MATH 255 TERM PAPER: FAY'S TRISECANT IDENTITY

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Introduction

The purpose of this term paper is to give an accessible exposition of *Fay's trisecant identity* [1]. Fay's identity is a relation between the *theta function* and the *cross-ratio function* associated to a compact Riemann surface [2]. Although the majority of the paper will consist of developing the machinery needed to define these objects, let us briefly say a few words about some familiar objects that they generalize. The cross-ratio function will be a higher genus generalization of the usual cross-ratio of an ordered set of four points in $\mathbb{P}^1$, defined by

$$\rho(p_1, p_2; q_1, q_2) = \frac{(p_1 - q_1)(p_2 - q_2)}{(p_1 - q_2)(p_2 - q_1)}$$

Some key properties of the $\mathbb{P}^1$ cross-ratio function are the following: it is invariant under Mobius transformations

$$\left(z \longmapsto \frac{az + b}{cz + d}\right) \in PGL_2(\mathbb{C})$$

and it satisfies the 'addition law'

$$\rho(p_1, p_2; q_1, q_2) + \rho(p_1, q_1; p_2, q_2) = 1 \tag{1}$$

Let us also say a few words about the theta function. Very roughly speaking, the theta function associated to a compact Riemann surface can be thought of as a generalization of the trigonometric function $\sin z$. Trigonometric functions also have addition laws: for example,

$$\begin{aligned}
&\sin(p_1 + p_2)\sin(p_1 - p_2)\sin(q_1 + q_2)\sin(q_1 - q_2) = \\
&\sin(p_1 + q_1)\sin(p_1 - q_1)\sin(p_2 + q_2)\sin(p_2 - q_2) - \\
&\sin(p_1 + q_2)\sin(p_1 - q_2)\sin(p_2 + q_1)\sin(p_2 - q_1)
\end{aligned}$$

We will see that Fay's trisecant identity is a generalization of this formula and formula (1). Moreover, the trisecant identity itself can easily be generalized to a determinantal "multisecant" identity, which can be thought of as a higher genus analog of the Cauchy determinant evaluation

$$\frac{\prod_{i<j}(p_i - p_j)(q_j - q_i)}{\prod_{i,j=1}^{N}(p_i - q_j)} = \det\left[\frac{1}{p_i - q_j}\right]_{i,j=1,\dots,N}$$

We will conclude the paper by making some brief remarks on the meaning of the word 'trisecant' and the application of Fay's identity to the Schottky problem in classical algebraic geometry.

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Meromorphic differentials on a compact Riemann surface

We begin by giving some important facts about differentials on compact Riemann surfaces and their periods. Let Σ be a compact Riemann surface of genus g. Then $H_1(\Sigma,\mathbb{Z})=\mathbb{Z}^{2g}$, and we can choose a basis $\{a_i,b_i\}$ on Σ which is symplectic with respect to the intersection pairing: that is,

$$(a_i,a_j)=0=(b_i,b_j),(a_i,b_j)=\delta_{ij}$$

The choice of such a basis of cycles is called a **Torelli marking** of Σ, and gives rise to a useful bilinear pairing $(\Omega_1\bullet\Omega_2)$ between meromorphic differentials on Σ as follows:

$$(\Omega_1\bullet\Omega_2)=\sum_{i=1}^{g}\left(\oint_{a_i}\Omega_1\oint_{b_i}\Omega_2-\oint_{a_i}\Omega_2\oint_{b_i}\Omega_1\right) \tag{2}$$

If we fix some basepoint $p\in\Sigma$, we can choose $2g$ closed curves $a_1,\dots,a_g,b_1,\dots,b_g$ on Σ passing through p whose homology classes form a symplectic basis for $H_1(\Sigma,\mathbb{Z})$. We can cut the compact Riemann surface Σ up along these closed curves to obtain a simply connected Riemann surface Σ° whose boundary is the curve $\gamma=a_1b_1a_1^{-1}b_1^{-1}\cdots a_gb_ga_g^{-1}b_g^{-1}$. Then given any meromorphic differential Ω on Σ, there exists a meromorphic function g on Σ° such that $\Omega=dg$.

