Quantum total positivity and continuous tensor categories

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The plan

I am going to talk about some recent joint work with **Alexander Shapiro** (University of Toronto)

Our work involves the application of theory of *quantum cluster algebras* and *quantum total positivity* to some problems in representation theory and topological field theory.

Let's begin by describing the kind of problems we want to solve using these techniques.

Quantum group $\mathrm{U_q}(\mathfrak{g})$

Let $\mathfrak{g}$ be any Lie algebra.

Its universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ is a Hopf algebra; i.e. an associative algebra with a co-associative algebra map

$$\Delta : \mathrm{U}(\mathfrak{g}) \to \mathrm{U}(\mathfrak{g}) \otimes \mathrm{U}(\mathfrak{g}).$$

called the coproduct.

The Hopf algebra $\mathrm{U}(\mathfrak{g})$ is *co-commutative*;

$$\begin{aligned} \Delta^{op} &:= \mathrm{Flip} \circ \Delta \\ &= \Delta. \end{aligned}$$

So modules for $\mathrm{U}(\mathfrak{g})$ form a symmetric tensor category: we have

$$\mathrm{Flip} : \; V \otimes W \simeq W \otimes V.$$

Quantum group $\mathrm{U_q}(\mathfrak{g})$

Now suppose $\mathfrak{g}$ is a simple Lie algebra over $\mathbb{C}$.

The *quantum group* $\mathrm{U_q}(\mathfrak{g})$ is a Hopf algebra deformation of the enveloping algebra $\mathrm{U}(\mathfrak{g})$.

Coproduct in $\mathrm{U_q}(\mathfrak{g})$ is no longer co-commutative; instead $\mathrm{U_q}(\mathfrak{g})$ is *quasi-triangular* : it has an R-matrix $R \in \mathrm{U_q}(\mathfrak{g}) \otimes \mathrm{U_q}(\mathfrak{g})$ satisfying

$$\Delta^{op} = \mathrm{Ad}_R \circ \Delta,$$
$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \in \mathrm{U_q},(\mathfrak{g})^{\otimes 3}$$

where Δ^{op} is the opposite coproduct.

This means $\mathrm{U_q}(\mathfrak{g})$ can be used to construct interesting braided tensor categories.

Jones-Reshetikhin-Turaev-Witten construction

One interesting class of such categories comes from finite dimensional representations of $\mathrm{U_q}(\mathfrak{g})$ when q is a root of unity.

Reshetikin and Turaev used this category to define a 3d-TQFT, which yields invariants of 3-manifolds and framed links. When $\mathfrak{g} = \mathfrak{sl}_2$, these recover the Jones polynomial.

Witten: construct these invariants from geometric quantization of Chern-Simons theory with compact gauge group K. If $K = SU_2$, again obtain Jones polynomials.

Question: Is there an analog of these relations in the case of a non-compact split real gauge group, e.g. $G = SL_n(\mathbb{R})$?

Principal series for $U_q(\mathfrak{sl}_2)$

Let $\hbar \in \mathbb{R}_{>0} \setminus \mathbb{Q}$, and set

$$q = e^{\pi i \hbar^2}.$$

Ponsot and Teschner, '99: The split real quantum group $U_q(\mathfrak{sl}_2)$ has a *principal series* of irreducible $*$-representations

$$\mathcal{P}_s \simeq L^2(\mathbb{R}), \ s \in \mathbb{R}_{\geq 0},$$

with the following properties:

- the Chevalley generators E, F, K of the quantum group act on $\mathcal{P}_s$ by positive essentially self-adjoint operators;
- $\mathcal{P}_s$ is a bimodule for the quantum group $U_q(\mathfrak{sl}_2)$ and its modular dual $U_{q^\vee}(\mathfrak{sl}_2)$, where $q, q^\vee$ are related by the modular S-duality:
$$q^\vee = e^{\pi i / \hbar^2}.$$

