Fay's Trisecant Identity

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Motivation

Fay's identity is a relation between *cross-ratio functions* and *theta functions* on a compact Riemann surface.

Definition

Let p_1, p_2, q_1, q_2 be four distinct points in $\mathbb{P}^1 = \mathbb{C} \cup \infty$. Their **cross-ratio** is defined to be

$$\rho(p_1, p_2; q_1, q_2) = \frac{(p_1 - q_1)(p_2 - q_2)}{(p_1 - q_2)(p_2 - q_1)}$$

The cross-ratio is a *projective invariant* of an (ordered) set of four points in $\mathbb{P}^1$: it is preserved by any projective (Mobius) transformations of $\mathbb{P}^1$

Motivation

The cross-ratio function satisfies an "addition law"

$$\rho(p_1, p_2; q_1, q_2) + \rho(p_1, q_1; p_2, q_2) = 1 \quad (1)$$

You can think of a theta function as a generalization of the trig function $\sin z$. Trig functions also have addition laws:

$$\begin{aligned} &\sin(p_1+p_2)\sin(p_1-p_2)\sin(q_1+q_2)\sin(q_1-q_2) = \\ &\sin(p_1+q_1)\sin(p_1-q_1)\sin(p_2+q_2)\sin(p_2-q_2) - \\ &\sin(p_1+q_2)\sin(p_1-q_2)\sin(p_2+q_1)\sin(p_2-q_1) \end{aligned} \quad (2)$$

Fay's identity combines and generalizes formulas (1,2).

Motivation

Problem: How can I write down meromorphic functions on a Riemann surface Σ?

If my Riemann surface came from an algebraic curve like $P(x, y) = 0$ this is pretty easy: just start writing down rational functions in x and y.

But what if I want to write down a meromorphic function with zeros and poles at certain prescribed points? Maybe this isn't so easy in general using x and y...

There is a systematic way to solve this problem using **theta functions** and the **Abel map**

The Jacobian

Let Σ be a compact Riemann surface of genus $g > 0$. Topologically, Σ is a sphere with g handles, so $H_1(\Sigma, \mathbb{Z}) = \mathbb{Z}^{2g}$. We can choose a basis $\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$ for $H_1(\Sigma, \mathbb{Z})$ which is *symplectic* with respect to the intersection form:

$$(a_i, a_j) = (b_i, b_j) = 0, (a_i, b_j) = \delta_{ij}$$

The choice of such a basis is called a **Torelli marking** of Σ.

The Jacobian

If we fix some basepoint $p \in \Sigma$, we can choose $2g$ closed curves on Σ passing through p whose homology classes give our canonical basis $a_1, \dots, a_g, b_1, \dots, b_g$. We can delete these closed curves from Σ to obtain a *simply connected* Riemann surface Σ°. Remember that simply connected spaces are nice to do integration on because *every holomorphic differential is exact*.

The Jacobian

Recall from the Riemann-Roch theorem that the holomorphic differentials on Σ form a g-dimensional $\mathbb{C}$-vector space $\mathcal{L}(K)$. You can check that the integration pairing between a-cycles and holomorphic differentials is non-degenerate. So we can choose a *normalized* basis $\omega_1, \ldots, \omega_g$ for $\mathcal{L}(K)$ such that

$$\oint_{a_i} \omega_j = \delta_{ij}$$

The Jacobian

Definition

The **matrix of $\mathcal{B}$-periods** of the Torelli marked Rieman surface Σ is the $g \times g$ complex matrix

$$\mathcal{B}_{ij} = \oint_{b_i} \omega_j$$

The matrix of $\mathcal{B}$ has the following important properties:

Lemma

The matrix of $\mathcal{B}$-periods is symmetric, and its imaginary part defines a positive definite quadratic form on $\mathbb{R}^g$.

This is easy to prove by finding primitives for the holomorphic differentials on the simply connected surface Σ°.

The Jacobian

Definition

The **period lattice** of the Torelli marked surface Σ is the free $\mathbb{Z}$-module

$$\Lambda = \mathbb{Z}^g + \mathcal{B}\mathbb{Z}^g \subset \mathbb{C}^g$$

In words, the $2g$ generators of the period lattice are the g standard basis vectors of $\mathbb{C}^g$, together with the g columns of the matrix of $\mathcal{B}$-periods.

Since the imaginary part of $\mathcal{B}$ is positive-definite (hence invertible), this lattice has maximal rank $2g$.

