g)v] \in \text{c-Ind}_K^G V$ under φ. If $\varphi(g)v \neq 0$ then $g \in Kh\mathcal{P}$ by Cor. 2.17. We verify that the natural map $\mathcal{P}\backslash\mathcal{P}h\mathcal{P} \to K\backslash Kh\mathcal{P}$ is a bijection. It is enough to show that $\mathcal{P} \cap K^h = \mathcal{P} \cap \mathcal{P}^h$ or equivalently that ${}^h\mathcal{P} \cap K = {}^h\mathcal{P} \cap \mathcal{P}$. Now note that ${}^h\mathcal{P} \cap K = ({}^h\mathcal{P}^-)\mathcal{P}^0\mathcal{P}^+ \subset \mathcal{P}$ by Lemma 2.15.

Thus we see that

$$\sum_{K\backslash KhK} [g^{-1}, \varphi(g)v] = \sum_{\mathcal{P}\backslash\mathcal{P}h\mathcal{P}} [g^{-1}, i^{\mathcal{P}}(\varphi)(g)v], \tag{2.27}$$

which shows that the left square commutes.

Now suppose that $\operatorname{supp}(\varphi) = KhK$ for some $h \in Z_M^{--}$. In this case $\overline{P_h} = P$. We will define $j(\varphi) \in \mathcal{H}_{K,\mathcal{P}}(V, V^{N(k)})$ as follows (see §6.1 for the notation). We let $j(\varphi)$ agree with φ on $\mathcal{P}hK$ and we let it vanish outside. To see that it is well defined, note that $\varphi(h)$ maps V to $V^{\overline{N_h}(k)} = V^{N(k)}$ and that $\mathcal{P}$ preserves $V^{N(k)}$. We check that the top left triangle commutes on $V^{N(k)}$. For $v \in V^{N(k)}$ the arrow to the right maps it to $v \in V$ and $j(\varphi)$ in turn maps it to $\sum_{K\backslash Kh\mathcal{P}}[g^{-1}, \varphi(g)v]$. Then we are done as in (2.27). Since the map at the top is surjective, the bottom triangle also commutes.

We now dualise the left half of the diagram and obtain

$$\begin{array}{ccc}
\text{c-Ind}_{K}^{G} V^* & \longrightarrow & \text{c-Ind}_{\mathcal{P}}^{G} (V^*)_{N(k)} \\
\downarrow{\scriptstyle \varphi'} & \swarrow & \downarrow{\scriptstyle i^{\mathcal{P}}(\varphi)'} \\
\text{c-Ind}_{K}^{G} V^* & \longrightarrow & \text{c-Ind}_{\mathcal{P}}^{G} (V^*)_{N(k)}
\end{array}$$

Since the natural maps $V^{N(k)} \to V$ and $V^* \to (V^*)_{N(k)}$ are dual, the top and bottom maps are the natural ones. By construction of $i_{\mathcal{P}}$, $i^{\mathcal{P}}(\varphi)' = i_{\mathcal{P}}(\varphi')$. By replacing V by V^* and by interchanging positive and negative roots, we obtain the right half of the diagram. ∎

Corollary 2.28. *Suppose V is M-regular and let $\overline{V}$ denote the representation $V^{N(k)} \xrightarrow{\sim} V_{\overline{N}(k)}$. Suppose that $\chi : \mathcal{H}_M(\overline{V}) \to \bar{k}$ is an algebra homomorphism. Then the maps of Prop. 2.26 induce isomorphisms*

$$\text{c-Ind}_{\mathcal{P}}^G V^{N(k)} \otimes_{\mathcal{H},\chi} \bar{k} \xrightarrow{\sim} \text{c-Ind}_K^G V \otimes_{\mathcal{H},\chi} \bar{k} \xrightarrow{\sim} \text{c-Ind}_{\overline{\mathcal{P}}}^G V_{\overline{N}(k)} \otimes_{\mathcal{H},\chi} \bar{k},$$

where $\mathcal{H} = \mathcal{H}(V) \subset \mathcal{H}_M(\overline{V})$.

Proof. Pick any $h \in Z_M^{--}$. Since $T_h^M T_{h^{-1}}^M = 1$, $\chi(T_h^M) \neq 0$. Let $\mathfrak{m} = \ker(\chi)$, an ideal of $\mathcal{H}$.

The first map is clearly surjective. Let $\varphi = T_h \in \mathcal{H}$ so that $\chi(\varphi) \neq 0$. Let us denote the corresponding diagonal map on the left-hand side of the diagram in Prop. 2.26 by φ_δ. Then φ_δ is $\mathcal{H}$-linear because it is so after composing with the surjective, $\mathcal{H}$-linear map $\xi : \text{c-Ind}_{\mathcal{P}}^G V^{N(k)} \to \text{c-Ind}_K^G V$. If $f \in \text{c-Ind}_{\mathcal{P}}^G V^{N(k)}$ and $\xi(f) \otimes 1 = 0$ then $\xi(f) \in \mathfrak{m}(\text{c-Ind}_K^G V)$. Then

$$\varphi(f) = \varphi_\delta(\xi(f)) \in \varphi_\delta(\mathfrak{m}(\text{c-Ind}_K^G V)) = \mathfrak{m}\, \varphi_\delta(\text{c-Ind}_K^G V),$$

so $\varphi(f) \otimes 1 = 0$. Thus $f \otimes 1 = \chi(\varphi)^{-1}(\varphi(f) \otimes 1) = 0$, so the first map is injective.

Suppose $y \otimes 1 \in \text{c-Ind}_{\overline{\mathcal{P}}}^G V_{\overline{N}(k)} \otimes_{\mathcal{H},\chi} \bar{k}$. Then $y \otimes 1 = \chi(\varphi)^{-1}(i_{\overline{\mathcal{P}}}(\varphi) y \otimes 1)$ comes from $\text{c-Ind}_K^G V$, by the last part of Prop. 2.26. This proves that the second map is surjective.

Suppose $x \otimes 1 \in \text{c-Ind}_K^G V \otimes_{\mathcal{H},\chi} \bar{k}$ maps to zero. We again use Prop. 2.26. Then $\eta(x) \in \mathfrak{m}(\text{c-Ind}_{\overline{\mathcal{P}}}^G V_{\overline{N}(k)})$, so

$$\eta(\varphi x) = \varphi \eta(x) \in \mathfrak{m}(\varphi\, \text{c-Ind}_{\overline{\mathcal{P}}}^G V_{\overline{N}(k)}) \subset \mathfrak{m}\, \eta(\text{c-Ind}_K^G V).$$

As η is injective, $\varphi x \in \mathfrak{m}(\text{c-Ind}_K^G V)$. Finally $x \otimes 1 = \chi(\varphi)^{-1}(\varphi x \otimes 1) = 0$. This proves that the second map is injective. ☐

3. Parabolic inductions and compact inductions

The following theorem is inspired by work of Barthel–Livné for GL_2 (see [BL94, Thm. 25]). One aspect of the proof, namely the comparison of parahoric and parabolic inductions, crucially use ideas of Schneider–Stuhler [SS91] and Vignéras [Vig04]. See also the comment after Cor. 3.4.

Theorem 3.1. *Let $P = MN$ be a standard parabolic in G and suppose that V is an M-regular Serre weight for G. Then for any algebra homomorphism $\chi : \mathcal{H}_M(V_{\overline{N}(k)}) \to \bar{k}$, there is a natural isomorphism of smooth G-representations,*

$$\big(\text{c-Ind}_K^G V\big) \otimes_{\mathcal{H}_G(V),\chi} \bar{k} \xrightarrow{\sim} \mathrm{Ind}_P^G \Big\{ \big(\text{c-Ind}_{M(\mathcal{O})}^M V_{\overline{N}(k)}\big) \otimes_{\mathcal{H}_M(V_{\overline{N}(k)}),\chi} \bar{k} \Big\}.$$

Note that χ becomes a character of $\mathcal{H}_G(V)$ by composing with the partial Satake homomorphism ${}'\mathcal{S}_G^M$.

Proof.

⋄ We begin by defining maps

$$\text{c-Ind}_K^G V \xrightarrow{\eta} \text{c-Ind}_{\overline{\mathcal{P}}}^G(V_{\overline{N}(k)}) \xrightarrow{\zeta} \text{Ind}_{\overline{P}}^G(\text{c-Ind}_{M(\mathcal{O})}^M V_{\overline{N}(k)}). \tag{3.2}$$

The map η is the natural one that we already used in Prop. 2.26.

We define ζ by the following formula, for $f \in \text{c-Ind}_{\overline{\mathcal{P}}}^G V_{\overline{N}(k)}$:

$$\zeta(f)(g) = \sum_{\overline{P}(\mathcal{O})\backslash\overline{P}} \overline{p}^{-1}[1, f(\overline{p}g)].$$

By using that $\overline{P}(\mathcal{O}) = M(\mathcal{O})\overline{N}(\mathcal{O})$ we see that the term in the sum only depends on $\overline{p} \in \overline{P}(\mathcal{O})\backslash\overline{P}$. We check that the sum only involves finitely nonzero terms. Without loss of generality f is supported on a single coset $\overline{\mathcal{P}}g$. Since $\overline{P} \cap \overline{\mathcal{P}} = \overline{P}(\mathcal{O})$, we see that the sum involves at most one term. Clearly $\zeta(f)$ is $\overline{P}$-equivariant and ζ is G-equivariant. It follows that $\zeta(f)$ is smooth, so $\zeta(f) \in \text{Ind}_{\overline{P}}^G(\text{c-Ind}_{M(\mathcal{O})}^M V_{\overline{N}(k)})$. Therefore ζ is well defined.

⋄ Let $\mathcal{H} = \{f \in \mathcal{H}_G(V) : \text{supp}(f) \subset KZ_MK\}$ as in §2.5. Recall that $\mathcal{H}$ is a subalgebra of $\mathcal{H}_G(V)$, and thus of $\mathcal{H}_M(V_{\overline{N}(k)})$, and that we have an algebra homomorphism $i_{\overline{\mathcal{P}}} : \mathcal{H} \hookrightarrow \mathcal{H}_{\overline{\mathcal{P}}}(V_{\overline{N}(k)})$ (Lemma 2.25). Therefore $\mathcal{H}$ naturally acts on all terms of (3.2).

⋄ Check that η and ζ are $\mathcal{H}$-equivariant. We already checked this for η in Prop. 2.26.

We introduce some useful shorthand for functions in $\text{Ind}_{\overline{P}}^G \sigma$ for any smooth M-representation σ. For any open subset $\Omega \subset N$ and $x \in \sigma$ we write $[\Omega, x]$ for the function that is supported on $\overline{P}\Omega$ and sends all $\nu \in \Omega$ to x. Since $\overline{P}\Omega$ is open in G this is well defined. Note that for $m \in M$ and $n \in N$,

$$m[\Omega, x] = [{}^m\Omega, mx], \quad n[\Omega, x] = [n\Omega, x].$$

We claim that $\zeta([1, \overline{v}]) = [\overline{\mathcal{P}}^+, [1, \overline{v}]]$. Clearly the left-hand side is supported on $\overline{P}\overline{\mathcal{P}} = \overline{P}\overline{\mathcal{P}}^+$. It is then an easy computation to check that both sides agree on $\overline{\mathcal{P}}^+$.

To check that ζ is $\mathcal{H}$-equivariant, we can immediately reduce to the case when $\varphi = T_h$ for some $h \in Z_M^-$. Then ${}'\mathcal{S}_G^M(T_h) = T_h^M$ by Cor. 2.17. Suppose $\overline{v} \in V_{\overline{N}(k)} \subset \text{c-Ind}_{\overline{\mathcal{P}}}^G V_{\overline{N}(k)}$. On the one hand, $T_h^M([1, \overline{v}]) = [h^{-1}, \overline{v}]$ as $h \in Z_M^-$ and $T_h^M(h) = 1$. Thus $({}'\mathcal{S}_G^M(\varphi) \circ \zeta)(\overline{v}) = [\overline{\mathcal{P}}^+, [h^{-1}, \overline{v}]]$. On the other hand,

$$\begin{aligned}(\zeta \circ i_{\overline{\mathcal{P}}}(\varphi))(\overline{v}) &= \sum_{\overline{\mathcal{P}}\backslash\overline{\mathcal{P}}h\overline{\mathcal{P}}} g^{-1}[\overline{\mathcal{P}}^+, [1, \varphi(g)\overline{v}]] \\ &= \sum_{(\overline{\mathcal{P}}\cap\overline{\mathcal{P}}^h)\backslash\overline{\mathcal{P}}} \overline{p}^{-1}h^{-1}[\overline{\mathcal{P}}^+, [1, T_h(h)\overline{p}\overline{v}]].\end{aligned}$$

Since $(\overline{\mathcal{P}}^+)^h\backslash\overline{\mathcal{P}}^+ \to (\overline{\mathcal{P}}\cap\overline{\mathcal{P}}^h)\backslash\overline{\mathcal{P}}$ is a bijection by Lemma 2.15 and taking into account that $\overline{\mathcal{P}}^+$ fixes $\overline{v}$ and that $T_h(h)$ is trivial on $V_{\overline{N}(k)}$, the sum above simplifies to

$$\sum_{(\overline{\mathcal{P}}^+)^h\backslash\overline{\mathcal{P}}^+} [\overline{p}^{-1}(\overline{\mathcal{P}}^+)^h, [h^{-1},\overline{v}]] = [\overline{\mathcal{P}}^+, [h^{-1},\overline{v}]].$$

$\diamond$ Check that $\zeta\circ\eta$ is $\mathcal{H}_G(V)$-equivariant (via partial Satake). We let $\theta = (\zeta\circ\eta)|_V$. Note that by definition of ζ,

$$\theta(v)(1) = \zeta(\eta(v))(1) = [1,\eta(v)(1)] = [1, p_{\overline{N}}(v)].$$

Thus, in the notation of (2.12), $\overline{\theta}\in \mathrm{Hom}_{M(\mathcal{O})}(V_{\overline{N}(k)}, \text{c-Ind}_{M(\mathcal{O})}^M V_{\overline{N}(k)})$ is the natural map. By Lemma 2.13, $\overline{\theta*\varphi} = \overline{\theta}*\varphi_M$. As the map in (2.12) is injective we get $\theta*\varphi = \varphi_M\circ\theta$, as required.

$\diamond$ Check that $\eta\otimes_{\mathcal{H},\chi}\bar{k}$ is an isomorphism. This is the content of Cor. 2.28.

$\diamond$ Check that $\zeta\otimes_{\mathcal{H},\chi}\bar{k}$ is surjective.