A continuous tensor category

Most importantly, Ponsot and Teschner showed that the principal series of $U_q(\mathfrak{sl}_2)$ form a "continuous tensor category": one has

$$\mathcal{P}_{s_1} \otimes \mathcal{P}_{s_2} = \int_{\mathbb{R}_{\geq 0}}^{\oplus} \mathcal{P}_s d\mu(s).$$

The measure on the Weyl chamber $\mathbb{R}_{\geq 0}$ is

$$d\mu(s) = 4 \sinh\left(2\pi\hbar s\right) \sinh\left(2\pi\hbar^{-1} s\right) ds.$$

Higher rank principal series

Frenkel and Ip, '11 : For $\mathfrak{g}$ of any finite Dyknin type, the split real quantum group $U_q(\mathfrak{g}, \mathbb{R})$ has a family of irreducible principal series representations $\mathcal{P}_\lambda$ labelled by points of a Weyl chamber $\lambda \in \mathcal{C}^+$.

As in the rank 1 case:

- the Chevalley generators of the quantum group act on $\mathcal{P}_\lambda$ by positive essentially self-adjoint operators;
- $\mathcal{P}_\lambda$ is a bimodule for the quantum group $U_q(\mathfrak{g})$ and its Langlands dual $U_{q^\vee}({}^L\mathfrak{g})$, where $q, q^\vee$ are related by the modular S-duality:

$$q^\vee = e^{\pi i/\hbar^2}.$$

Conjecture (Frenkel and Ip, '11) : The principal principal series representations $\mathcal{P}_\lambda$ of $U_q(\mathfrak{g})$ are closed under tensor product, and thus form a continuous tensor category.

In recent joint work with Alexander Shapiro, we have proved this conjecture for $\mathfrak{g} = \mathfrak{sl}_{n+1}$.

Our proof is based on a geometric reformulation of the conjecture that allows us to study the principal series using the tools of *quantum cluster algebras* and *quantum total positivity.*

Moduli spaces of framed local systems

A *marked surface* $\widehat{S}$ is a compact oriented surface S with a finite set $\{x_1, \ldots, x_k\} \subset \partial S$ of marked points on the boundary.

Its punctured boundary is $\partial\widehat{S} := \partial S \setminus \{x_1, \ldots, x_k\}$.

Let G be a complex semisimple Lie group of adjoint type, and $B \subset G$ a Borel subgroup. A *framed G-local system* on $\widehat{S}$ is:

1. a G-local system $\mathcal{L}$ on S, together with
2. a flat section of the restriction of the associated flag bundle $(\mathcal{L} \times_G G/B)|_{\partial\widehat{S}}$.

Definition

$$\mathcal{X}_{G,\widehat{S}} := \text{moduli of framed } G\text{-local systems on } \widehat{S}$$

Concrete description of framing data

More concretely: The punctured boundary $\partial \widehat{S}$ is a union of intervals and circles. The framing data consists of

- For each interval component of $\partial \widehat{S}$, a choice of flag $\mathcal{F} \in G/B$;
- For each S^1 component of $\partial \widehat{S}$, a choice of flag $\mathcal{F} \in G/B$ such that $\mathcal{F}$ is preserved by the monodromy around S^1.

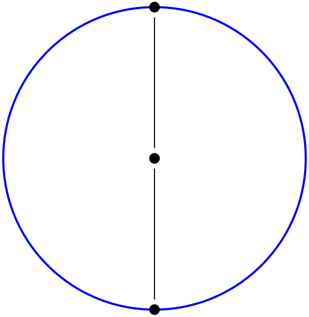
Example

Let $\widehat{S}$ be a punctured disk with 2 marked points. Then

$$\mathcal{X}_{G,\widehat{S}} = \{(g, F_1, F_2, F) \,|\, gF = F\} \,/\, G$$

where

$$g \in G, \qquad F, F_1, F_2 \in G/B.$$



$\mathcal{X}_{G,\widehat{S}}$ is a cluster variety

Fock-Goncharov: For $G = PGL_n(\mathbb{C})$, $\mathcal{X}_{G,\widehat{S}}$ is a *cluster Poisson variety:* it is covered up to codimension 2 by an atlas of toric charts

$$\mathcal{T}_\Sigma \colon (\mathbb{C}^*)^d \longrightarrow \mathcal{X}_{G,\widehat{S}},$$

labelled by quivers Q_Σ. The Poisson brackets are determined by the adjacency matrix ϵ_{jk} of Q_Σ:

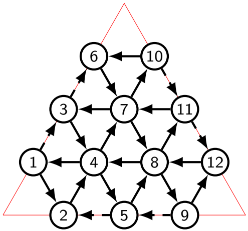
$$\{X_j, X_k\} = \epsilon_{kj} X_j X_k.$$

Different charts are related by subtraction-free birational transformations called *cluster mutations*.