Proposition 1. *(Riemann's bilinear identity) Suppose g_1 is a function defined on the simply connected cut Riemann surface Σ° satisfying $\Omega_1=dg_1$. Then*

$$(\Omega_1\bullet\Omega_2)=2\pi i\sum_{p\in\Sigma^\circ}Res\,(g_1\Omega_2) \tag{3}$$

Proof. We compute $\oint_\gamma g_1\Omega_2$ where $\gamma=a_1b_1a_1^{-1}b_1^{-1}\cdots a_gb_ga_g^{-1}b_g^{-1}$ is the boundary of the simply connected cut Riemann surface Σ°. By the residue theorem, we have

$$\oint_\gamma g_1\Omega_2=2\pi i\sum_{p\in\Sigma^\circ}\mathrm{Res}\,(g_1\Omega_2)$$

But since the values of the function g_1 on opposite sides of the cut a_j differ by $\oint_{b_j}\Omega_1$, and its values on opposite sides of the cut b_j differ by $\oint_{a_j}\Omega_1$, we find

$$\oint_\gamma g_1\Omega_2=\sum_{i=1}^{g}\left(\oint_{a_i}\Omega_1\oint_{b_i}\Omega_2-\oint_{a_i}\Omega_2\oint_{b_i}\Omega_1\right)$$

which proves the claim. □

Now suppose ω is a holomorphic differential on Σ. Then its complex conjugate $\overline{\omega}$ is a smooth complex-valued 1-form on Σ, and in a coordinate chart $z=x+iy$ on Σ where $\omega=h(z)dz$ we have $\omega\wedge\overline{\omega}=-2i|h|^2dx\wedge dy$. Hence

$$i\int_\Sigma\omega\wedge\overline{\omega}\geq 0$$

with equality if and only if $\omega\equiv 0$. Let us write $A_k=\oint_{a_k}\omega, B_k=\oint_{b_k}\omega$ for the a- and b-periods of ω. Then applying Stokes' theorem to $\omega\wedge\overline{\omega}=d\,(g\overline{\omega})$, we find

$$i\int_{\Sigma^\circ}\omega\wedge\overline{\omega}=i\left(\sum_{k=1}^{g}A_k\overline{B}_k-\overline{A}_kB_k\right)\geq 0 \tag{4}$$

This formula has some important corollaries. If we denote the set of globally defined holomorphic differentials on Σ by $\mathcal{L}(K)$, recall from the Riemann-Roch theorem that $\mathcal{L}(K)$ has dimension g as a complex vector vector space.

As a consequence of formula (4), we have

Corollary 1. *The intersection pairing between a-cycles and holomorphic differentials is nondegenerate: there is no holomorphic differential whose a-periods are all zero.*

In view of this fact, we apply the Gram-Schmidt procedure to find a basis $\omega_1, \ldots, \omega_g$ which is normalized with respect to the cycles $a_1, \ldots, a_g$ in the sense that

$$\oint_{a_i} \omega_j = \delta_{ij} \tag{5}$$

Definition 1. *(Matrix of $\mathcal{B}$-periods)* Let (a_i, b_i) be a canonical basis of cycles on Σ and let ω_i be a basis for $\mathcal{L}(K)$ normalized according to (5) with respect to the a-cycles. The **matrix of $\mathcal{B}$-periods** corresponding to this data is the $g \times g$ matrix

$$\mathcal{B}_{ij} = \int_{b_i} \omega_j$$

The matrix of $\mathcal{B}$-periods has the following two essential properties which will allow us to use it to build theta functions:

Corollary 2. *The matrix of $\mathcal{B}$-periods is symmetric and its imaginary part is a positive definite real symmetric matrix.*

Proof. If we apply the bilinear identity (3) to $(\omega_i \bullet \omega_j)$, the symmetry of $\mathcal{B}$ follows from the normalization (5) and the fact that the sum of residues on the RHS is zero since the differentials are holomorphic. The positive-definiteness follows from applying formula (4) to $\omega = \sum_{i=1}^{g} c_i \omega_i$ where c_i is any g-tuple of real numbers; we find

$$i \int_{\Sigma} \omega \wedge \overline{\omega} = 2 \sum_{k,l} \mathrm{Im}(B_{kl}) c_k c_l \geq 0$$

with equality if and only if $c \equiv 0$. □

The period lattice and the Jacobian

We are now ready to construct the Jacobian of a compact Riemann surface.

Definition 2. *(Period lattice)* Let Σ be a genus g compact Riemann surface with symplectic homology basis (a_i, b_i) and corresponging normalized basis of holomorphic differentials $\omega_1, \ldots, \omega_g$. The **period lattice** associated to this data is the free $\mathbb{Z}$-module $\Lambda \subset \mathbb{C}^g$ spanned by the $2g$ vectors

$$e_i = \left(\oint_{a_i} \omega_1, \ldots, \oint_{a_i} \omega_g \right) \in \mathbb{C}^g$$

$$f_i = \left(\oint_{b_i} \omega_1, \ldots, \oint_{b_i} \omega_g \right) \in \mathbb{C}^g$$

By the normalization of the basis (ω_i), the vector e_i in the period lattice is just the i-th standard basis vector in $\mathbb{C}^g$, while $f_i = \mathcal{B} e_i$, where $\mathcal{B}$ is the matrix of $\mathcal{B}$-periods. Since the imaginary part of $\mathcal{B}$ is positive definite and in particular invertible, we see that the period lattice $\Lambda = \mathbb{Z}^g + \mathcal{B}\mathbb{Z}^g \subset \mathbb{C}^g$ has full rank $2g$ inside $\mathbb{C}^g \simeq \mathbb{R}^{2g}$.