Fix any non-zero $\overline{v}\in V_{\overline{N}(k)}$. We claim that $\zeta([1,\overline{v}])\otimes 1 = [\overline{\mathcal{P}}^+,[1,\overline{v}]]\otimes 1$ generates $\mathrm{Ind}_{\overline{P}}^G(\text{c-Ind}_{M(\mathcal{O})}^M V_{\overline{N}(k)})\otimes_{\mathcal{H},\chi}\bar{k}$. Pick $h_0\in Z_M^{--}$ and let $x_0 = [1,\overline{v}]\in \text{c-Ind}_{M(\mathcal{O})}^M V_{\overline{N}(k)}$. Since

$$[\overline{\mathcal{P}}^+, h_0^n x_0]\otimes 1 = [\overline{\mathcal{P}}^+, T^M_{h_0^{-n}}(x_0)]\otimes 1 = [\overline{\mathcal{P}}^+, x_0]\otimes\chi(T^M_{h_0^{-n}}),$$

it is enough to show that $[\overline{\mathcal{P}}^+, h_0^n x_0]$ ($n\in\mathbb{Z}$) generate $\mathrm{Ind}_{\overline{P}}^G(\text{c-Ind}_{M(\mathcal{O})}^M V_{\overline{N}(k)})$.

Note that $\sigma = \text{c-Ind}_{M(\mathcal{O})}^M V_{\overline{N}(k)}$ is generated by x_0 as M-representation. (This is the only property of σ that we will use.) We want to show that any $f\in\mathrm{Ind}_{\overline{P}}^G\sigma$ is contained in the G-representation generated by the $[\overline{\mathcal{P}}^+, h_0^n x_0]$, $n\in\mathbb{Z}$. By writing $\overline{P}\backslash G$ as a (finite) disjoint union of compact open subsets, each of which is contained in a G-translate of $\overline{P}\backslash\overline{P}N$, we can reduce to the case that $\mathrm{supp}(f)\subset\overline{P}N$. Since f is locally constant and compactly supported, we can moreover assume that $f = [\Omega, x]$, for some compact open subset $\Omega\subset N$ and $x\in\sigma$. We can write $x = \sum_i\lambda_i m_i x_0$ for some $\lambda_i\in\bar{k}$, $m_i\in M$, so

$$f = \sum_i \lambda_i m_i[\Omega^{m_i}, x_0].$$

Thus we may assume that $f = [\Omega, x_0]$, for some compact open $\Omega\subset N$. By Lemma 2.15 there is an $n\gg 0$ such that $\Omega = \coprod_j(\overline{\mathcal{P}}^+)^{h_0^n}\cdot\nu_j$. Therefore

$$f = \sum_j \nu_j^{-1}[(\overline{\mathcal{P}}^+)^{h_0^n}, x_0] = \sum_j \nu_j^{-1}h_0^{-n}[\overline{\mathcal{P}}^+, h_0^n x_0],$$

which completes the argument.

$\diamond$ Check that $\zeta\otimes_{\mathcal{H},\chi}\bar{k}$ is injective.

Let $f' \in \text{c-Ind}_{\overline{\mathcal{P}}}^{G} V_{\overline{N}(k)}$ such that $\zeta(f') \otimes 1 = 0$. Note that for $h \in Z_M^-$ we have $\text{supp}(i_{\overline{\mathcal{P}}}(T_h)f') \subset \overline{\mathcal{P}}h\,\text{supp}(f')$ since $i_{\overline{\mathcal{P}}}(T_h)$ is supported on $\overline{\mathcal{P}}h\overline{\mathcal{P}}$. Thus by Lemma 2.19 there is an $h \in Z_M^-$ such that $\text{supp}(i_{\overline{\mathcal{P}}}(T_h)f') \subset \overline{\mathcal{P}}T^-K$. Since $i_{\overline{\mathcal{P}}}(T_h)f' \otimes 1 = f' \otimes \chi(T_h)$ and $\chi(T_h) \neq 0$, we may assume that $\text{supp}(f') \subset \overline{\mathcal{P}}T^-K$.

Lemma 2.15 shows that for $t \in T^-$, ${}^t(\overline{\mathcal{P}}^-) \subset \overline{\mathcal{P}}^-$ and $(\overline{\mathcal{P}}^+)^t \subset \overline{\mathcal{P}}^+$. Thus for $h \in Z_M^-$ and $t \in T^-$, $\overline{\mathcal{P}}h\overline{\mathcal{P}}tK = \overline{\mathcal{P}}(h\overline{\mathcal{P}}^-\overline{\mathcal{P}}^0)(\overline{\mathcal{P}}^+t)K \subset \overline{\mathcal{P}}htK$. We can write $f' = \sum_{i=1}^r f'_i$ such that $\text{supp}(f'_i) = \overline{\mathcal{P}}t'_ik_i$ ($t'_i \in T$, $k_i \in K$) and that these are all disjoint. Since any element of Z_M can be written in the form $h'h^{-1}$ for h', $h \in Z_M^-$, we can find $h_i \in Z_M^-$ such that $h_it'_i = h_jt'_j$ whenever $Z_Mt'_i = Z_Mt'_j$. Let $f_i = \chi(T_{h_i})^{-1} i_{\overline{\mathcal{P}}}(T_{h_i})f'_i$, $f = \sum_i f_i$, and $t_i = h_it'_i \in T^-$. Then $f' \otimes 1 = f \otimes 1$, moreover $\text{supp}(f_i) = \overline{\mathcal{P}}t_ik_i$ and $t_i = t_j$ whenever $Z_Mt_i = Z_Mt_j$. By combining those f_i that have identical support, we may assume moreover that the $\overline{\mathcal{P}}t_ik_i$ are all disjoint.

We now show that f vanishes on $\overline{\overline{\mathcal{P}}P}$. Since we may apply the same argument to all K-translates of f, this will show that $f = 0$ and complete the argument. All we will use is that the image of $\zeta(f)(1)$ in $\text{c-Ind}_{M(\mathcal{O})}^{M} V_{\overline{N}(k)} \otimes_{\mathcal{H},\chi} \bar{k}$ is zero. Let us show that this latter space is naturally isomorphic to $\text{c-Ind}_{M(\mathcal{O})Z_M}^{M} V_{\overline{N}(k)}$. We let Z_M act on $V_{\overline{N}(k)}$ by declaring that $h\overline{v} = \chi(T_{h^{-1}}^M)\overline{v}$. This is compatible with the $M(\mathcal{O})$-action: for $h_0 \in Z_M(\mathcal{O})$, $T_{h_0^{-1}}^M = \omega(h_0)T_1^M$ by (2.8), where $\omega : Z_M(k) \to k^\times$ is the central character of $V_{\overline{N}(k)}$, so $\chi(T_{h_0^{-1}}^M) = \omega(h_0)$. The map

$$\text{c-Ind}_{M(\mathcal{O})}^{M} V_{\overline{N}(k)} \to \text{c-Ind}_{M(\mathcal{O})Z_M}^{M} V_{\overline{N}(k)}$$

is the obvious one induced by Frobenius reciprocity. It sends $[m, \overline{v}]$ to $[m, \overline{v}]_{Z_M}$, where the subscript is used to distinguish between the two induced representations. In particular it is surjective. For $h \in Z_M$ it sends $T_h^M[m, \overline{v}] = h^{-1}[m, \overline{v}]$ to $[h^{-1}m, \overline{v}]_{Z_M} = [m, h^{-1}\overline{v}]_{Z_M} = \chi(T_h^M)[m, \overline{v}]_{Z_M}$. The induced map

$$\text{c-Ind}_{M(\mathcal{O})}^{M} V_{\overline{N}(k)} \otimes_{\mathcal{H},\chi} \bar{k} \twoheadrightarrow \text{c-Ind}_{M(\mathcal{O})Z_M}^{M} V_{\overline{N}(k)} \tag{3.3}$$

is injective: we can lift any element on the left-hand side to one of the form $\sum [m_i, \overline{v}_i] \in \text{c-Ind}_{M(\mathcal{O})}^{M} V_{\overline{N}(k)}$, where the m_i lie in distinct $M(\mathcal{O})Z_M$-cosets. By considering its image on the right-hand side we see that all $\overline{v}_i$ are zero.

Since $\zeta(f)(1) \otimes 1$ depends only on $f|_{\overline{\overline{\mathcal{P}}P}}$, we will assume from now on that f is supported on $\overline{\overline{\mathcal{P}}P} = \overline{\mathcal{P}}^+\overline{P}$. Thus $k_i \in (\overline{\mathcal{P}}^+)^{t_i} \cdot \overline{P} \cap K = (\overline{\mathcal{P}}^+)^{t_i} \cdot \overline{P}(\mathcal{O})$ for all i. Thus we may assume that $k_i \in \overline{P}(\mathcal{O}) = \overline{\mathcal{P}}^-\overline{\mathcal{P}}^0$ without changing $\overline{\mathcal{P}}t_ik_i$. Since $t_i \in T^-$ shrinks $\overline{\mathcal{P}}^-$ we may even assume that $k_i \in \overline{\mathcal{P}}^0 = M(\mathcal{O})$. Let us write $f_i = [k_i^{-1}t_i^{-1}, \overline{v}_i]$. As $k_i^{-1}t_i^{-1} \in M$, we see that $\zeta(f_i)(1) \otimes 1 = [k_i^{-1}t_i^{-1}, \overline{v}_i]_{Z_M}$. By showing that $M(\mathcal{O})Z_Mt_ik_i$ are all disjoint we will deduce that $\overline{v}_i = 0$ for all i which will complete the argument.

Suppose $M(\mathcal{O})Z_M t_i k_i = M(\mathcal{O})Z_M t_j k_j$. By the Cartan decomposition for M, $Z_M t_i = Z_M t_j$, so by our assumption on the t_i above, $t_i = t_j$. Thus $M(\mathcal{O})t_i k_i = M(\mathcal{O})t_j k_j$ which implies $\overline{\mathcal{P}}t_i k_i = \overline{\mathcal{P}}t_j k_j$, so $i = j$ and we are done.

$\diamond$ Conclude: showed that $(\zeta \circ \eta) \otimes_{\mathcal{H},\chi} \bar{k}$ is an $\mathcal{H}_G(V)$-linear isomorphism (as $\mathcal{H}_G(V)$ commutative). By tensoring over $\mathcal{H}_G(V)$ with χ we see that $(\zeta \circ \eta) \otimes_{\mathcal{H}_G(V),\chi} \bar{k}$ is an isomorphism. The target of that isomorphism is isomorphic to $\mathrm{Ind}_P^G(\text{c-Ind}_{M(\mathcal{O})}^M V_{\overline{N}(k)}) \otimes_{\mathcal{H}_M(V_{\overline{N}(k)}),\chi} \bar{k}$, since ${}'\mathcal{S}_G^M : \mathcal{H}_G(V) \hookrightarrow \mathcal{H}_M(V_{\overline{N}(k)})$ is a localisation map. Finally, as $\mathcal{H}_M(V_{\overline{N}(k)})$ is noetherian and Ind_P^G is exact, $\mathrm{Ind}_P^G \sigma \otimes_{\mathcal{H}_M(V_{\overline{N}(k)}),\chi} \bar{k} \cong \mathrm{Ind}_P^G(\sigma \otimes_{\mathcal{H}_M(V_{\overline{N}(k)}),\chi} \bar{k})$ for any smooth M-representation σ. This completes the proof. $\square$

We record the following corollary to the proof.

Corollary 3.4. *Suppose that $\overline{V}$ is a Serre weight for M and that $\chi_M : Z_M \to \bar{k}^\times$ such that $\chi_M|_{Z_M(\mathcal{O})}$ is the central character of $\overline{V}$. Then there is an algebra homomorphism $\chi : \mathcal{H} \to \bar{k}$ such that $\chi(T_h^M) = \chi_M(h)^{-1}$ for all $h \in Z_M^-$ and we have*

$$\text{c-Ind}_{\mathcal{P}}^G \overline{V} \otimes_{\mathcal{H},\chi} \bar{k} \xrightarrow{\sim} \mathrm{Ind}_P^G(\text{c-Ind}_{M(\mathcal{O})Z_M}^M \overline{V}).$$

On the right-hand side we let $h \in Z_M$ act on $\overline{V}$ by $\chi_M(h)$.

Proof. This isomorphism was obtained in the above proof, in case $\overline{V}$ is of the form $V_{\overline{N}(k)}$, where V is M-regular, and χ is the restriction of an algebra homomorphism $\mathcal{H}_M(V_{\overline{N}(k)}) \to \bar{k}$. But it did not matter that $\overline{V}$ was of that form. (It is anyway, by Lemma 2.5.) Moreover, by Prop. 4.1 and Cor. 4.2 (or directly) the pair (M, χ_M) gives rise to $\chi : \mathcal{H}_M(\overline{V}) \to \bar{k}$ such that $\chi(T_h^M) = \chi_M(h)^{-1}$ for $h \in Z_M$. $\square$

If $G = \mathrm{GL}_n$ and $P = B$ is the Borel, this recovers the results of Schneider–Stuhler [SS91, Prop. 11] ($\chi = 1$) and Vignéras [Vig04, Thm. 4.10] (χ arbitrary). In those cases the right-hand side simplifies to the principal series $\mathrm{Ind}_B^G(\chi_T)$.

4. Hecke eigenvalues and supersingularity

The following proposition allows one to compare Hecke eigenvalues between different Serre weights. It is analogous to the classical result parameterising unramified Hecke eigenvalues by unramified characters of the torus [Car79, Cor. 4.2]. It will be convenient to define a *standard Levi* to be the unique Levi subgroup containing T of a standard parabolic subgroup.

Proposition 4.1. *There is a natural bijection between $\chi \in \mathrm{Hom}_{\bar{k}\text{-}alg}(\mathcal{H}_G(V), \bar{k})$ and pairs (M, χ_M), where M is a standard Levi and $\chi_M : Z_M \to \bar{k}^\times$ a character such that $\chi_M|_{Z_M(\mathcal{O})}$ is the central character of $V_{\overline{N}(k)}$.*

Given such a pair (M, χ_M), complete M to a standard parabolic $P = MN$. The corresponding set of eigenvalues is the composite of algebra homomorphisms

$$\chi : \mathcal{H}_G(V) \xrightarrow{{}'\mathcal{S}_G} \mathcal{H}_T^-(V_{\overline{U}(k)}) \xrightarrow{\chi'} \bar{k},$$

where $\chi'(\varphi) = \sum_{Z_M(\mathcal{O})\backslash Z_M} \varphi(z)\chi_M^{-1}(z)$.

We will say in the following that $\chi \in \mathrm{Hom}_{\bar{k}\text{-alg}}(\mathcal{H}_G(V), \bar{k})$ *is parameterised by the pair* (M, χ_M) if they correspond under the bijection in the proposition. Suppose now that (M, χ_M) consists of a standard Levi M and an arbitrary smooth character $\chi_M : Z_M \to \bar{k}^\times$. Then there may be more than one Serre weight V such that the central character of $V_{\overline{N}(k)}$ equals $\chi_M|_{Z_M(\mathcal{O})}$; for each one we obtain a corresponding algebra homomorphism $\mathcal{H}_G(V) \to \bar{k}$. In §9 we will see that if $G = \mathrm{GL}_n$ and π is an irreducible admissible G-representation, then all Hecke eigenvalues in all Serre weights of π are identified in this manner.

We will see below that M is in fact the smallest standard Levi such that χ factors through ${}'\mathcal{S}_G^M$ (as an algebra homomorphism). We have the following immediate consequence.