Clusters from ideal triangulations

Shrink closed components of $\partial\widehat{S}$ to punctures. A triangulation of $\widehat{S}$ is *ideal* if its vertices are at punctures or marked points.

Each ideal triangulation provides a cluster chart. Fill each triangle with the following quiver (here $G = PGL_4$):



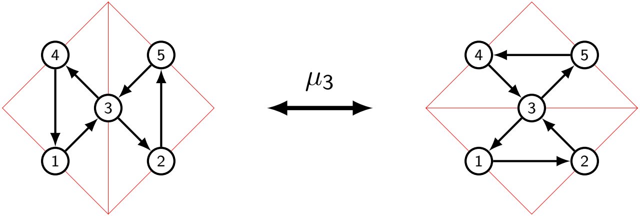
When two triangles share an edge, we "amalgamate" them, i.e. identify the quiver vertices lying on that edge, and replace the corresponding pairs of cluster variables by their products.

Cluster mutations

Mutation at a vertex X_i proceeds in 3 steps:

1. reverse all incident edges;
2. for each pair of edges $j \to i$ and $i \to k$ create an edge $k \to j$;
3. delete pairs of opposite edges;

For example, for $G = PGL_2$:



Mutations for $G = PGL_2$ correspond to flips of triangulation. For $G = PGL_n$, one flip is realized by a sequence of $\binom{n+1}{3}$ mutations.

Total positivity

Let X be any cluster variety.

Since the gluing maps between cluster charts are *subtraction–free* , there is a well–defined notion of the *totally positive* points $X^+ \subset X$.

The totally positive locus X^+ consists of points in X whose toric coordinates in every cluster chart are positive.

So we have a well–defined totally positive subset $\mathcal{X}^+_{G,\widehat{S}} \subset \mathcal{X}_{G,\widehat{S}}$.

If $G = PGL_2(\mathbb{C})$, the variety $\mathcal{X}^+_{G,\widehat{S}}$ can be identified with a component in the moduli space $\mathcal{M}_{flat}(\widehat{S}, PSL_2(\mathbb{R}))$, isomorphic to a decorated Teichmüller space.

So the positive loci $\mathcal{X}^+_{G,\widehat{S}}$ are higher rank analogs of Teichmüller spaces.

Promote each cluster chart to a quantum torus algebra

$$\mathcal{T}_{\Sigma}^{q} = \langle X_1, \ldots, X_d \rangle / \{X_j X_k = q^{2\epsilon_{kj}} X_k X_j\}.$$

They are "glued" by quantum mutations, which are the algebra automorphisms of conjugation by the *quantum dilogarithm* $\Gamma_q(X_k)$, where

$$\Gamma_q(X) = \prod_{n=1}^{\infty} \frac{1}{1 + q^{2n+1} X}.$$

Quantum mutation

Example: If $X_2X_1 = q^2X_1X_2$, then

$$\begin{aligned}\mu_2(X_1) &= \Gamma_q(X_2)\cdot X_1\cdot\Gamma_q(X_2)^{-1}\\ &= X_1\cdot\Gamma_q(q^2X_2)\cdot\Gamma_q(X_2)^{-1}\\ &= X_1\frac{(1+qX_2)(1+q^3X_2)\dots}{(1+q^3X_2)(1+q^5X_2)\dots}\\ &= X_1(1+qX_2).\end{aligned}$$

Positivity for quantum cluster varieties?