The Jacobian

Definition

The **Jacobian** of Σ is the g-dimensional complex manifold

$$J(\Sigma) = \mathbb{C}^g/\Lambda$$

As a smooth manifold, the Jacobian is the $2g$-dimensional torus $\mathbb{R}^{2g}/\mathbb{Z}^{2g}$. But there are lots of different isomorphism classes of complex tori (c.f. genus 1 where we have $\mathcal{H}_+/SL(2,\mathbb{Z})$ many)

Exercise: different choices of Torelli marking and basis for $\mathcal{L}(K)$ on the same Riemann surface define isomorphic complex tori.

Abel map

There is an important map from Σ to $J(\Sigma)$ called the **Abel map**. The Abel map with basepoint $q \in \Sigma$ is defined by the formula

$$\mathcal{A}(p) = \left(\int_q^p \omega_1, \ldots, \int_q^p \omega_g \right) \tag{3}$$

If we modify the integration path by an *a* or *b*-cycle, we add a vector in the period lattice to the resulting vector in $\mathbb{C}^g$. Hence the Abel map is well-defined, and doesn't depend on the choice of integration path. But it **does** depend on the choice of basepoint q.

Genus 1

These constructions look much less scary if $\Sigma = \mathbb{C}/\{1,\tau\}$ has genus 1. Define a Torelli marking with the *a*-cycle being $[0,1] \subset \mathbb{C}$, and the *b*-cycle being $\{t \cdot \tau | t \in [0,1]\} \subset \mathbb{C}$. Then $\omega = dz$ is a normalized basis of holomorphic differentials, and the *b*-period is

$$\int_0^\tau dz = \tau$$

so the Riemann surface and its Jacobian are actually the same torus! The Abel map with basepoint $[0] \in \Sigma$ is just the identity map:

$$\mathcal{A}(p) = \int_0^p dz = p$$

Theta functions

Definition

Let $\mathcal{B}$ be a symmetric $g \times g$ matrix with complex entries whose imaginary part is positive definite. The **theta function** associated to $\mathcal{B}$ is the holomorphic function $\theta : \mathbb{C}^g \to \mathbb{C}$ defined by the multidimensional Fourier series

$$\theta(z) = \sum_{n \in \mathbb{Z}^g} e^{\pi i n^t \mathcal{B} n} e^{2\pi i n^t z}$$

Hence the matrix of $\mathcal{B}$-periods lets us build a theta function from a Torelli marked Riemann surface Σ.

Theta functions

Although the theta function is not invariant with respect to translation by the period lattice $\Lambda = \mathbb{Z}^g + \mathcal{B}\mathbb{Z}^g$, in view of the symmetry of $\mathcal{B}$ it transforms by a simple multiplicative factor under such a translation: if $l \in \mathbb{Z}^g$, we have

$$\theta(z + l) = \theta(z) \tag{4}$$

$$\theta(z + \mathcal{B}l) = e^{-\pi i l^t \mathcal{B} l - 2\pi i l^t z}\theta(z) \tag{5}$$

Theta functions

Now we can combine the Abel map with the theta function to build meromorphic functions on Σ. Note that although it is not single-valued on Σ, the function

$$f(p) = \theta(\mathcal{A}(p) + c)$$

defines a meromorphic function on the simply connected Riemann surface Σ°.

Theorem (Riemann's theorem)

The function $f(p) = \theta(\mathcal{A}(p) + c)$ either vanishes identically in $p \in \Sigma$ or has exactly g zeroes $p_1, \ldots, p_g \in \Sigma$ such that the following equality holds in $J(\Sigma)$:

$$\mathcal{A}(p_1) + \ldots + \mathcal{A}(p_g) + c + \mathcal{K} = 0 \tag{6}$$

where $\mathcal{K} \in J(\Sigma)$ is called the **Riemann point***; this point depends on the curve Σ, its Torelli marking, and the basepoint P_0 of the Abel map, but not on the vector $c \in \mathbb{C}^g$.*

Theta functions

To prove this you use the *argument principle* from complex analysis: to count and locate zeros respectively, integrate $\frac{df}{f}$ and $\frac{\mathcal{A}(p)df}{f}$ around the boundary of Σ°.

Reassuring aside: for 'almost all' $c \in \mathbb{C}^g$, the function $f(p) = \theta(\mathcal{A}(p) + c)$ is not identically zero.