Corollary 4.2. *In the situation of Prop. 4.1, we have for* $\lambda \in X_*(T)_-$,

$$\chi'(\tau_\lambda) = \begin{cases} \chi_M(\lambda(\varpi))^{-1} & \text{if } \lambda(\varpi) \in Z_M, \\ 0 & \text{otherwise.} \end{cases}$$

Given $\chi \in \mathrm{Hom}_{\bar{k}\text{-alg}}(\mathcal{H}_G(V), \bar{k})$ we can consider its Satake transform $\chi' \in \mathrm{Hom}_{\bar{k}\text{-alg}}(\mathcal{H}_T^-(V_{\overline{U}(k)}), \bar{k})$. We will say that χ' vanishes on an open subset $X \subset T^-$ if χ' vanishes on all elements of $\mathcal{H}_T^-(V_{\overline{U}(k)})$ that are supported on X. Then χ' has a well-defined *support*, namely the complement in T^- of the biggest open subset on which it vanishes.

Lemma 4.3. *Suppose $P = MN$ is a standard parabolic. A $\bar{k}$-algebra homomorphism $\chi : \mathcal{H}_G(V) \to \bar{k}$ factors through ${}'\mathcal{S}_G^M$ if and only if* $\operatorname{supp} \chi' \supset Z_M^- T(\mathcal{O})$.

Proof. As we noted in the proof of Prop. 2.11, $\mathcal{H}_M(V_{\overline{N}(k)})$ is the localisation of $\mathcal{H}_G(V)$ at any non-zero element φ such that ${}'\mathcal{S}_G(\varphi)$ is supported on a single $T(\mathcal{O})$-coset in Z_M^{--}. If $\operatorname{supp} \chi' \supset Z_M^- T(\mathcal{O})$ then $\chi'({}'\mathcal{S}_G(\varphi)) \neq 0$ and so χ factors through $\mathcal{H}_M(V_{\overline{N}(k)})$.

Conversely, if χ factors through ${}'\mathcal{S}_G^M$, χ' extends to the subalgebra of $\mathcal{H}_T(V_{\overline{U}(k)})$ consisting of those elements whose support in T is antidominant for M. Since the subgroup $Z_M T(\mathcal{O}) \subset T$ consists of elements that are antidominant for M, $\operatorname{supp} \chi' \supset Z_M^- T(\mathcal{O})$. ☐

Proof of Proposition 4.1. By the proof of Cor. 1.5 in [Her], $\operatorname{supp} \chi'$ is of the form $Z_M^- T(\mathcal{O})$ for some standard parabolic $P = MN$. We will show that characters with $\operatorname{supp} \chi' = Z_M^- T(\mathcal{O})$ biject with pairs (M, χ_M) as in the proposition. By Lemma 4.3, χ factors through an algebra homomorphism $\widetilde{\chi} : \mathcal{H}_M(V_{\overline{N}(k)}) \to \bar{k}$. By replacing $(M, V_{\overline{N}(k)}, \widetilde{\chi})$ by (G, V, χ), we are reduced to the case $\operatorname{supp} \chi' = ZT(\mathcal{O})$.

Let us write $\mathcal{H}_T^-(V_{\overline{U}(k)}) = \mathcal{H} \oplus \mathcal{I}$, where $\mathcal{H}$ (resp., $\mathcal{I}$) consists of those elements φ whose support is contained in $ZT(\mathcal{O})$ (resp., disjoint from $ZT(\mathcal{O})$). Clearly $\mathcal{H}$ is a subalgebra and $\mathcal{I}$ is an ideal, so the χ' with support $ZT(\mathcal{O})$ biject with $\operatorname{Hom}_{\bar{k}\text{-alg}}(\mathcal{H}, \bar{k})$. By restricting functions to Z, $\mathcal{H}$ is isomorphic to

$$\{\varphi : Z \to \bar{k} : \varphi(z_0 z) = \omega(z_0)\varphi(z)\ \forall z_0 \in Z(\mathcal{O}), z \in Z;\ \operatorname{supp} \varphi \text{ cpt.}\}$$

(as algebra under convolution), where ω is the central character of V. The linear dual of this space consists of functions $f : Z \to \bar{k}$ such that $f(z_0 z) = \omega(z_0)^{-1} f(z)$ under the pairing $\langle \varphi, f \rangle = \sum_{Z(\mathcal{O})\backslash Z} \varphi(z) f(z)$. A simple argument shows that the linear map $\mathcal{H} \to \bar{k}$ induced by f is an algebra homomorphism if and only if $f : Z \to \bar{k}$ is a homomorphism. Finally we take the inverse of this homomorphism. □

Recall that $\mathcal{H}_G(V)$ is commutative (§2.2).

Definition 4.4. We say that an algebra homomorphism $\chi : \mathcal{H}_G(V) \to \bar{k}$ is *supersingular* if the corresponding pair is of the form (G, χ_G) for some character $\chi_G : Z \to \bar{k}^\times$. An irreducible admissible representation π is *supersingular* if for all Serre weights V and for all Hecke eigenvalues χ on $\operatorname{Hom}_K(V, \pi)$, χ is supersingular.

Equivalently, χ is supersingular if it does not factor through ${}'\mathcal{S}_G^M$ for any standard parabolic $P = MN \subsetneq G$. When $G = \mathrm{GL}_n$ we will see in Section 9 that π is supersingular if and only if it is supercuspidal. Moreover when $G = \mathrm{GL}_n$ the supersingularity condition only has to be checked for one Serre weight and for one χ.

Lemma 4.5. *Suppose $P = MN$ is a standard parabolic and that σ is an irreducible admissible M-representation. Then for V a Serre weight for G, the Hecke eigenvalues of V in $\operatorname{Ind}_{\overline{P}}^G \sigma$ and of $V_{\overline{N}(k)}$ in σ are parameterised by the same set of pairs (L, χ_L). In particular, $L \subset M$ in each case.*

Proof. By Lemma 2.13 the possible $\mathcal{H}_G(V)$-eigenvalues in $\operatorname{Hom}_K(V, \operatorname{Ind}_{\overline{P}}^G \sigma)$ are obtained from the possible $\mathcal{H}_M(V_{\overline{N}(k)})$-eigenvalues in $\operatorname{Hom}_{M(\mathcal{O})}(V_{\overline{N}(k)}, \sigma)$ by composing with ${}'\mathcal{S}_G^M$. By construction, the pair associated to $\chi : \mathcal{H}_M(V_{\overline{N}(k)}) \to \bar{k}$ is the same as the pair associated to $\chi \circ {}'\mathcal{S}_G^M$. □

Lemma 4.6. *Suppose π is a smooth G-representation. If $V \hookrightarrow \pi$ is an $\mathcal{H}_G(V)$-eigenvector with supersingular Hecke eigenvalues, these are parameterised by the pair (G, χ_π), where χ_π is the central character of π.*

Proof. Let ω_π be the central character of π. It is enough to show that $\chi_\pi(z) = \omega_\pi(z)$ for all $z = \lambda(\varpi)$, with $\lambda \in X_*(T)$ orthogonal to all roots. First, note that $\mathcal{S}_G(T_\lambda) = \tau_\lambda$ (this follows, for example, directly from the definition of $\mathcal{S}_G$). By Cor. 4.2, $\chi(T_\lambda) = \chi_G(z)^{-1}$. Finally, consider the induced G-linear map $\text{c-Ind}_K^G V \to \pi$. It follows from (2.9) that $\chi(T_\lambda) = \omega_\pi(z)^{-1}$. □

Lemma 4.7. *Suppose that π is a smooth G-representation and that $f : V \hookrightarrow \pi$ is an $\mathcal{H}_G(V)$-eigenvector. Suppose that $\eta : G \to \bar{k}^\times$ is a smooth character. If the Hecke eigenvalues of f are parameterised by (M, χ_M), then the Hecke eigenvalues of $f \otimes \eta$ are parameterised by $(M, \chi_M \eta|_{Z_M})$.*

Note that $\eta|_K$ factors through $G(k)$ (since $K(1)$ is pro-p).

Proof. Since $\text{c-Ind}_K^G(V \otimes \eta) \cong \text{c-Ind}_K^G V \otimes \eta$, we have a natural isomorphism $\mathcal{H}_G(V) \xrightarrow{\sim} \mathcal{H}_G(V \otimes \eta)$, $\varphi \mapsto \varphi^\eta$, under which the Hecke eigenvalues of f and $f \otimes \eta$ are identified. Note that $\varphi^\eta(g) = \eta(g)\varphi(g)$, which implies that $\mathcal{S}_G(\varphi^\eta)(t) = \eta(t)\mathcal{S}_G(\varphi)(t)$. (Note also that η is trivial on U, as it kills all pro-p subgroups.) The claim now follows from Cor. 4.2. □

5. Computing the Satake transform

In this section we determine the inverse of the Satake transform completely explicitly. The key tool is the Lusztig–Kato formula. The final result has a very simple shape: since $q = 0$ in $\bar{k}$, only the constant terms of the intervening Kazhdan–Lusztig polynomials matter.

If M is a standard Levi, we will denote by $\geq_M$ the usual partial order on $X_*(T)$ with respect to M, i.e., $\lambda \geq_M \mu$ means that $\lambda - \mu$ is a non-negative integral linear combination of the simple coroots of M. We also write $\geq$ for $\geq_G$.

Proposition 5.1. *Suppose that the derived subgroup of G is simply connected. Let V be a Serre weight. Let M be the standard Levi subgroup such that* $\text{Stab}_W(V^{U(k)}) = W_M$. *Then for all $\mu \in X_*(T)_-$,*

$$\tau_\mu = \sum_{\substack{\lambda \in X_*(T)_- \\ \lambda \geq_M \mu}} \mathcal{S}_G(T_\lambda).$$

The assumption on G may be unnecessary.

Proof. We can immediately reduce to the case when $M = G$ by Cor. 2.17(ii).

Next we will reduce to the case when V is the trivial Serre weight, by showing that the coefficients of $\mathcal{S}_G(T_\lambda)$ in the basis $(\tau_\mu)_\mu$ do not depend on V. We can write $V = F(\nu)$ for some q-restricted weight ν. We have $\nu \in X^0(T)$ since $\text{Stab}_W(\nu) = W$. The irreducible representation of $G_{/\mathcal{O}}$ of highest weight ν is one-dimensional (for example by the Weyl character formula [Jan03, Prop. II.5.10]). In particular there is a character $K = G(\mathcal{O}) \to \mathcal{O}^\times$ that agrees with ν on $T(\mathcal{O})$, that extends to a character $\widetilde{\nu} : G \to F^\times$, and whose reduction modulo ϖ is V. Let λ, $\mu \in X_*(T)_-$ and put $t := \lambda(\varpi)$, $t' := \mu(\varpi)$. For $g = k_1 t k_2 \in KtK$ we see that $T_\lambda(g) = k_1 k_2 \in \text{End}_{\bar{k}}(V) = \bar{k}$, i.e., $T_\lambda(g) = \overline{\widetilde{\nu}(g)\widetilde{\nu}(t)^{-1}} \in \bar{k}^\times$. Now we follow a standard argument (e.g., [Gro98, §3]). By the Bruhat decomposition we can write $KtK = \coprod_{i=1}^r t_i u_i K$ with $t_i = \lambda_i(\varpi)$, $\lambda_i \in X_*(T)$, $u_i \in U$. For each i, $t_i u_i K \cap t'U \neq \varnothing$ if and only if $\lambda_i = \mu$, in which case the intersection equals $t'u_i U(\mathcal{O})$. Thus the coefficient of τ_μ in $\mathcal{S}_G(T_\lambda)$ equals

$$\sum_{\lambda_i=\mu} T_\lambda(t'u_i) = \#\{i : \lambda_i = \mu\} \cdot \overline{\widetilde{\nu}(t')\widetilde{\nu}(t)^{-1}} = \#\{i : \lambda_i = \mu\},$$

where the last equality follows since $\mu \geq_{\mathbb{R}} \lambda$ if $KtK \cap t'U \neq \varnothing$ [Her, Lemma 3.6]. It is therefore indeed independent of ν.

We now prove the proposition in case V is trivial (so $M = G$). Recall the Lusztig–Kato formula [HKP, Thm. 7.8.1] (see also [Gro98, §4]). This is an identity in $\mathbb{Z}[q^{1/2}, q^{-1/2}][X_*(T)]$ which, when the variable q is specialised to a complex square root of $q = \#k$, gives the following in $\mathbb{C}[X_*(T)]$:

$$\operatorname{ch} V_\mu = \sum_{\lambda \leq \mu} q^{-\langle \mu, \rho \rangle} P_{w_\lambda, w_\mu}(q) 1^\vee_{K\lambda(\varpi)K}.$$

Here λ, μ denote any *dominant* coweights, V_μ is the irreducible complex representation of highest weight μ of the dual group, ρ is the half-sum of all positive roots, w_λ is the element μw_0 in the extended affine Weyl group $\widetilde{W} = X_*(T) \rtimes W$, and $1^\vee_{K\lambda(\varpi)K}$ denotes the classical (i.e., normalised) Satake transform of the characteristic function of $K\lambda(\varpi)K$. Besides, for any elements $w \leq w'$ in $\widetilde{W}$, $P_{w,w'}(q) \in 1 + q\mathbb{Z}[q]$ denotes the Kazhdan–Lusztig polynomial. We remark that $\lambda \leq \mu$ if and only if $w_\lambda \leq w_\mu$ in $\widetilde{W}$.

For any $\mu' \in X_*(T)$ take the coefficient of μ' in the above formula and rescale:

$$q^{\langle \mu - w_0\mu', \rho \rangle} \dim V_\mu(\mu') = \sum_{\lambda \leq \mu} P_{w_\lambda, w_\mu}(q) \sum_{U/U(\mathcal{O})} 1_{K\lambda(\varpi)K}(\mu'(\varpi)u).$$

Here $V_\mu(\mu')$ denotes the μ'-weight space in V_μ. Also note that $\delta_B^{1/2}(\mu'(\varpi)) = q^{\langle w_0\mu', \rho \rangle}$ for the modulus character of B.

Consider the left-hand side. We have $V_\mu(\mu') = 0$ unless $w_0\mu \leq \mu' \leq \mu$. If these inequalities hold, then $\langle \mu - w_0\mu', \rho \rangle \geq 0$ with equality if and only if $\mu = w_0\mu'$. Therefore we may reduce both sides modulo p and obtain

$$\tau_{w_0\mu} = \sum_{\lambda \leq \mu} \mathcal{S}_G(T_{w_0\lambda}),$$

noting again that the Kazhdan–Lusztig polynomials have constant coefficient 1. Finally we interchange λ with $w_0\lambda$ and μ with $w_0\mu$. □

6. Maps between compact inductions

6.1. **Generalities.** Suppose that we are given compact open subgroups H_i of G and finite-dimensional smooth H_i-representations V_i ($i = 1, 2$). We define

$$\mathcal{H}_{H_1,H_2}(V_1, V_2) := \operatorname{Hom}_G(\text{c-Ind}_{H_1}^G V_1, \text{c-Ind}_{H_2}^G V_2).$$

If moreover V_3 is a finite-dimensional smooth representation of a compact open subgroup H_3, we have a natural bilinear map given by composition

$$\mathcal{H}_{H_2,H_3}(V_2,V_3)\times\mathcal{H}_{H_1,H_2}(V_1,V_2)\to\mathcal{H}_{H_1,H_3}(V_1,V_3). \tag{6.1}$$

In particular $\mathcal{H}_{H_1,H_2}(V_1,V_2)$ is a Hecke bimodule, with $\mathcal{H}_{H_1}(V_1)$ acting on the right and $\mathcal{H}_{H_2}(V_2)$ acting on the left.