Now let's suppose that $q = e^{\pi i \hbar^2}$, with $\hbar \in \mathbb{R}_{>0} \setminus \mathbb{Q}$.

We can embed a quantum cluster chart $\mathcal{T}^q$ into a Heisenberg algebra $\mathcal{H}$ generated by $x_1, \ldots, x_d$ with relations

$$[x_j, x_k] = \frac{i}{2\pi} \epsilon_{jk},$$

by the homomorphism

$$X_j \mapsto e^{2\pi \hbar x_j}.$$

The algebra $\mathcal{H}$ has a family of irreducible Hilbert space representations V_χ parameterized by central characters $\chi \in \mathrm{Hom}(\ker \epsilon, \mathbb{R})$, in which the generators X_j act by **positive** essentially self-adjoint operators.

Modular duality

Now consider $q^\vee = e^{\pi i/\hbar^2}$, obtained from $q = e^{\pi i \hbar^2}$ by the transformation $\hbar \mapsto 1/\hbar$.

We also have an embedding of $\mathcal{T}^{q^\vee}$ into the Heisenberg algebra $\mathcal{H}$ given by

$$\tilde{X}_j = e^{2\pi\hbar^{-1}x_j},$$

so

$$\tilde{X}_k\tilde{X}_j = (q^\vee)^{2\epsilon_{kj}}\tilde{X}_j\tilde{X}_k.$$

Note that the generators $\tilde{X}_j$ commute with the original ones $X_j = e^{2\pi\hbar x_k}$:

$$X_j\tilde{X}_k = e^{2\pi i\epsilon_{jk}}\tilde{X}_kX_j = \tilde{X}_kX_j,$$

since $\epsilon_{jk} \in \mathbb{Z}$.

Now we've defined a positive representation of a single quantum cluster chart, we need the quantum analog of 'subtraction-free gluing maps'.

i.e. We want to promote quantum mutations to *unitary equivalences* between the Hilbert spaces for mutation-equivalent quantum torus algebras.

Problem: The formal power series $\Gamma_q(X)$ does not make sense when $|q| = 1$.

Non-compact quantum dilogarithm

Faddeev's solution: We can replace $\Gamma_q(X)$ by the *non-compact quantum dilogarithm*

$$\Phi^{\hbar}(z) = \exp\left(-\int_{\mathbb{R}+i0} \frac{e^{tz}}{(e^{\hbar t}-1)(e^{\hbar^{-1}t}-1)}\frac{dt}{t}\right).$$

The function $\Phi^{\hbar}(z)$ is well-defined for $\hbar \in \mathbb{R}$, and satisfies the difference equations

$$\Phi^{\hbar}(z+i\hbar) = (1+qe^{2\pi\hbar z})\Phi^{\hbar}(z),$$

$$\Phi^{\hbar}(z+i\hbar^{-1}) = (1+q^{\vee}e^{2\pi\hbar^{-1}z})\Phi^{\hbar}(z).$$

Non-compact quantum dilogarithm

Therefore cluster mutation in direction k can be realized by conjugation by $\Phi^\hbar(x_k)$.

Moreover, we have

$$\overline{\Phi^\hbar(\overline{z})} = \frac{1}{\Phi^\hbar(z)}.$$

So since x_k is self-adjoint, the operator $\Phi^\hbar(x_k)$ is a unitary operator on V_χ.

Thus the mutated operators $\mu_k(X_j)$ are also positive self-adjoint, and we get a unitary representation of the groupoid of cluster transformations.

Next: I want to present a theorem that lets us apply quantum total positivity to study the principal series for $U_q(\mathfrak{sl}_n)$.

Cluster realization of $U_q(\mathfrak{sl}_n)$

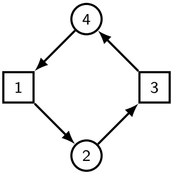
Theorem (S.-Shapiro '16)

Let $\widehat{S}$ be a punctured disk with two marked points, and let $\mathcal{X}^q_{PGL_n,\widehat{S}}$ be the corresponding quantum cluster algebra. Then there is an embedding of algebras

$$U_q(\mathfrak{sl}_n) \hookrightarrow \mathcal{X}^q_{PGL_n,\widehat{S}},$$

with the property that for each Chevalley generator of the quantum group, there is a cluster in which that generator is a cluster monomial.