Zeros of θ in genus 1

In genus 1: if x is the basepoint of the Abel map, check the Riemann point is

$$\mathcal{K} = \frac{1+\tau}{2} - x$$

The function $\theta(z+c)$ is never identically zero; it always has one zero z_0 with the property that

$$z_0 + c + \mathcal{K} = 0$$

This equation allows us to interpret the zeros of the genus 1 theta function geometrically in terms of the plane cubic model $C \subset \mathbb{P}^2$ of Σ.

Zeros of θ in genus 1

Let $\pi : \Sigma \to C$ be the isomorphism

$$z \longmapsto [\wp(z) : \wp'(z) : 1]$$

Let $E = [0 : 0 : 1] = \pi(0)$ be the identity element of C and let $R = \pi(\mathcal{K})$ be the image of the Riemann point. Let $\overline{ER}$ be the line spanned by E and R. Then the zero of the theta function $\theta(z)$ is the third point of the intersection of the line $\overline{ER}$ with C. More generally, for any point $P \in C$, the zero of $\theta(z + \pi^{-1}(P))$ is given by the third intersection point of the line $\overline{PR}$ with C.

Writing meromorphic functions with θ

Riemann's theorem lets us build functions on Σ. Suppose we want to write down a meromorphic function $f(p)$ such that

$$(f) \geq q^{+} - q^{-} - p_1 - \ldots - p_g$$

By Riemann-Roch, there is *generically* a 1-dimensional space of such functions.

Writing meromorphic functions with θ

In this generic case,

$$f(p) = \text{constant} \times \frac{\theta(\mathcal{A}(p) - w^+)\theta(\mathcal{A}(p) - w^0)}{\theta(\mathcal{A}(p) - w^-)\theta(\mathcal{A}(p) - w)}$$

where

$$w = \sum_{k=1}^{g} \mathcal{A}(p_k) + \mathcal{K}$$

$$w^\pm = \mathcal{A}(q^\pm) + \sum_{k=2}^{g} \mathcal{A}(p_k) + \mathcal{K}$$

$$w^0 = w + w^- - w^+$$

The cross-ratio function on a Riemann surface

Theta functions can also be used to define an analog of the $\mathbb{P}^1$ cross-ratio function

$$\frac{(p_1 - q_1)(p_2 - q_2)}{(p_1 - q_2)(p_2 - q_1)}$$

on a higher genus Riemann surface. The key ingredient is the following: there exists a vector $\alpha \in \mathbb{C}^g$ called an **odd, non-singular characteristic** such that the function

$$e(p, q) = \theta(\mathcal{A}(p) - \mathcal{A}(q) + \alpha)$$

is not identically zero and satisfies $e(p, q) = -e(q, p)$. As a function of p, $e(p, q)$ has g zeros

$$q, y_1, \ldots, y_{g-1}$$

where the points y_i are *independent of* q.

The cross-ratio function on a Riemann surface

Hence the formula

$$\rho(p_1, p_2; q_1, q_2) = \frac{e(p_1, q_1)e(p_2, q_2)}{e(p_1, q_2)e(p_2, q_1)} \tag{7}$$

defines a function on $(\Sigma^\circ)^4$ called the **cross-ratio** which has the following properties as a function of p_1:

1. As a function of p_1, ρ has divisor $q_1 - q_2$, and $\rho = 1$ when $q_1 = q_2$.
2. ρ is invariant under after passing around any a-cycle, but after passing around the cycle b_j transforms via

$$\rho \longmapsto e^{2\pi i \mathcal{A}_j(q_1 - q_2)} \rho$$

where $\mathcal{A}_j$ denotes the j-th component of the Abel map.

Fay's identity

The higher genus crosss-ratio function no longer satisfies the addition law that held in genus zero. Instead, it satisfies a relation involving the theta funtion:

Theorem (Fay's trisecant identity)

Let $c \in \mathbb{C}^g$ be such that $\theta(c) \neq 0$. Then the following identity holds:

$$\begin{aligned}
&\theta\left(\int_{q_1}^{p_1}\omega + c\right)\theta\left(\int_{q_2}^{p_2}\omega + c\right)\rho(p_1, q_1; q_2, p_2) \\
&+ \theta\left(\int_{q_2}^{p_1}\omega + c\right)\theta\left(\int_{q_1}^{p_2}\omega + c\right)\rho(p_1, q_2; q_1, p_2) \\
&= \theta(c)\theta\left(\int_{q_1+q_2}^{p_1+p_2}\omega + c\right)
\end{aligned} \tag{8}$$