By Frobenius reciprocity, $\mathcal{H}_{H_1,H_2}(V_1,V_2)$ is isomorphic to $\mathrm{Hom}_{H_1}(V_1,\text{c-Ind}_{H_2}^G V_2)$ and thus, by thinking of it inside the space of functions on $G\times V_1\to V_2$, to

$$\{\varphi:G\to\mathrm{Hom}_{\bar{k}}(V_1,V_2):\operatorname{supp}\varphi\text{ compact},\\ \varphi(h_2gh_1)=h_2\varphi(g)h_1\ \forall h_i\in H_i, g\in G\}.$$

In this language the composition (6.1) is given convolution: $(\varphi'*\varphi)(g)=\sum_{G/H_2}\varphi'(gx)\varphi(x^{-1})$.

If $H_1=H_2=K$ and the V_i are Serre weights, we just write $\mathcal{H}_G(V_1,V_2)$ for $\mathcal{H}_{K,K}(V_1,V_2)$.

Proposition 6.2. *Suppose that V_1, V_2 are two Serre weights. Then $\mathcal{H}_G(V_1,V_2)$ is non-zero if and only if $V_1^{U(k)}\cong V_2^{U(k)}$ as $T(k)$-representations.*

If this is satisfied, the Hecke algebras $\mathcal{H}_G(V_1)$ and $\mathcal{H}_G(V_2)$ can naturally be identified via $\mathcal{S}_G$. Under this identification the actions of the Hecke two algebras on $\mathcal{H}_G(V_1,V_2)$ agree. If moreover the centre of G is connected and the derived subgroup of G is simply connected then $\mathcal{H}_G(V_1,V_2)$ is a free module of rank one under $\mathcal{H}_G(V_1)\cong\mathcal{H}_G(V_2)$.

This follows immediately from the following proposition and its proof.

Proposition 6.3. *Suppose that V_1, V_2 are two Serre weights. The Satake transform*

$$\mathcal{S}_G:\mathcal{H}_G(V_1,V_2)\to\mathcal{H}_T(V_1^{U(k)},V_2^{U(k)})\\ \varphi\mapsto\left(t\mapsto\sum_{u\in U/U(\mathcal{O})}\varphi(tu)\Big|_{V_1^{U(k)}}\right)$$

is injective. The transform is compatible with compositions:

$$\begin{array}{ccccc} \mathcal{H}_G(V_2,V_3) & \times & \mathcal{H}_G(V_1,V_2) & \longrightarrow & \mathcal{H}_G(V_1,V_3)\\ \downarrow{\scriptstyle \mathcal{S}_G} & & \downarrow{\scriptstyle \mathcal{S}_G} & & \downarrow{\scriptstyle \mathcal{S}_G}\\ \mathcal{H}_T(V_2^{U(k)},V_3^{U(k)}) & \times & \mathcal{H}_T(V_1^{U(k)},V_2^{U(k)}) & \longrightarrow & \mathcal{H}_T(V_1^{U(k)},V_3^{U(k)}) \end{array} \tag{6.4}$$

If $V_1^{U(k)}\cong V_2^{U(k)}$ as $T(k)$-representation, G has connected centre, and the derived subgroup of G is simply connected, the image of $\mathcal{S}_G$ is a free module of rank one under the compatible actions of $\mathcal{H}_T^-(V_1^{U(k)})\cong\mathcal{H}_T^-(V_2^{U(k)})$.

Proof. That the transform is well defined and compatible with compositions is formal using the Iwasawa decomposition and follows exactly the same steps as in the case when $V_1 = V_2$ (see [Her]).

Suppose $\lambda \in X_*(T)_-$. By Step 1 of the proof of [Her, Thm. 1.2], we see that the vector space of $\varphi \in \mathcal{H}_G(V_1, V_2)$ that are supported on $K\lambda(\varpi)K$ is one-dimensional if $V_1^{N_{-\lambda}(k)} \cong V_2^{N_{-\lambda}(k)}$ as $M_\lambda(k)$-representations and zero otherwise. Since $N_{-\lambda}(k) \subset U(k)$ it follows that $\mathcal{H}_G(V_1, V_2) \neq 0$ implies $V_1^{U(k)} \cong V_2^{U(k)}$ as $T(k)$-representation. To see the converse, suppose $V_1^{U(k)} \cong V_2^{U(k)}$ and choose any $\lambda \in X_*(T)_-$ such that $\langle \lambda, \alpha \rangle < 0$ for all $\alpha \in \Delta$. Then $N_{-\lambda} = U$, so that there exists a non-zero Hecke operator supported on $K\lambda(\varpi)K$.

We pick $\varphi \neq 0$ in $\mathcal{H}_G(V_1, V_2)$ and show that $\mathcal{S}_G(\varphi)$ is non-zero. Suppose first that φ is supported on $K\lambda(\varpi)K$ for some $\lambda \in X_*(T)_-$. For $\mu \in X_*(T)$ the same argument as in Step 3 of [Her, Thm. 1.2] shows that $\mathcal{S}_G(\varphi)(\mu(\varpi)) \neq 0$ implies $\mu \geq_{\mathbb{R}} \lambda$ and that $\mathcal{S}_G(\varphi)(\lambda(\varpi)) \neq 0$. For general φ, picking a minimal $\lambda \in X_*(T)_-$ for $\geq_{\mathbb{R}}$ such that φ is non-zero on $K\lambda(\varpi)K$, we see that $\mathcal{S}_G(\varphi)(\lambda(\varpi)) \neq 0$. Thus $\mathcal{S}_G$ is injective. (This gives an alternative proof that $\mathcal{H}_G(V_1, V_2) = 0$ unless $V_1^{U(k)} \cong V_2^{U(k)}$.)

Let us now assume that $V_1^{U(k)} \cong V_2^{U(k)}$ and that G has connected centre and simply connected derived subgroup. We first show that there is a $\lambda_0 \in X_*(T)_-$ such that there is a non-zero $\varphi \in \mathcal{H}_G(V_1, V_2)$ supported on $K\lambda(\varpi)K$ if and only if $\lambda - \lambda_0 \in X_*(T)_-$. We can write $V_i \cong F(\nu_i)$ with ν_i q-restricted. As $V_1^{U(k)} \cong V_2^{U(k)}$, we have $\nu_1 - \nu_2 \in (q-1)X(T)$. There is a non-zero $\varphi \in \mathcal{H}_G(V_1, V_2)$ supported on $K\lambda(\varpi)K$ if and only if $V_1^{N_{-\lambda}(k)} \cong V_2^{N_{-\lambda}(k)}$ if and only if $\langle \nu_1 - \nu_2, \alpha \rangle = 0$ for all roots α of M_λ, i.e., for all $\alpha \in \Phi$ such that $\langle \lambda, \alpha \rangle = 0$ (the last step uses [Her, Lemma 2.5] and Prop. 1.3 in the appendix of [Her09]). Since the centre of G is connected we may choose $\lambda_0 \in X_*(T)_-$ such that for all simple roots α, $\langle \lambda_0, \alpha \rangle = 0$ if $\langle \nu_1 - \nu_2, \alpha \rangle = 0$ and $\langle \lambda_0, \alpha \rangle = -1$ otherwise. This clearly satisfies the desired property.

We can canonically identify $\mathcal{H}_T(V_1^{U(k)})$, $\mathcal{H}_T(V_2^{U(k)})$ and we denote them simply by $\mathcal{H}_T$. By Thm. 2.6 and (6.4) we see that $\mathcal{H}_T^-$ preserves the image of $\mathcal{S}_G$. Choose $\varphi_0 \in \mathcal{H}_G(V_1, V_2)$ that is non-zero and supported on $K\lambda_0(\varpi)K$. We show below that $\mathcal{H}_T^- \mathcal{S}_G(\varphi_0) = \mathrm{im}(\mathcal{S}_G)$. Since $\mathcal{H}_T \cong \bar{k}[X_*(T)]$ is an integral domain, this completes the proof.

Claim: For all $\lambda \in \lambda_0 + X_*(T)_-$ there is a $\psi_\lambda \in \mathcal{H}_T^- \mathcal{S}_G(\varphi_0)$ such that for all $\mu \in \lambda_0 + X_*(T)_-$, $\psi_\lambda(\mu(\varpi)) \neq 0$ if and only if $\mu = \lambda$.

We prove the claim by induction with respect to $\geq_{\mathbb{R}}$, noting that $\{\mu \in X_*(T)_- : \mu \geq_{\mathbb{R}} \lambda\}$ is finite. Consider $\tau_{\lambda - \lambda_0} \mathcal{S}_G(\varphi_0) \in \mathcal{H}_T^- \mathcal{S}_G(\varphi_0)$. By the above this is zero on $\mu(\varpi)$ unless $\mu \geq_{\mathbb{R}} \lambda$ and non-zero on $\lambda(\varpi)$. If there is no $\mu >_{\mathbb{R}} \lambda$ in $\lambda_0 + X_*(T)_-$ we are done. Otherwise by induction we can subtract off multiples of ψ_μ for such μ to find a ψ_λ.

Finally given $\psi \in \mathrm{im}(\mathcal{S}_G)$, by the claim we can subtract off multiples of the $\psi_\lambda \in \mathcal{H}_T^- \mathcal{S}_G(\varphi_0)$ to assume without loss of generality that $\psi(\lambda(\varpi)) = 0$

for all $\lambda \in \lambda_0 + X_*(T)_-$. Then the above argument for the injectivity of $\mathcal{S}_G$ shows that $\psi = 0$. ☐

Remark 6.5. Considering $G = SL_3$ shows that the freeness result can break down if the centre of G is disconnected.

Corollary 6.6. *Suppose V is a Serre weight. Then* $\text{c-Ind}_K^G V$ *is a torsion-free* $\mathcal{H}_G(V)$*-module.*

Proof. If $\text{c-Ind}_K^G V$ has torsion, there is a non-zero $\varphi : \text{c-Ind}_K^G V \to \text{c-Ind}_K^G V$ in $\mathcal{H}_G(V)$ that has non-zero kernel. As a non-zero smooth G-representation, the kernel has to contain a Serre weight V'. By Frobenius reciprocity we get a non-zero map $\text{c-Ind}_K^G V' \to \text{c-Ind}_K^G V$ whose composite with φ is zero. More generally suppose that a composite of non-zero maps $\varphi_1 : \text{c-Ind}_K^G V_1 \to \text{c-Ind}_K^G V_2$ and $\varphi_2 : \text{c-Ind}_K^G V_2 \to \text{c-Ind}_K^G V_3$ is zero. By Prop.6.2, the $T(k)$-representations $V_1^{U(k)}$, $V_2^{U(k)}$, and $V_3^{U(k)}$ are isomorphic, and we identify them (non-canonically). Then for all i, j the bimodules $\mathcal{H}_T(V_i^{U(k)}, V_j^{U(k)})$ are all naturally identified with the integral domain $\mathcal{H}_T(V_1^{U(k)})$ such that moreover for any triple (i,j,k) the bimodule multiplication corresponds to the ring multiplication. Since $\mathcal{S}_G(\varphi_2) * \mathcal{S}_G(\varphi_1) = 0$ and $\mathcal{S}_G$ is injective, one of φ_1, φ_2 has to be zero. ☐

6.2. **The minuscule case.** Let V, V' denote distinct Serre weights such that $V^{U(k)} \cong (V')^{U(k)}$. As we just saw, in this case we can identify $\mathcal{H}_G(V)$ and $\mathcal{H}_G(V')$, and we will denote them simply by $\mathcal{H}_G$. The goal of this subsection is to find an explicit criterion when

$$\text{c-Ind}_K^G V \otimes_{\mathcal{H}_G,\chi} \bar{k} \cong \text{c-Ind}_K^G V' \otimes_{\mathcal{H}_G,\chi} \bar{k}, \tag{6.7}$$

in case V and V' differ only minimally (in the sense that $V^{N(k)} \cong (V')^{N(k)}$ as $M(k)$-representations for some maximal parabolic $P = MN$) and provided there exists a corresponding minuscule fundamental coweight. We remark that when $G = \mathrm{GL}_n$, any simple root admits a minuscule fundamental coweight.

An isomorphism as in (6.7) is extremely useful because it allows us to "change the weight" in a smooth G-representation π. The point is that $\mathrm{Hom}_G(\text{c-Ind}_K^G V \otimes_{\mathcal{H}_G,\chi} \bar{k}, \pi) \neq 0$ is equivalent to saying that V occurs in π with Hecke eigenvalues χ. The results in this subsection will play a key role in the proofs of Theorems 8.1 and 9.10.

Suppose that G has simply connected derived subgroup. This implies simple coroots possess fundamental weights. Fix a simple root α. We denote by $\omega_{\alpha^\vee}$ a fundamental weight for $\alpha^\vee$. We make the following assumption.

There exists a minuscule fundamental coweight $-\lambda$ associated to α.

Recall that this means that $\langle \beta^\vee, \omega_{\alpha^\vee} \rangle = \delta_{\alpha\beta}$ for all simple roots β, respectively that $\langle -\lambda, \gamma \rangle \in \{0, 1\}$ for all $\gamma \in \Phi^+$.

Suppose that ν is a q-restricted weight satisfying $\langle \nu, \alpha^\vee \rangle = 0$. We let $\nu' := \nu + (q-1)\omega_\alpha$ (again q-restricted) and define Serre weights $V := F(\nu)$,

$V' := F(\nu')$. Note that $V^{U(k)} \cong (V')^{U(k)}$ as $T(k)$-representations. There exist Hecke operators $\varphi_\lambda^+ \in \mathcal{H}_G(V, V')$ and $\varphi_\lambda^- \in \mathcal{H}_G(V', V)$ whose support is $K\lambda(\varpi)K$. (Note that λ is precisely the λ_0 in the proof of Prop. 6.3.)

Proposition 6.8. *With the above notation, $\mathcal{S}_G(\varphi_\lambda^- * \varphi_\lambda^+) \in \mathcal{H}_T^-$ has support $\lambda(\varpi)^2 T(\mathcal{O}) \cup \lambda(\varpi)^2\alpha^\vee(\varpi)T(\mathcal{O})$. The values at $\lambda(\varpi)^2$ and $\lambda(\varpi)^2\alpha^\vee(\varpi)$ add to zero.*

Proof. We first compute $\varphi_\lambda^- * \varphi_\lambda^+ \in \mathcal{H}_G(V)$. Let $t = \lambda(\varpi)$. We show that $K = \bigcup_W U(\mathcal{O})\dot{w}t(K \cap {}^tK)$, where $\dot{w} \in K$ denotes a representative of $w \in W$.