Example: $U_q(\mathfrak{sl}_2)$



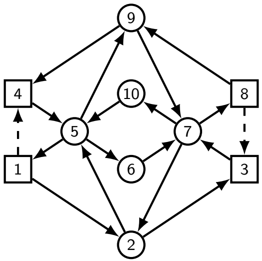
$$E \mapsto X_1(1 + qX_2), \qquad K \mapsto q^2 X_1 X_2 X_3,$$
$$F \mapsto X_3(1 + qX_4), \qquad K' \mapsto q^2 X_3 X_4 X_1.$$

$$KK' \quad \text{is central.}$$

Note that

$$E = \mu_2(X_1) \qquad \text{and} \qquad F = \mu_4(X_3).$$

Example: $U_q(\mathfrak{sl}_3)$



$$
\begin{aligned}
E_1 &\mapsto X_1(1+qX_2), & K_1 &\mapsto q^2 X_1 X_2 X_3,\\
E_2 &\mapsto X_4(1+qX_5(1+qX_6(1+qX_7))), & K_2 &\mapsto q^4 X_4 X_5 X_6 X_7 X_8,\\
F_1 &\mapsto X_3(1+qX_7(1+qX_{10}(1+qX_5))), & K_1' &\mapsto q^4 X_3 X_7 X_{10} X_5 X_1,\\
F_2 &\mapsto X_8(1+qX_9), & K_2' &\mapsto q^2 X_8 X_9 X_4.
\end{aligned}
$$

$K_1 K_1'$ and $K_2 K_2'$ are central.

Example: positive representations of $U_q(\mathfrak{sl}_2)$

Consider the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$, and the self-adjoint, unbounded operators

$$\hat{p} = \frac{1}{2\pi i}\frac{\partial}{\partial x}, \quad \hat{x} = x.$$

Then for all $s \in \mathbb{R}$, we have positive self-adjoint operators

$$X_1 = e^{2\pi\hbar(\hat{p}-\frac{s}{2})}, \quad X_2 = e^{2\pi\hbar(\hat{x}+s)}$$

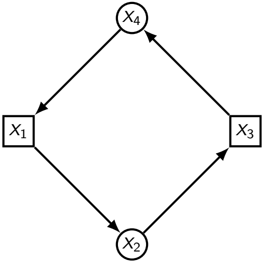
$$X_3 = e^{2\pi\hbar(-\hat{p}-\frac{s}{2})}, \quad X_4 = e^{2\pi\hbar(-\hat{x}+s)}$$

satisfying the cyclic quiver relations

$$q^2 X_k X_{k+1} = X_{k+1} X_k, \quad k \in \mathbb{Z}/4\mathbb{Z},$$

where $q = e^{\pi i \hbar^2}$.

Example: positive representations of $U_q(\mathfrak{sl}_2)$



Chevalley generators E, F of $U_q(\mathfrak{sl}_2)$ act by positive, self-adjoint operators

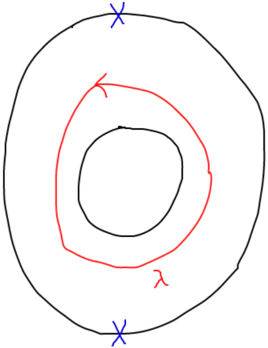
$$E \mapsto \mu_2(X_1), \qquad F \mapsto \mu_4(X_3).$$

The $U_q(\mathfrak{sl}_2)$ Casimir element Ω acts by

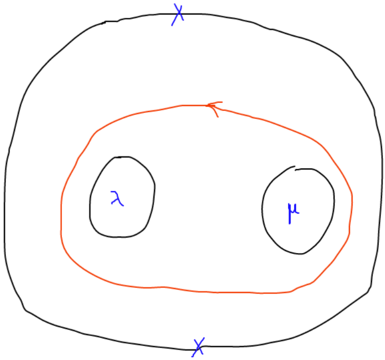
$$\Omega \mapsto e^{2\pi\hbar s} + e^{-2\pi\hbar s}.$$

Geometric approach to Frenkel and Ip's conjecture

First observation: central character of $\mathcal{P}_\lambda$ is determined by eigenvalues of holonomy around the puncture.