Fay's identity

Theorem (Fay's multisecant identity)

Let $c \in \mathbb{C}^g$ be such that $\theta(c) \neq 0$. Then the following identity holds:

$$\theta(c)^{N-1}\theta\left(\int_{q_1+\ldots+q_N}^{p_1+\ldots+p_N}\omega + c\right) = \tag{9}$$

$$\frac{\prod_{i,j=1}^{N} e(p_i, q_j)}{\prod_{i<j} e(p_i, p_j)e(q_j, q_i)} \det\left[\frac{\theta\left(\int_{q_j}^{p_i}\omega + c\right)}{e(p_i, q_j)}\right]_{i,j=1,\ldots,N} \tag{10}$$

where we use the compact notation

$$\int_q^p \omega = \mathcal{A}(p) - \mathcal{A}(q)$$

for the Abel map.

Fay's identity

Why do the Fay identities hold?

Warmup: How do we know

$$\frac{\prod_{i<j\leq N}(p_i - p_j)(q_j - q_i)}{\prod_{i,j=1}^{N}(p_i - q_j)} = \det\left[\frac{1}{p_i - q_j}\right]_{i,j=1,\ldots,N}$$

Answer: Think of both sides as rational functions of $p_1 \in \mathbb{P}^1$. Up to a constant factor, such a function is determined by its divisor of zeros and poles.

Fay's identity

We can apply similar reasoning to prove Fay's identities. Consider both sides as functions of p_1.

1. Check both sides transform the same way when we go around an a-or b-cycle. This means LHS/RHS is a single-valued meromorphic function on Σ.
2. Observe the LHS has a positive divisor D of exactly g zeroes, coming from the theta function $\theta\left(\int_{q_1+\ldots+q_N}^{p_1+\ldots+p_N} \omega + c\right)$, and no poles. The RHS has no poles, so it too must have a positive divisor E of exactly g zeros.
3. Hence the divisor of the quotient LHS/RHS is greater than $-E$. By Riemann-Roch, if p_i, q_i are in general position the only such functions are constants. (i.e. D=E)
4. Letting $p_1 \to q_1$, we check this constant is 1.

Back to genus 1

Let's consider the case $g = 1$. The **Weierstrass sigma function** is an analog of the theta function which is related to the $\wp$ function by

$$\wp(z) = -\frac{d^2}{dz^2} \log \sigma(z)$$

Nice fact: The sigma function is an odd function of z.

Back to genus 1

Now let's consider the degeneration of the multisecant formula as the points q_i merge to $0 \in \mathbb{C}/\Lambda$. Written in terms of the sigma function, this becomes

$$C_N \sigma\left(\sum_{k=1}^{N} z_k\right) \frac{\prod_{i<j} \sigma(z_i - z_j)}{\prod_{i=1}^{N} \sigma^{N-1}(z_i)} = \begin{vmatrix} 1 & \wp(z_1) & \wp'(z_1) & \cdots & \wp^{(N-1)}(z_1) \\ 1 & \wp(z_2) & \wp'(z_2) & \cdots & \wp^{(N-1)}(z_2) \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & \wp(z_N) & \wp'(z_N) & \cdots & \wp^{(N-1)}(z_N) \end{vmatrix}$$

where the constant $C_N = (-1)^{\frac{1}{2}N(N-1)} 1!2! \cdots N!$
Since $\sigma(0) = 0$, this formula is really a generalization of problem 3 (addition formula for $\wp$) on Math 255 Homework 3.

Why 'trisecant'?

Given a Jacobian $J(\Sigma)$, you can use the "second order theta functions" (which I haven't defined) to embed $J(\Sigma)/\{\pm 1\}$ into $\mathbb{P}^{2^g-1}$; the image is called the **Kummer variety**. Fay's identity says that the images of three points

$$\mathcal{A}(p_1 + p_2 - q_1 - q_2),\ \mathcal{A}(p_1 + q_1 - p_2 - q_2),\ \mathcal{A}(p_1 + q_2 - p_2 - q_1)$$

in the Kummer variety are **collinear**. So the Kummer variety of a Jacobian has a four dimensional family of trisecant lines. This property turns out to characterize Jacobians among all Abelian varieties...

The End

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