Sublemma 6.9. *As $-\lambda$ is minuscule, we have $K \cap {}^tK \supset \ker(G(\mathcal{O}) \to G(k))$.*

Thus by Prop. 2.14, $K \cap {}^tK$ is the preimage of $P_\lambda(k)$ under the reduction map $G(\mathcal{O}) \to G(k)$. It follows that

$$K/(K \cap {}^tK) \cong G(k)/P_\lambda(k) \cong \coprod_{W/W_\lambda} U(k)\dot{w}P_\lambda(k)/P_\lambda(k),$$

using the rational Bruhat decomposition in the last step and where W_λ denotes $W_{M_\lambda} = \mathrm{Stab}_W(\lambda)$. Therefore $K = \bigcup_W U(\mathcal{O})\dot{w}t(K \cap {}^tK)$.

We claim that the support of $\varphi_\lambda^- * \varphi_\lambda^+$ is Kt^2K. Let $\lambda' \in X_*(T)_-$ and let $t' = \lambda'(\varpi)$. When we compute $(\varphi_\lambda^- * \varphi_\lambda^+)(t')$, by the above all terms are of the form $\varphi_\lambda^-(u\dot{w}t)\varphi_\lambda^+(t^{-1}\dot{w}^{-1}u^{-1}t')$, where $u \in U(\mathcal{O})$ and $w \in W$. As $t' \in T^-$, $(t')^{-1}u^{-1}t' \in U(\mathcal{O})$, so if the term is non-zero then so is $\varphi_\lambda^-({}^wt)\varphi_\lambda^+({}^wt^{-1}t')$ (where we also pulled a $\dot{w}$ from the left to the right). By the refined Cartan decomposition we need that $-w\lambda + \lambda' = w'\lambda$ for some $w' \in W$. Moreover $\varphi_\lambda^-({}^wt)\varphi_\lambda^+({}^{w'}t) \neq 0$, which implies that $\varphi_\lambda^-(t)\dot{w}^{-1}\dot{w}'\varphi_\lambda^+(t) \neq 0$. This in turn implies that $p_{N_\lambda}(\dot{w}^{-1}\dot{w}'(V')^{N_{-\lambda}(k)}) \neq 0$. The proof of Lemma 2.16 then shows that $w^{-1}w' \in W_\lambda$. (Note that $\mathrm{Stab}_W(\nu') \subset W_\lambda$ since the stabiliser is generated by simple reflections and since $\langle \nu', \alpha^\vee \rangle = q - 1 > 0$.) Thus $w'\lambda = w\lambda$. As $-w\lambda + \lambda' = w'\lambda$, we obtain $\lambda' = 2w\lambda$. Taking into account that λ' and λ are antidominant coweights, we see that $w\lambda = \lambda$ and that $\lambda' = 2\lambda$. The latter equation shows that the support of $\varphi_\lambda^- * \varphi_\lambda^+$ is contained in Kt^2K. But if we take $\lambda' = 2\lambda$ the former equation shows that $u\dot{w}t = t(t^{-1}ut)\dot{w} \in tK$, so that only the trivial term $\varphi_\lambda^-(t)\varphi_\lambda^+(t)$ contributes to $(\varphi_\lambda^- * \varphi_\lambda^+)(t^2)$, but that term is clearly non-zero.

To complete the proof, we will show that $\mathcal{S}_G(T_{2\lambda}) = \tau_{2\lambda} - \tau_{2\lambda+\alpha^\vee}$. Let M be the standard Levi with $W_M = \mathrm{Stab}_W(\nu)$. Note that $\alpha \in \Delta_M$. By Prop. 5.1, it suffices to show the following:

Claim: Suppose $\mu \in X_*(T)_-$. Then $\mu \geq_M 2\lambda$ if and only if $\mu = 2\lambda$ or $\mu \geq_M 2\lambda + \alpha^\vee$.

By passing to the Levi, it suffices to prove the claim when $M = G$. Suppose $\mu \geq 2\lambda$, so $\mu - 2\lambda = \sum_i \beta_i^\vee$, where the β_i are simple roots. Let α_0 be the sum of the longest roots of all irreducible components of the root system (it need not itself be a root). Then $\alpha_0 = \sum_{\beta\in\Delta} n_\beta \beta$ with $n_\beta \geq 1$. Moreover α_0 is dominant and $\langle \lambda, \alpha_0 \rangle = -1$ (as $-\lambda$ is minuscule). As μ is antidominant, we find

$$\langle \mu, \alpha \rangle \geq \sum_{\beta \in \Delta} n_\beta \langle \mu, \beta \rangle = \langle \mu, \alpha_0 \rangle \geq \langle 2\lambda, \alpha_0 \rangle = -2.$$

If $\langle \mu, \alpha \rangle > -2$ then $\sum_i \langle \beta_i^\vee, \alpha \rangle > 0$, so α has to occur among the simple roots β_i and thus $\mu \geq 2\lambda + \alpha^\vee$. If $\langle \mu, \alpha \rangle = -2$, then $\langle \mu, \beta \rangle = 0$ for all $\beta \in \Delta - \{\alpha\}$. Then $\mu - 2\lambda = 0$ since it is orthogonal to all roots and contained in $\mathbb{Z}\Phi^\vee$. □

Question 6.10. Suppose that the centre of G is connected so that every simple root α possesses a fundamental coweight λ. Is the result of Prop. 6.8 true when λ is not assumed to be minuscule? It seems that this is the case when $G = \mathrm{GSp}_4$.

Corollary 6.11. *We keep the above notation. Suppose* $\chi \in \mathrm{Hom}_{\bar{k}\text{-}alg}(\mathcal{H}_G, \bar{k})$ *is parameterised by* (M, χ_M)*. Assume that* $\alpha \notin \Delta_M$*. Then*

$$\text{c-Ind}_K^G V \otimes_{\mathcal{H}_G, \chi} \bar{k} \cong \text{c-Ind}_K^G V' \otimes_{\mathcal{H}_G, \chi} \bar{k}, \tag{6.12}$$

provided either $\alpha^\vee(\varpi) \notin Z_M$ *or* $\chi_M(\alpha^\vee(\varpi)) \neq 1$.

Proof. By the above, φ_λ^+, φ_λ^- induce G-linear maps between the two representations in (6.12). If we can show that $\chi(\varphi_\lambda^- * \varphi_\lambda^+) = \chi(\varphi_\lambda^+ * \varphi_\lambda^-)$ is non-zero, we will be done. Let χ' be the Satake transform of χ. By Prop. 6.8 it suffices to show that $\chi'(\tau_{2\lambda}) \neq \chi'(\tau_{2\lambda + \alpha^\vee})$.

Since $\langle \lambda, \beta \rangle = 0$ for all simple roots $\beta \in \Delta_M$ (as $\alpha \notin \Delta_M$), we have $\lambda(\varpi)^2 \in Z_M^-$; moreover $\lambda(\varpi)^2 \alpha^\vee(\varpi) \in Z_M^-$ if and only if $\alpha^\vee(\varpi) \in Z_M$. We now use Cor. 4.2. If $\alpha^\vee(\varpi) \notin Z_M$ then $\chi'(\tau_{2\lambda}) \neq 0 = \chi'(\tau_{2\lambda+\alpha^\vee})$. Otherwise, $\chi'(\tau_{2\lambda}) \chi'(\tau_{2\lambda+\alpha^\vee})^{-1} = \chi_M(\alpha^\vee(\varpi)) \neq 1$. □

Remark 6.13. Equation (6.12) also holds if $\alpha \in \Delta_M$ and $\langle \alpha^\vee, \beta \rangle = 0$ for all $\beta \in \Delta_M - \{\alpha\}$.

◇ Example for GL_2, remark on Kisin's map

7. Generalised Steinberg representations

In this section we will determine the Jordan–Hölder factors of $\mathrm{Ind}_{\overline{B}}^G 1$ (completing the work of Grosse-Klönne) as well as their Serre weights and Hecke eigenvalues. As usual, $P = MN$ and Q denote standard parabolic subgroups of G throughout.

Recall that the *generalised Steinberg representations* are defined as follows:

$$\mathrm{Sp}_P = \frac{\mathrm{Ind}_{\overline{P}}^G 1}{\sum_{Q \supsetneq P} \mathrm{Ind}_{\overline{Q}}^G 1}.$$

Theorem 7.1. *For any standard parabolic subgroup* P*, the generalised Steinberg representation* Sp_P *is irreducible and admissible.*

For the Steinberg representation Sp_B this was proved by Vignéras [Vig08, §4]. The general case was proved by Grosse-Klönne [GK, Cor. 4.3], under the assumption that G is of type A, B, C, or D. We recall Cor. 4.4 of [GK], which now holds without any restriction on the root system:

Corollary 7.2. *The generalised Steinberg representations* Sp_P *are pairwise non-isomorphic. They form the irreducible constituents of* $\mathrm{Ind}_{\overline{B}}^{G} 1$*, each occurring with multiplicity one. In particular,* $\mathrm{Ind}_{\overline{B}}^{G} 1$ *is of finite length* $2^{\#\Delta}$*.*

We deduce Thm. 7.1 from the work of Grosse-Klönne with the help of Thm. 3.1. More precisely we show that Sp_P contains a unique Serre weight, that it lifts to a Serre weight of $\mathrm{Ind}_{\overline{P}}^{G} 1$, and finally that the lifted Serre weight generates $\mathrm{Ind}_{\overline{P}}^{G} 1$.

Proposition 7.3. *For any standard parabolic subgroup P, there is a unique Serre weight V_P occurring in* Sp_P*. It occurs with multiplicity one. The Hecke eigenvalues occurring in Serre weight V_P are parameterised by $(T, 1)$.*

We will see in the proof that V_P is the unique M-regular Serre weight such that $(V_P)_{\overline{N}(k)}$ is trivial.

Proof of Theorem 7.1 and Proposition 7.3. Let $\overline{\mathrm{Sp}}_P$ denote the generalised Steinberg representation of $G(k)$, i.e.,

$$\overline{\mathrm{Sp}}_P = \frac{\mathrm{Ind}_{\overline{P}(k)}^{G(k)} 1}{\sum_{Q \supsetneq P} \mathrm{Ind}_{\overline{Q}(k)}^{G(k)} 1}.$$

Then by [GK, §3], there is a natural K-linear embedding $\overline{\mathrm{Sp}}_P \hookrightarrow \mathrm{Sp}_P$ such that

$$(\overline{\mathrm{Sp}}_P)^{B(k)} = (\mathrm{Sp}_P)^I = (\mathrm{Sp}_P)^{I(1)}. \tag{7.4}$$

Since $I(1)$ is a pro-p group, we see in particular that Sp_P is admissible.

Step 1: We show that $\mathrm{soc}_K(\mathrm{Sp}_P)$ is irreducible. Let $V \subset \mathrm{Sp}_P$ be any Serre weight. By (7.4), $V^{U(k)} \subset (\overline{\mathrm{Sp}}_P)^{B(k)}$, so $V^{U(k)} = V^{B(k)}$ is a one-dimensional subspace of $(\overline{\mathrm{Sp}}_P)^{B(k)}$, which is stable under the action of the Hecke algebra $\bar{k}[B(k)\backslash G(k)/B(k)]$. In [GK, Prop. 3.4] it is shown that any non-zero $\bar{k}[B(k)\backslash G(k)/B(k)]$-submodule of $(\overline{\mathrm{Sp}}_P)^{B(k)}$ contains the element g_P, which is the image of $1_{\overline{P}(k)B(k)} \in \mathrm{Ind}_{\overline{P}(k)}^{G(k)} 1$ in $\overline{\mathrm{Sp}}_P$. This shows that V is generated by g_P as $G(k)$-module.

Step 2: By Lemma 2.5, there is a unique M-regular Serre weight V_P such that $(V_P)_{\overline{N}(k)}$ is trivial. Let $\chi : \mathcal{H}_M(1) \to \bar{k}$ denote the Hecke eigenvalues of the trivial Serre weight in the trivial M-representation. It is parameterised by the pair $(T, 1)$ by Lemma 4.6. From Thm. 3.1 we get a surjective G-linear map $\text{c-Ind}_K^G V_P \otimes_{\mathcal{H}_G(V_P),\chi} \bar{k} \twoheadrightarrow \mathrm{Ind}_{\overline{P}}^{G} 1$. Thus the image of V_P generates $\mathrm{Ind}_{\overline{P}}^{G} 1$ as G-representation; a fortiori, its image in Sp_P is non-zero and generates. By Step 1, V_P is the unique Serre weight of Sp_P.

Step 3: Suppose that $\pi \subset \mathrm{Sp}_P$ is a non-zero subrepresentation. Then π contains a Serre weight. By the previous two steps, we know that this is V_P and that V_P generates Sp_P. Thus $\pi = \mathrm{Sp}_P$.

The final statement of Prop. 7.3 follows from Lemma 4.5. □

8. Irreducibility of parabolic inductions

The goal of this section is to construct many irreducible admissible G-representations by parabolically inducing an irreducible admissible representation of a Levi subgroup. The most precise results are obtained for GL_n.

8.1. The case of GL_n.

Theorem 8.1. *Suppose that $G = \mathrm{GL}_n$. Let P be the standard parabolic with Levi $\prod_i \mathrm{GL}_{n_i}$ (where $\sum_i n_i = n$). Suppose that σ_i is an irreducible admissible supersingular representation of $\mathrm{GL}_{n_i}(F)$ ($i = 1, \dots, r$). Then $\mathrm{Ind}_{\overline{P}}^{G}(\sigma_1 \otimes \cdots \otimes \sigma_r)$ is irreducible (and admissible) if and only if there is no i such that $n_i = n_{i+1} = 1$ and $\sigma_i \cong \sigma_{i+1}$.*

When $n_1 = \cdots = n_r = 1$, this was proved by Ollivier [Oll06]. Her method relied on a detailed knowledge of the $I(1)$-Hecke algebra and the study of intertwinings of certain induced $I(1)$-Hecke modules. Henniart (unpublished) found a different proof in that case, which utilises the structure of a principal series as B-representation and which works, under some conditions, for quasi-split groups. By contrast, our strategy is to show that every non-zero subrepresentation of an induced representation satisfying the criterion in Thm. 8.1 has to contain a sufficiently regular Serre weight (by "changing the weight" as in §6.2), and that such a Serre weight has to generate the whole representation (by using the surjectivity of the map in Thm. 3.1).

Lemma 8.2. *Suppose that G is a product of split reductive groups G_i ($i = 1, \dots, r$). The irreducible admissible G-representations are the $\sigma = \bigotimes_i \sigma_i$, where each σ_i runs through the irreducible admissible G_i-representations.*

The Serre weights of σ are of the form $V = \bigotimes_i V_i$ where each V_i runs through the Serre weights of σ_i. The Hecke eigenvalues of V in σ are of the form $(\prod_i M_i, \prod \chi_{M_i})$, where each (M_i, χ_{M_i}) runs through the Hecke eigenvalues of V_i in σ_i.