Next, we realize $P_\lambda \otimes P_\mu$ as a positive representation as follows:

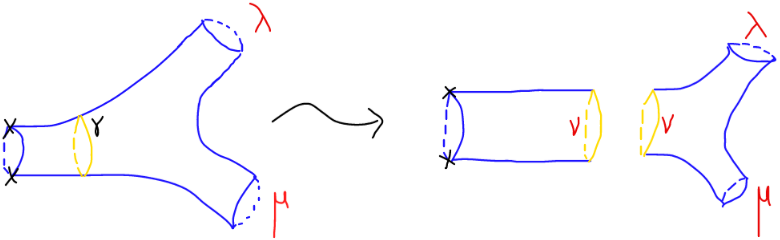


Action of the center of $\Delta U_q(\mathfrak{sl}_\mathfrak{n})$ is determined by eigenvalues of holonomy around red loop.

Cutting and gluing isomorphisms

Teichmüller theory interpretation of the decomposition of the tensor product of positive representations $P_\lambda \otimes P_\mu$ into positive representations P_ν:

$$P_\lambda \otimes P_\mu = \int_\nu P_\nu \otimes M^{\nu}_{\lambda,\mu} d\nu.$$



Cutting and gluing isomorphisms

On the right hand side: we have a "fiber product" of two quantum Teichmüller spaces:

- the quantum Teichmüller space for a for the punctured disk with two marked points on its boundary, corresponding to the diagonally acting copy of $U_q(\mathfrak{sl}_n)$; *and*
- the quantum Teichmüller space for a *thrice-punctured sphere*, where the eigenvalues of two monodromies are specified to λ and μ respectively.

The fiber product is taken over the spectrum of the unspecified monodromy ν around the loop associated to the remaining third puncture of the sphere, which parameterizes the central characters of $U_q(\mathfrak{sl}_n)$.

Multiplicities and the thrice-punctured sphere

The appearance of the 'multiplicity space' corresponding to the thrice-punctured sphere is a new feature in higher rank.

Indeed, when $\mathfrak{g} = \mathfrak{sl}_2$, the positive representation of the quantum Teichmüller space for the thrice-punctured sphere degenerates to a one-dimensional representation.

Algebraically: the picture let us read off a natural sequence of cluster transformations, which identifies the traces of holonomies that determine the action of the center of $U_q(\mathfrak{sl}_n)$ with the Hamiltonians of the q-difference *open Toda lattice.*

In particular, these operators form a **quantum integrable system.**

q–Whittaker functions

Next step: The eigenfunctions of the quantum Toda Hamiltonians have been determined by Kharchev-Lebedev-Semenov-Tian-Shansky: they are the *q-Whittaker functions.*

So passing to the basis of q–Whittaker functions diagonalizes the action of the center of $U_q(\mathfrak{sl}_n)$ on $\mathcal{P}_\lambda \otimes \mathcal{P}_\mu$.

The orthogonality and completeness relations for the Whittaker functions yields the desired the direct integral decomposition

$$P_\lambda \otimes P_\mu = \int_\nu P_\nu \otimes M^\nu_{\lambda,\mu} d\nu.$$

Towards an infinite dimensional modular functor?

This cutting and gluing result is the first step towards constructing an infinite dimensional analog of a modular functor from the quantization of higher Teichmüller spaces.

When $\mathfrak{g} = \mathfrak{sl}_2$, the construction of such a modular functor has been carried out by Teschner. On the 'loop group' side, closely related to the one coming from Virasoro conformal blocks.

Suggests connections with 4D gauge theory and higher rank AGT (IR line operators $\sim$ Fock-Goncharov coordinates)

Will be interesting to complete the construction in higher ranks and see if it can shed any light on the story for $\mathcal{W}$-algebra conformal blocks.