Proof. The first part follows just as in the case of finite groups (using a proof that does not use character theory). Similarly, the Serre weights for G are of the form $\bigotimes V_i$ where each V_i is a Serre weight for G_i. Next note that $\bigotimes \mathrm{Hom}_{G_i(\mathcal{O})}(V_i, \sigma_i) \cong \mathrm{Hom}_{G(\mathcal{O})}(V, \sigma)$. Finally we have a natural isomorphism $\bigotimes \mathcal{H}_{G_i}(V_i) \cong \mathcal{H}_G(V)$ which is induced by the map $\bigotimes \varphi_i \mapsto \varphi$ with $\varphi(g_1, \dots, g_r) = \bigotimes \varphi_i(g_i)$. (To see that it is a bijection, note that it sends $\bigotimes T_{\lambda_i}$ to $T_{\prod \lambda_i}$.) One also verifies that this is compatible with the Satake isomorphisms (for G and the G_i). The lemma follows easily from Prop. 4.1. □

Proof of Theorem 8.1. Let $M_i = \mathrm{GL}_{n_i}$. We have $P = MN$ with $M = \prod_i \mathrm{GL}_{n_i}$. We let $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_r$, an irreducible admissible representation of M. Note first that the criterion is clearly necessary since otherwise σ extends to a representation of a larger parabolic subgroup.

To show that the criterion is sufficient, assume that $\pi \subset \mathrm{Ind}_{\overline{P}}^G \sigma$ is a non-zero subrepresentation. We verify first that $\mathrm{Ind}_{\overline{P}}^G \sigma$ is admissible by showing that $(\mathrm{Ind}_{\overline{P}}^G \sigma)^{K(1)}$ is finite-dimensional. This follows from the admissibility of σ since

$$(\mathrm{Ind}_{\overline{P}}^G \sigma)^{K(1)} = (\mathrm{Ind}_{\overline{P}(\mathcal{O})}^K \sigma)^{K(1)} = \mathrm{Ind}_{\overline{P}(k)}^{G(k)}(\sigma^{M(1)}),$$

where $M(1)$ is the kernel of $M(\mathcal{O}) \to M(k)$.

Pick a Serre weight $V \subset \pi$. As $\mathrm{Ind}_{\overline{P}}^G \sigma$ is admissible, $\mathrm{Hom}_K(V, \pi)$ is finite-dimensional. Since $\mathcal{H}_G(V)$ is moreover commutative we can pick an $\mathcal{H}_G(V)$-eigenvector $f \in \mathrm{Hom}_K(V, \pi)$. Lemma 2.13 gives us the following natural maps of $\mathcal{H}_G(V)$-modules

$$\bar{k} f \hookrightarrow \mathrm{Hom}_K(V, \pi) \hookrightarrow \mathrm{Hom}_K(V, \mathrm{Ind}_{\overline{P}}^G \sigma) \xrightarrow{\sim} \mathrm{Hom}_{M(\mathcal{O})}(V_{\overline{N}(k)}, \sigma).$$

Note that $\mathcal{H}_G(V)$ acts on the final term via $'\mathcal{S}_G^M$. Any vector space automorphism of the right-hand side that preserves the line $\bar{k} f$ obviously acts invertibly on $\bar{k} f$. We apply this observation, together with the fact that $'\mathcal{S}_G^M : \mathcal{H}_G(V) \to \mathcal{H}_M(V_{\overline{N}(k)})$ is a localisation map (Prop. 2.11), to deduce that $\mathcal{H}_G(V) \to \bar{k}$, the system of eigenvalues of f, is of the form $\chi \circ {'\mathcal{S}_G^M}$ for some $\chi : \mathcal{H}_M(V_{\overline{N}(k)}) \to \bar{k}$. The map f therefore gives rise to a surjection

$$\text{c-Ind}_{M(\mathcal{O})}^M V_{\overline{N}(k)} \otimes_{\mathcal{H}_M(V_{\overline{N}(k)}), \chi} \bar{k} \twoheadrightarrow \sigma.$$

Case 1: V is M-regular. Then Thm. 3.1 applies. Since $\mathrm{Ind}_{\overline{P}}^G$ is an exact functor, we obtain

$$\text{c-Ind}_K^G V \otimes_{\mathcal{H}_G(V), \chi} \bar{k} \twoheadrightarrow \mathrm{Ind}_{\overline{P}}^G \sigma.$$

Since this map is naturally induced by $V \xrightarrow{f} \pi \to \mathrm{Ind}_{\overline{P}}^G \sigma$, it follows that $\mathrm{Ind}_{\overline{P}}^G \sigma$ is generated by $f(V)$ as G-representation. In other words, $\pi = \mathrm{Ind}_{\overline{P}}^G \sigma$.

Case 2: V is not M-regular. We show by induction that π has to contain an M-regular Serre weight (in fact the one corresponding to $V^{N(k)}$ by Lemma 2.5). Then Case 1 implies that $\pi = \mathrm{Ind}_{\overline{P}}^G \sigma$ and concludes the proof that $\mathrm{Ind}_{\overline{P}}^G \sigma$ is irreducible.

Since the derived subgroup of G is simply connected, we can write $V \cong F(\nu)$. As $\mathrm{Stab}_W(\nu) \not\subset W_M$ and since the left-hand side is generated by simple reflections, there is a simple root $\alpha \in \Delta - \Delta_M$ such that $s_\alpha(\nu) = \nu$. Since the centre of G is connected, there is a fundamental coweight $-\lambda$ associated to α, which is minuscule as $G = \mathrm{GL}_n$. Just as in §6.2 we now put $\nu' = \nu + (q-1)\omega_\alpha$ and $V' = F(\nu')$. The map f above gives rise to a non-zero map $\text{c-Ind}_K^G V \otimes_{\mathcal{H}_G, \chi} \bar{k} \to \pi$ (we also write χ for $\chi \circ {'\mathcal{S}_G^M}$). If we can

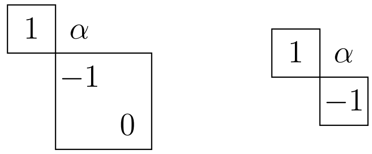

FIGURE 8.1. Final part of the proof of Thm. 8.1: α and $\alpha^\vee$.

show that Cor. 6.11 applies, there is a non-zero map $\text{c-Ind}_K^G V' \otimes_{\mathcal{H}_{G,\chi}} \bar{k} \to \pi$, in particular V' is a Serre weight of π. Since $\text{Stab}_W(\nu')$ is strictly smaller than $\text{Stab}_W(\nu)$, by induction we eventually find that π has to contain a M-regular Serre weight.

Finally we show that the criterion in Thm. 8.1 implies that the assumptions in Cor. 6.11 are satisfied. By Lemmas 8.2 and 4.6, the Hecke eigenvalues χ are parameterised by $(M, \prod \chi_i)$, where χ_i is the central character of σ_i. Certainly $\alpha \notin \Delta_M$. Suppose first that there exists $\beta \in \Delta_M$ that is adjacent to α in the Dynkin diagram. In this case $\langle \alpha^\vee, \beta \rangle \neq 0$, so $\alpha^\vee(\varpi) \notin Z_M$ and we are done. See the left part of Fig. 8.1. Otherwise, we have $n_i = n_{i+1} = 1$ for the two Levi blocks (M_i, σ_i) and (M_{i+1}, σ_{i+1}) that α "sits between." (We work with the diagonal torus and the upper triangular Borel.) See the right part of Fig. 8.1. In particular $\sigma_i = \chi_i$, $\sigma_{i+1} = \chi_{i+1}$. Since $\chi_M(\alpha^\vee(\varpi)) = \chi_i \chi_{i+1}^{-1} \neq 1$ by assumption, we are done. $\square$

Theorem 8.3. *Suppose that $G = \text{GL}_n$. Let P be the standard parabolic with Levi $\prod_i \text{GL}_{n_i}$ (where $\sum_i n_i = n$). Suppose that σ_i is an irreducible admissible supersingular representation of $\text{GL}_{n_i}(F)$ ($i = 1, \dots, r$). Then $\text{Ind}_P^G(\sigma_1 \otimes \cdots \otimes \sigma_r)$ has finite length.*

By the transitivity of parabolic induction, we can rewrite such a representation in the form $\text{Ind}_Q^G(\tau_1 \otimes \cdots \otimes \tau_s)$ with Q the standard parabolic with Levi $\prod_i \text{GL}_{m_i}$, such that for all i either

- τ_i is supersingular and $m_i > 1$, or
- $\tau_i \cong \text{Ind}_{B_i}^{\text{GL}_{m_i}} \eta_i$ for some $\eta_i : F^\times \to \bar{k}^\times$, where B_i is the Borel in GL_{m_i}.

(Here and in the following, we often write η when we really mean $\eta \circ \det$.) We can moreover demand without loss of generality that $\eta_i \neq \eta_{i+1}$ whenever both τ_i and τ_{i+1} fall in the second case.

By Cor. 7.2, the irreducible constituents of $\text{Ind}_{B_i}^{\text{GL}_{m_i}} \eta_i$ are of the form $\text{Sp}_{Q_i} \otimes \eta_i$, where Q_i runs through the standard parabolics of GL_{m_i}.

We have thus reduced the proof of Thm. 8.3 to the following generalisation of Thm. 8.1. It shows that $\text{Ind}_Q^G(\tau_1 \otimes \cdots \otimes \tau_s)$ has length $2^{\sum_i d_i}$, where $d_i = 0$ or $m_i - 1$ depending on whether τ_i falls in the first or in the second case.

Theorem 8.4. *Suppose that $G = \text{GL}_n$. Let P be the standard parabolic with Levi $\prod_i \text{GL}_{n_i}$ (where $\sum_i n_i = n$). Suppose that σ_i is an irreducible* *admissible representation of* $\mathrm{GL}_{n_i}(F)$ $(i = 1, \dots, r)$ *such that for all* i *either*

- σ_i *is supersingular and* $n_i > 1$*, or*
- $\sigma_i \cong \mathrm{Sp}_{Q_i} \otimes \eta_i$ *for some* η_i *and some standard parabolic* $Q_i \subset \mathrm{GL}_{n_i}$.

Assume that $\eta_i \neq \eta_{i+1}$ *whenever both* σ_i *and* σ_{i+1} *fall in the second case. Then* $\mathrm{Ind}_P^G(\sigma_1 \otimes \cdots \otimes \sigma_r)$ *is irreducible and admissible.*

Proof. The proof differs from the one of Thm. 8.1 only in the final part. This time the Hecke eigenvalues are parameterised by a pair $(M' = \prod M_i', \prod \chi_i)$, where $M_i' = M_i$ and χ_i is the central character of σ_i if σ_i falls in the first case, whereas M_i' is the torus of M_i and $\chi_i = \eta_i \circ \det$ if σ_i falls in the second case. It follows that $\alpha^\vee(\varpi) \notin Z_{M'}$ unless the blocks adjacent to α, (M_i, σ_i) and (M_{i+1}, σ_{i+1}), both fall in the second case and the argument goes through since $\eta_i \eta_{i+1}^{-1} \neq 1$. □

Theorem 8.5. *Suppose that the final condition on the* η_i *is dropped in Thm. 8.4. Then* $\mathrm{Ind}_P^G(\sigma_1 \otimes \cdots \otimes \sigma_r)$ *is of finite length with explicit Jordan–Hölder factors. All Jordan–Hölder factors occur with multiplicity one.*

Proof. By the transitivity of parabolic inductions, by twisting, and by Thm. 8.4, we reduce to the following statement.

The representation $\mathrm{Ind}_P^G(\mathrm{Sp}_{Q_1} \otimes \mathrm{Sp}_{Q_2})$ *has length* 2 *and its Jordan–Hölder factors consist of two distinct* (*and explicit*) *generalised Steinberg representations.*

The disjoint union of the constituents as Q_1, Q_2 run over the standard parabolics of the Levi blocks equals $\mathrm{JH}(\mathrm{Ind}_P^G(\mathrm{Ind}\, 1 \otimes \mathrm{Ind}\, 1)) = \mathrm{JH}(\mathrm{Ind}_B^G 1)$. Thus the constituents of $\mathrm{Ind}_P^G(\mathrm{Sp}_{Q_1} \otimes \mathrm{Sp}_{Q_2})$ are generalised Steinberg representations and occur with multiplicity one. By counting standard parabolics, we see that it suffices to exhibit two distinct constituents for each (Q_1, Q_2). By (2.12) the representation contains two distinct Serre weights and so we are done by Prop. 7.3. □

8.2. **General results.** We show that parabolic inductions of irreducible admissible representations are usually irreducible, even for general G. As we do not have the results of §6.2 available in general, we need to put stronger hypotheses on the Serre weights that are allowed to occur. On the other hand, the representation of the Levi does not have to be supersingular.

Theorem 8.6. *Let* $P = MN$ *be a standard parabolic and suppose that* σ *is an irreducible admissible* M*-representation satisfying:*

(REG) *For all simple roots* $\alpha \in \Delta - \Delta_M$*, the restriction to* $k^\times$ *via the cocharacter* $\alpha^\vee$ *of the* $T(k)$*-representation* $(\mathrm{soc}_{M(\mathcal{O})}\, \sigma)_{(\overline{U} \cap M)(k)}$ *does not contain the trivial representation.*

Then $\mathrm{Ind}_P^G \sigma$ *is irreducible and admissible.*

Proof. This follows by the proof of Thm. 8.1, since we now show that (REG) implies that all Serre weights V of $\mathrm{Ind}_P^G \sigma$ are M-regular. First assume that the derived subgroup of G is simply connected. In this case, $V \cong F(\nu)$ for some q-restricted weight ν and $V_{\overline{N}(k)} \cong F^M(\nu)$. Assumption (REG) shows that for all simple roots $\alpha \in \Delta - \Delta_M$, $\langle \nu, \alpha^\vee \rangle \not\equiv 0 \pmod{q-1}$ (as $V_{\overline{N}(k)}$ is a direct summand of $\mathrm{soc}_{M(\mathcal{O})}\,\sigma$ and $(V_{\overline{N}(k)})_{(\overline{U}\cap M)(k)}$ is isomorphic to $\nu|_{T(k)}$). In particular, $\langle \nu, \alpha^\vee \rangle > 0$ for such α. Since $\mathrm{Stab}_W(\nu)$ is generated by simple reflections, we see that $\mathrm{Stab}_W(\nu) \subset W_M$, so V is indeed M-regular.

For general G, we reduce to the previous case using a z-extension $\widetilde{G} \twoheadrightarrow G$, just as in [Her, Lemma 2.5]. For a Serre weight V of $\mathrm{Ind}_P^G\,\sigma$, we have that the restriction of $V_{\overline{U}(k)}$ to $k^\times$ via $\alpha^\vee$ is non-trivial for $\alpha \in \Delta - \Delta_M$ by (REG). Since coroots are compatible in z-extensions, we get the analogous statement for V as $\widetilde{G}(k)$-representation. By the previous paragraph, V is $\widetilde{M}$-regular and therefore M-regular. □

Remark 8.7. The condition (REG) is best possible in the sense that it is equivalent to the condition that all Serre weights of $\mathrm{Ind}_P^G\,\sigma$ are M-regular (at least when the derived subgroup of G is simply connected).

9. Classification results

9.1. **The case of GL_n.** In this subsection, $G = \mathrm{GL}_n$. We first establish an important preliminary result.

Proposition 9.1. *Suppose π is a smooth G-representation such that π contains the trivial Serre weight with Hecke eigenvalues parameterised by $(T, 1)$. Then either π contains the trivial G-representation or there exists a proper standard parabolic P such that $\mathrm{Ord}_P\,\pi \neq 0$.*

In fact we will see in Cor. 9.13 below that if π is moreover irreducible and admissible, then π is the trivial G-representation.

Remark 9.2. The first statement just means that π^K contains a non-zero element v such that for all $1 \le i \le n$ we have $T_i v = [Kt_iK]v = v$, where $t_i = \mathrm{diag}(\varpi, \dots, \varpi, 1, \dots, 1)$ (i copies of ϖ).

Notation. We use the more classical notation $[KgK]$ for a Hecke operator in the unramified Hecke algebra. The action on π^K is the natural left action: write $KgK = \coprod g_\alpha K$, then $[KgK]v := \sum g_\alpha v \in \pi^K$. Note that $[KgK] = 1_{Kg^{-1}K} \in \mathcal{H}_G(1)$. [This ties in with the fact that $\mathcal{H}_G(1)$ naturally acts on π^K on the *right*.]

Similarly we use the double coset notation for other, not necessarily commutative Hecke algebras.

Proof. Pick $v \in \pi^K$, a Hecke eigenvector with Hecke eigenvalues parameterised by $(T, 1)$. This just means that $T_i v = v$, $1 \le i \le n$. [This follows from Cor. 4.2, since $\mathcal{S}_G(T_\lambda) = \tau_\lambda$ for minuscule λ; see the displayed formula after Thm. 2.6 for that. I.e., "$\mathcal{S}_G(T_i) = \tau_i$."]

For the first step I'll leave $n = 3$, because it is equally clear in general. We consider the two parahorics $\mathcal{P}$, $\mathcal{Q}$ corresponding to the standard parabolics

P, Q of type $(2,1)$ and $(1,2)$. [So $\mathcal{P} = \mathrm{red}^{-1}(P(k))$, etc.] We also have the Iwahori I contained in both of them. Note that we are using the upper triangular Borel.

Step 1: Suppose that the Hecke operator $[\mathcal{P}\begin{pmatrix}\varpi & & \\ & \varpi & \\ & & 1\end{pmatrix}\mathcal{P}]$ has a non-zero eigenvalue on $\pi^{\mathcal{P}}$. Then Prop. 9.17 applies with $\overline{V} = 1$ and we see that $\mathrm{Ord}_P\,\pi \neq 0$. [Note that $[\mathcal{P}h^{-1}\mathcal{P}] = E_h \in \mathcal{H}_{\mathcal{P}}(1)$ for any $h \in Z_M^-$ in the notation of Lemma 2.23. And this is naturally identified with T_h^M (see Lemma 2.25). In particular, $\chi_M : Z_M \to \bar{k}^\times$ in Prop. 9.17 is the unramified character sending $\begin{pmatrix}\varpi & & \\ & \varpi & \\ & & 1\end{pmatrix}$ to the non-zero eigenvalue of $[\mathcal{P}\begin{pmatrix}\varpi & & \\ & \varpi & \\ & & 1\end{pmatrix}\mathcal{P}]$ and $\begin{pmatrix}\varpi & & \\ & \varpi & \\ & & \varpi\end{pmatrix}$ to the non-zero eigenvalue of $[\mathcal{P}\begin{pmatrix}\varpi & & \\ & \varpi & \\ & & \varpi\end{pmatrix}\mathcal{P}]$ (central character of π).]

So without loss of generality, we may assume that $[\mathcal{P}\begin{pmatrix}\varpi & & \\ & \varpi & \\ & & 1\end{pmatrix}\mathcal{P}]$ is nilpotent on $\pi^{\mathcal{P}}$. But $[\mathcal{P}\begin{pmatrix}\varpi & & \\ & \varpi & \\ & & 1\end{pmatrix}\mathcal{P}] = [I\begin{pmatrix}\varpi & & \\ & \varpi & \\ & & 1\end{pmatrix}I]$ on $\pi^{\mathcal{P}} \subset \pi^I$, so $U_2 := [I\begin{pmatrix}\varpi & & \\ & \varpi & \\ & & 1\end{pmatrix}I]$ is nilpotent on v.

Similarly, by considering $\mathcal{Q}$ we may assume without loss of generality that $U_1 := [I\begin{pmatrix}\varpi & & \\ & 1 & \\ & & 1\end{pmatrix}I]$ is nilpotent on v.

For general n, we have reduced to the case when $U_i = [It_iI]$ is nilpotent for all $1 \le i \le n-1$.

Step 2: Setup and some lemmas.

Let us translate what $T_i v = v$ means. $T_n v = v$ means that $\begin{pmatrix}\varpi & & \\ & \ddots & \\ & & \varpi\end{pmatrix}$ acts trivially on v and therefore on π (central character).

We use a standard presentation of the Iwahori Hecke algebra: $S_1 = \left[I\begin{pmatrix} & 1 & & \\ 1 & & & \\ & & \ddots & \\ & & & 1\end{pmatrix}I\right], \dots, S_{n-1} = \left[I\begin{pmatrix}1 & & & \\ & \ddots & & \\ & & & 1 \\ & & 1 & \end{pmatrix}I\right], \Pi = \left[I\begin{pmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ \varpi & & & \end{pmatrix}I\right]$ [i.e., the S_i are defined by the permutation matrices corresponding to *simple* reflections]. We have the following relations:

$$\boxed{S_i^2 = -S_i} \quad \text{for all } i,$$
$$\boxed{S_iS_j = S_jS_i} \quad \text{whenever } |i-j| > 1,$$
$$\boxed{S_kS_{k+1}S_k = S_{k+1}S_kS_{k+1}} \quad \text{for all } k < n-1,$$
$$\boxed{S_k\Pi = \Pi S_{k+1}} \quad \text{for all } k < n-1.$$

The quadratic relations take a simple form, since we work in characteristic p. We also have $\boxed{\Pi^n = 1}$ on π^I (central character). [Note that Π is just a single left or right coset, as the matrix defining it normalises I.]

Recall that

$$T_1 v = \sum \begin{pmatrix}\varpi & a_2 & a_3 & \cdots \\ & 1 & & \\ & & 1 & \\ & & & \ddots\end{pmatrix} v + \sum \begin{pmatrix}1 & & & \\ & \varpi & a_3 & \cdots \\ & & 1 & \\ & & & \ddots\end{pmatrix} v + \cdots,$$

where in each sum, the a_i run over representatives of $\mathcal{O}$ modulo ϖ.

It is not hard to express this in terms of the Iwahori Hecke action [for example, using the same strategy as in the proof of Lemma 9.6]. Since $T_1 v = v$, we get:

$$v = \sum_{i=1}^{n} S_{i(i+1)\cdots(n-1)} \Pi v, \tag{9.3}$$

where we abbreviate $S_{i(i+1)\cdots(n-1)} := S_i S_{i+1} \cdots S_{n-1}$. [We will also use such abbreviations for ascending sequences of consecutive integers in the following.]

Lemma 9.4. *Suppose $i \le k \le \ell \le j$. Then*

$$S_{i(i+1)\cdots j} S_{k(k+1)\cdots(\ell-1)} = S_{(k+1)(k+2)\cdots\ell} S_{i(i+1)\cdots j}$$

Proof. It suffices to consider the case $\ell - 1 = k$. In that case, $i \le k \le j-1$. By the braid relations,

$$\begin{aligned} S_{i(i+1)\cdots j} S_k &= S_{i(i+1)\cdots(k-1)k(k+1)} S_k S_{(k+2)\cdots j} \\ &= S_{i(i+1)\cdots(k-1)} S_{(k+1)} S_{k(k+1)(k+2)\cdots j} \\ &= S_{(k+1)} S_{i(i+1)\cdots j}. \end{aligned}$$

□

Lemma 9.5. *We have $S_{n-1}\Pi^2 v = 0$.*

Proof. We first show that for all i we have $S_i v = 0$. This is immediate from the fact that $v \in \pi^K$ and that the double coset defining S_i is a disjoint union of $q = \#k$ one-sided cosets.

Then we get $S_{n-1}\Pi^2 v = \Pi^2 S_1 v = 0$. [E.g., since $\Pi^2 = \Pi^{-n+2}$.] □

Lemma 9.6. *For all i, we have $U_i = (S_{i\cdots(n-1)}\Pi)^i$.*

Proof. If we multiply out the double cosets on the right-hand side, we certainly find the matrix $\mathrm{diag}(\varpi, \dots, \varpi, 1, \dots, 1)$ among the products (i copies of ϖ). [We just multiply the defining matrices of all these double cosets.] To see that the product of the double cosets yields only the double coset of this diagonal matrix, it's enough to count the number of one-sided cosets. On the left-hand side the number of cosets is $q^{i(n-i)}$. Since each S_i contains q cosets and Π contains just one, the result follows. □

Step 3: We show that $U_1 v = S_{12\cdots(n-1)}\Pi v = 0$.

Lemma 9.6 tells us that $U_1 = S_{12\cdots(n-1)}\Pi$. We now see that

$$(S_{12\cdots(n-1)}\Pi)^2 v = S_{12\cdots(n-1)}\Pi \left(v - \sum_{i\ge 2} S_{i\cdots(n-1)}\Pi v \right)$$

by (9.3),

$$= S_{12\cdots(n-1)}\Pi v - \sum_{i\geq 2} S_{12\cdots(n-1)}S_{(i-1)\cdots(n-2)}\Pi^2 v$$

by pushing the Π to the right,

$$= S_{12\cdots(n-1)}\Pi v - \sum_{i\geq 2} S_{i\cdots(n-1)}S_{12\cdots(n-1)}\Pi^2 v$$

by Lemma 9.4,

$$= S_{12\cdots(n-1)}\Pi v,$$

by Lemma 9.5. In other words, $U_1^2 v = U_1 v$. So by Step 1, we have $U_1 v = 0$.

Step 4: We show that $U_2 v = S_{23\cdots(n-1)}\Pi v = 0$.

By Step 3, the first term in (9.3) vanishes and we get

$$v = \sum_{i=2}^{n} S_{i(i+1)\cdots(n-1)}\Pi v. \tag{9.7}$$

We repeat the calculation we did in the previous step, using (9.7) instead of (9.3); we find:

$$(S_{23\cdots(n-1)}\Pi)^2 v = S_{23\cdots(n-1)}\Pi v.$$

It follows from Lemma 9.6 that

$$U_2 v = (S_{23\cdots(n-1)}\Pi)^2 v = S_{23\cdots(n-1)}\Pi v \tag{9.8}$$

and moreover that $U_2^2 v = U_2 v$. By Step 1 we see that $U_2 v = 0$.

$\vdots$

Step $n+2$: We show that $v \in \pi^G$.

From Step $i+2$, we know that $U_i v = S_{i(i+1)\cdots(n-1)}\Pi v = 0$ $(1 \leq i \leq n-1)$. Equation (9.3) thus simplifies to $v = \Pi v$. Since G is generated by K and $\begin{pmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ \varpi & & & \end{pmatrix} \in N_G(I)$ [for example by the Cartan decomposition], the claim follows. This completes the proof. □

Remark 9.9. We did not use that $T_i v = v$ for $1 < i < n$. This may seem strange, but the point is that we assume in the calculation that the U_i are nilpotent. This yields $U_i = 0$, a very strong fact. In fact it's easy to see that, for example, $U_1 = 0$ implies $T_2 v = v$.

Theorem 9.10. *Let π be any irreducible admissible G-representation. Then there exists a standard parabolic P with Levi $\prod \mathrm{GL}_{n_i}$ (and $\sum n_i = n$) and irreducible admissible representations σ_i of $\mathrm{GL}_{n_i}(F)$ such that $\pi \cong \mathrm{Ind}_P^G(\sigma_1 \otimes \cdots \otimes \sigma_r)$ and such that for all i either*

- *σ_i is supersingular and $n_i > 1$, or*
- *$\sigma_i \cong \mathrm{Sp}_{Q_i} \otimes \eta_i$ for some η_i and some standard parabolic $Q_i \subset \mathrm{GL}_{n_i}$.*

Moreover $\eta_i \neq \eta_{i+1}$ whenever both σ_i and σ_{i+1} fall in the second case.

Together with Thm. 8.4, this achieves the classification of irreducible admissible $\mathrm{GL}_n(F)$-representations.

Proof. We argue by induction on n. For $n = 1$, there is nothing to show. For $n > 1$, we may assume that there is at least one Serre weight V that occurs in π with non-supersingular Hecke eigenvalues χ parameterised by (M, χ_M) (otherwise π is supersingular and we are done). This means that $M \neq G$.

Let $\Delta_V = \{\alpha \in \Delta : s_\alpha \in \mathrm{Stab}_W(V^{U(k)})\}$. Thus V is M-regular if and only if $\Delta_V \subset \Delta_M$.

Case 1: $\Delta_V \cup \Delta_M \neq \Delta$. Pick $\alpha \in \Delta - (\Delta_V \cup \Delta_M)$. Let $Q = LN'$ be the standard, maximal parabolic defined by α, so $\Delta_L = \Delta - \{\alpha\} \supset \Delta_M$. Then V is L-regular and χ factors through $\mathcal{H}_L(V_{\overline{N}'(k)})$. Thm. 3.1 gives rises to

$$\mathrm{Ind}_{\overline{Q}}^{G}(\text{c-Ind}_{L(\mathcal{O})}^{L} V_{\overline{N}'(k)} \otimes_{\mathcal{H}_L(V_{\overline{N}'(k)}),\chi} \bar{k}) \twoheadrightarrow \pi. \tag{9.11}$$

Let τ denote the L-representation that is being induced on the left-hand side. It has a central character (namely $h \mapsto \chi(T^L_{h^{-1}})$). We claim that as τ is locally Z_L-finite and π is smooth, the map (9.11) gives rise to a non-zero map $\tau \to \mathrm{Ord}_Q \pi$. This follows from the injectivity of the first map in [Eme, (4.4.7)] (a geometric fact) and the isomorphism in [Eme, Cor. 4.2.8]. In particular, $\mathrm{Ord}_Q \pi \neq 0$. As $\mathrm{Ord}_Q \pi$ is admissible ([Eme, Thm. 3.3.3]), it has an irreducible and admissible subrepresentation $\sigma \hookrightarrow \mathrm{Ord}_Q \pi$ (the dual of an admissible L-representation is a finitely-generated module over the noetherian ring $\bar{k}[[L(\mathcal{O})]]$, see [Eme, Lemma 2.2.11]). By the adjunction [Eme, Thm. 4.4.6] we get a G-linear map $\mathrm{Ind}_{\overline{Q}}^{G} \sigma \twoheadrightarrow \pi$.

We can now decompose σ as a tensor product of irreducible admissible representations of the Levi blocks (Lemma 8.2). By induction, each of them is of the desired form. By transitivity of parabolic induction we see that π is a quotient of a parabolic induction of the desired form, except that consecutive η_i might be equal. But by §8.1 we know that the Jordan–Hölder factors of this parabolic induction are all of the desired form.

Case 2: $\Delta_V \cup \Delta_M = \Delta$. Suppose first that there is an $\alpha \in \Delta - \Delta_M$ such that $\alpha^\vee(\varpi) \notin Z_M$ or $\chi_M(\alpha^\vee(\varpi)) \neq 1$. Then by Cor. 6.11, there is a Serre weight V' that occurs in π with the same Hecke eigenvalues χ and such that $\Delta_{V'} = \Delta_V - \{\alpha\}$. We are thus reduced to Case 1.

Otherwise, for all $\alpha \in \Delta - \Delta_M$ we have $\alpha^\vee(\varpi) \in Z_M$ and $\chi_M(\alpha^\vee(\varpi)) = 1$. The first condition implies that $M = T$ and then the second condition implies that $\chi_M = \eta \circ \det$, for some smooth character η. By twisting (Lemma 4.7), we may assume that $\eta = 1$. Thus $V_{\overline{U}(k)}$ is trivial. Since we are in Case 2, we also have $\Delta_V = \Delta$. Putting together these two facts and using Lemma 2.3, we see that V is the trivial Serre weight. By Prop. 9.1, either π is trivial (and we are done) or $\mathrm{Ord}_Q \pi \neq 0$ for some proper standard parabolic Q (and we induct as in Case 1). ∎

Theorem 9.12. *Suppose that π is an irreducible admissible G-representation.*

(i) *All Hecke eigenvalues of Serre weights of* π *are parameterised by the same pair* $(M', \chi_{M'})$.

(ii) *There is a unique datum* $(P, (\sigma_i)_{i=1}^r)$ *as in Thm. 9.10 such that* $\pi \cong \mathrm{Ind}_P^G(\sigma_1 \otimes \cdots \otimes \sigma_r)$. *In other words, there are no non-trivial intertwinings between the representations in Thm. 8.4.*

Proof. By Thm. 9.10 we know that $\pi \cong \mathrm{Ind}_P^G(\sigma_1 \otimes \cdots \otimes \sigma_r)$ with $(P = MN, (\sigma_i)_{i=1}^r)$ satisfying the conditions in that theorem.

As in Thm. 8.4 we write $M = \prod M_i$ as product of Levi blocks. By Lemmas 4.5 and 8.2 we see that the Hecke eigenvalues of Serre weights of π are parameterised by $(\prod M_i', \prod \chi_i)$, where (M_i', χ) runs over the Hecke eigenvalues of Serre weights of σ_i. If σ_i is supersingular, we have $M_i' = M_i$ and χ_i is the central character of σ_i (use Lemma 4.6). If $\sigma_i \cong \mathrm{Sp}_{Q_i} \otimes \eta_i$, M_i' is the torus in M_i and $\chi_i = \eta_i \circ \det$ (use Prop. 7.3 and Lemma 4.7). Part (i) follows.

Since consecutive η_i are distinct, these common Hecke eigenvalues determine M. Part (ii) follows since $\mathrm{Ord}_P(\mathrm{Ind}_P^G \sigma) \cong \sigma$ [Eme, Prop. 4.3.4]. □

We can now prove a converse to Prop. 7.3.

Corollary 9.13. *Suppose that* π *is an irreducible admissible* G*-representation. Suppose that* P *is a standard parabolic and that the Serre weight* V_P *occurs in* π *with Hecke eigenvalues parameterised by* $(T, 1)$. *Then* $\pi \cong \mathrm{Sp}_P$.

Proof. If $\pi \cong \mathrm{Ind}_P^G(\sigma_1 \otimes \cdots \otimes \sigma_r)$ as in Thm. 9.10, then the analysis of Hecke eigenvalues in the proof of Thm. 9.12 shows that $P = B$, $r = 1$, and that $\sigma_1 = 1$. Then Cor. 7.2 and Prop. 7.3 show that $\pi \cong \mathrm{Sp}_P$. □

Definition 9.14. Suppose that π is an irreducible admissible G-representation. We say that π is *supercuspidal* if it does not occur as subquotient in $\mathrm{Ind}_P^G \sigma$, where $P = MN$ is any proper standard parabolic and σ any irreducible admissible M-representation.

Theorem 9.15. *Suppose* π *is an irreducible admissible* G*-representation. Suppose* $P = MN$ *is a standard parabolic and* σ *an irreducible admissible* M*-representation.*

(i) $\mathrm{Ind}_P^G \sigma$ *is of finite length. All Hecke eigenvalues of all Serre weights of all constituents are parameterised by the same pair* $(M', \chi_{M'})$.

(ii) π *is supersingular if and only if* π *is supercuspidal.*

Proof. For part (i), we first note that this is true when σ is supersingular (using Theorems 8.3 and 9.12). In general, note that the constituents are a subset of the constituents of a parabolic induction of a supersingular (by Thm. 9.10).

If π is supercuspidal, it follows from Thm. 9.10 that π is supersingular. Conversely, suppose that π occurs in $\mathrm{Ind}_P^G \sigma$ for some proper parabolic P. By (i), the Hecke eigenvalues of Serre weights of π are parameterised by $(M', \chi_{M'})$ with $M' \subset M \neq G$. Thus π cannot be supersingular. □

9.2. **Some general results.** We show for general G that any irreducible admissible G-representation is parabolically induced from a supersingular representation, provided that it does not contain certain Serre weights at the boundary.

Theorem 9.16. *Suppose that π is an irreducible and admissible G-representation. Then there exists a standard parabolic $P = MN$ and an irreducible admissible M-representation σ such that $\mathrm{Ind}_P^G \sigma \twoheadrightarrow \pi$ and such that all T-regular Serre weights of σ are supersingular.*

Proof. We will even find a σ such that for all Serre weights $\overline{V}$ and all Hecke eigenvalues $(M', \chi_{M'})$ occurring in $\mathrm{Hom}_{M(\mathcal{O})}(\overline{V}, \sigma)$, we have $\Delta_{\overline{V}} \cup \Delta_{M'} = \Delta_M$ (same notation as in Thm. 9.10).

Suppose there is a Serre weight V and any Hecke eigenvalues (M, χ_M) on $\mathrm{Hom}_K(V, \pi)$ such that $\Delta_V \cup \Delta_M \neq \Delta$. (Otherwise we are done.) Then by Case 1 of the proof of Thm. 9.10, there is a *proper* standard parabolic $Q = LN'$, an irreducible admissible L-representation σ, and a G-linear surjection $\mathrm{Ind}_Q^G \sigma \twoheadrightarrow \pi$. By induction we are done. □

The following proposition allows us to determine Serre weights of ordinary parts.

Proposition 9.17. *Suppose that $\overline{V}$ is a Serre weight for M. Suppose $\chi_M : Z_M \to \bar{k}^\times$ is a homomorphism such that $\chi_M|_{Z_M(\mathcal{O})}$ is the central character of $\overline{V}$. Then there is an algebra homomorphism $\chi : \mathcal{H}(\overline{V}) \to \bar{k}$ such that $\chi(T_h^M) = \chi_M(h)^{-1}$ for all $h \in Z_M^-$ and for any smooth G-representation π we have a natural isomorphism*

$$\mathrm{Hom}_{M(\mathcal{O})}(\overline{V}, (\mathrm{Ord}_P \pi)^{Z_M = \chi_M}) \cong \mathrm{Hom}_G(\text{c-}\mathrm{Ind}_{\mathcal{P}}^G \overline{V} \otimes_{\mathcal{H}(\overline{V}),\chi} \bar{k}, \pi). \tag{9.18}$$

Proof. We let $h \in Z_M$ act on $\overline{V}$ by $\chi_M(h)$. As this agrees with the usual action on $Z_M(\mathcal{O})$, $\overline{V}$ becomes an $M(\mathcal{O})Z_M$-representation. Thus by the definition of $\mathrm{Ord}_P \pi$, the left-hand side of (9.18) is naturally isomorphic to

$$\mathrm{Hom}_{M(\mathcal{O})Z_M}(\overline{V}, \mathrm{Hom}_{Z_M^+}(Z_M, \pi^{N(\mathcal{O})})) \cong \mathrm{Hom}_{M(\mathcal{O})Z_M^+}(\overline{V}, \pi^{N(\mathcal{O})}).$$

We verify that any $M(\mathcal{O})Z_M^+$-linear map $\overline{V} \to \pi^{N(\mathcal{O})}$ factors through $\pi^{\mathcal{P}(1)}$, where $\mathcal{P}(1) = \ker(\mathcal{P} \to M(k))$. By the smoothness of π and by Lemma 2.15, there is an $h \in Z_M^{--}$ such that $f(\overline{V})$ is fixed by ${}^h(\mathcal{P}^-)$. Then $h^{-1}f(\overline{V})$ is fixed by $\mathcal{P}^-(M \cap K(1))(\mathcal{P}^+)^h = \mathcal{P}(1) \cap \mathcal{P}(1)^h$. Note that the natural map $N(\mathcal{O})/N(\mathcal{O})^h \to \mathcal{P}(1)/(\mathcal{P}(1) \cap \mathcal{P}(1)^h)$ is a bijection. By the definition of the Hecke Z_M^+-action on $\pi^{N(\mathcal{O})}$, we have for all $\overline{v} \in \overline{V}$:

$$\chi_M(h^{-1})f(\overline{v}) = \sum_{N(\mathcal{O})/N(\mathcal{O})^h} nh^{-1}f(\overline{v}) = \sum_{\mathcal{P}(1)/(\mathcal{P}(1)\cap\mathcal{P}(1)^h)} ph^{-1}f(\overline{v}).$$

This implies that $f(\overline{v}) \in \pi^{\mathcal{P}(1)}$. In particular f is $\mathcal{P}$-linear.

By Prop. 4.1 and Cor. 4.2 (or directly) the pair (M, χ_M) gives rise to $\chi : \mathcal{H}_M(\overline{V}) \to \bar{k}$ such that $\chi(T_h^M) = \chi_M(h)^{-1}$ for $h \in Z_M$, and we restrict it to $\mathcal{H}(\overline{V})$. We verify that the $\mathcal{P}$-linear map $f : \overline{V} \to \pi^{N(\mathcal{O})} \hookrightarrow \pi$ is an $\mathcal{H}(\overline{V})$-eigenvector with eigenvalues χ. For $h \in Z_M^-$,

$$(f * E_h)(\overline{v}) = \sum_{\mathcal{P}/(\mathcal{P}\cap\mathcal{P}^h)} ph^{-1} f(E_h(hp^{-1})\overline{v}) = \sum_{N(\mathcal{O})/N(\mathcal{O})^h} nh^{-1} f(\overline{v}) = \chi_M(h^{-1}) f(\overline{v}),$$

since $N(\mathcal{O})/N(\mathcal{O})^h \to \mathcal{P}/(\mathcal{P} \cap \mathcal{P}^h)$ is a bijection and by definition of the Z_M^+-Hecke action on $\pi^{N(\mathcal{O})}$. (Note that E_h was defined in the proof of Lemma 2.23.) Now note that $\chi(E_h) = \chi(T_h^M) = \chi_M(h^{-1})$.

Thus we have a natural map

$$\mathrm{Hom}_{M(\mathcal{O})Z_M^+}(\overline{V}, \pi^{N(\mathcal{O})}) \to \mathrm{Hom}_G(\text{c-Ind}_{\mathcal{P}}^G \overline{V} \otimes_{\mathcal{H}(\overline{V}),\chi} \bar{k}, \pi).$$

Reversing the above argument we obtain an inverse map. This completes the proof. □

Theorem 9.19. *Suppose that π is an irreducible admissible G-representation such that the following condition is satisfied:*

(REG′) *For all simple roots α, the restriction to $k^\times$ via the cocharacter $\alpha^\vee$ of the $T(k)$-representation $(\mathrm{soc}_K \pi)^{U(k)}$ does not contain the trivial representation.*

Then there exists a standard parabolic $P = MN$ and an irreducible admissible supersingular M-representation σ such that $\pi \cong \mathrm{Ind}_P^G \sigma$.

Of course, in this situation all Hecke eigenvalues on all Serre weights of π are parameterised by (M, χ_M), where χ_M is the central character of σ (just as in Thm. 9.12).

Proof. By Thm. 9.16 there exists an irreducible and admissible M-representation σ such that $\mathrm{Ind}_P^G \sigma \twoheadrightarrow \pi$ and such that all T-regular Serre weights of σ are supersingular. It will suffice to show that condition (REG′) implies that $\mathrm{Ind}_P^G \sigma$ is irreducible and that all Serre weights of σ are T-regular.

We first show that for any Serre weight $\overline{V}$ of σ the unique M-regular Serre weight V of G such that $V^{N(k)} \cong \overline{V}$ (see Lemma 2.5) is a Serre weight of π. Let $\chi_M : Z_M \to \bar{k}^\times$ be the central character of σ, so we have an $M(\mathcal{O})$-linear map $\overline{V} \hookrightarrow \sigma \hookrightarrow (\mathrm{Ord}_P \pi)^{Z_M = \chi_M}$. By Prop. 9.17 we obtain an algebra homomorphism $\chi : \mathcal{H}_M(\overline{V}) \to \bar{k}$ and a G-linear surjection $\text{c-Ind}_{\mathcal{P}}^G \overline{V} \otimes_{\mathcal{H}(\overline{V}),\chi} \bar{k} \twoheadrightarrow \pi$. As V is M-regular, we may apply Cor. 2.28 and get a G-linear surjection $\text{c-Ind}_K^G V \otimes_{\mathcal{H}(\overline{V}),\chi} \bar{k} \twoheadrightarrow \pi$. Thus V is a Serre weight of π.

To show that $\mathrm{Ind}_P^G \sigma$ is irreducible it suffices to show that σ satisfies condition (REG) of Thm. 8.6. Suppose that $\overline{V}$ is a Serre weight of σ. By what we just showed, $\overline{V} \cong V^{N(k)}$ for some Serre weight V of π. So $\overline{V}_{(\overline{U} \cap M)(k)} \cong \overline{V}^{(U\cap M)(k)} \cong V^{U(k)}$. Thus the condition follows from (REG′).

Finally we deduce that all Serre weights of σ are T-regular. The point is that we just established that condition (REG) holds for σ, but even for *all* simple roots α. The condition for simple roots $\alpha \in \Delta_M$ implies the T-regularity of all Serre weights of σ. (Note that $\mathrm{Stab}_W \overline{V}^{(U \cap M)(k)}$ is generated by simple reflections and if s_α fixes $\overline{V}^{(U \cap M)(k)}$, then $\overline{V}^{(U \cap M)(k)}$ composed with $\alpha^\vee$ is trivial on $k^\times$.) □

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Department of Mathematics, 2033 Sheridan Road, Evanston IL 60208-2730, USA

E-mail address: herzig@math.northwestern.edu