Definition 3. (*Jacobian of a compact Riemann surface)* The *Jacobian* of Σ is the g-dimensional complex manifold

$$J(\Sigma) = \mathbb{C}^g / \Lambda$$

where Λ is any period lattice of Σ.

Since the lattice Λ has full rank, as a smooth manifold the Jacobian is a torus $\mathbb{R}^{2g}/\mathbb{Z}^{2g}$.

Remark: Since any two choices of canonical basis are related by an integral symplectic matrix $J \in \mathrm{Sp}(2g, \mathbb{Z})$, the Jacobians constructed from different Torelli markings are isomorphic as complex tori [1].

We now describe an important mapping from a Riemann surface to its Jacobian which will play a fundamental role in constructing meromorphic functions.

Definition 4. *(Abel map)* Let Σ be a genus g compact Riemann surface with canonical homology basis (a_i, b_i) and normalized basis of holomorphic differentials $\omega_1, \ldots, \omega_g$. If q is a point of Σ, the **Abel map with basepoint** q is the mapping $\mathcal{A} : \Sigma \to J(\Sigma)$ defined by

$$\mathcal{A}(p) = \left[\left(\int_q^p \omega_1, \ldots, \int_q^p \omega_g \right) \right] \in J(\Sigma)$$

where the integration is taken over *any* path from P to Q in Σ.

Note that since the forms ω_i are holomorphic and thus closed, their integral along a path only depend on the homology class of that path. Since modifying a path by a cycle $\gamma \in H_1(\Sigma, \mathbb{Z})$ has the effect of adding a vector in Λ to the equivalence class representative $\mathcal{A}(p)$, we see that the Abel map does indeed give a well-defined map[1] to the Jacobian. The Abel map can be extended from points to arbitrary divisors on Σ by declaring

$$\mathcal{A}\left(\sum n_p p\right) = \sum n_p \mathcal{A}(p)$$

The fundamental theorem about the Abel map is

Theorem 1. *(Abel's theorem) Let D be a divisor of degree zero on Σ. Then D is the divisor of a meromorphic function if and only if $\mathcal{A}(D) = 0$ in $J(\Sigma)$.*

For the proof we refer the reader to Farkas and Kra's book [6].

Special divisors

We now make a brief digression about the notion of special divisors on a compact Riemann surface. Suppose Σ is a compact Riemann surface of genus $g > 0$ with canonical class K, and let D be a divisor on Σ. We will use the usual notation

$$l(D) = \dim_{\mathbb{C}} \mathcal{L}(D) = \dim_{\mathbb{C}} \{f \text{ meromorphic on } \Sigma | (f) + D \geq 0\}$$

Given an effective divisor D of degree $\leq g$, we say that D is **special** if $l(D) > 1$. The special divisors on Σ are non-generic in the following sense: if $D = p_1 + \ldots + p_k$ is a special divisor, there exists a non-special divisor $D' = q_1 + \ldots + q_k$ where the points q_i can be chosen to be arbitrarily close to p_i. Indeed, note that any divisor $D_1 = p$ is non-special, since a meromorphic function with a single pole would define an isomorphism of Σ with $\mathbb{P}^1$. Hence $l(p) = 1$ for all $p \in \Sigma$. Now if a divisor $D_2 = p_1 + p_2$ is special, then by Riemann-Roch we have

$$l(K - p_1 - p_2) = g - 3 + l(p_1 + p_2) > g - 2$$

Since $l(p_1) = 1$ and D_2 is special, we have[2] $l(p_1 + p_2) = 2$, so actually

$$l(K - p_1 - p_2) = g - 1 = l(K - p_1)$$

[1] Although it certainly depends on the choice of basepoint

[2] Recall dimension can grow by at most one when we allow an extra pole

Now since the zeroes of a meromorphic function form a discrete set, we can choose a point q_2 arbitrarily close to p_2 at which not all elements of $\mathcal{L}(K - p_1)$ vanish. Then the divisor $D_2' = p_1 + q_2$ is non-special, since

$$l(K - p_1 - q_2) < l(K - p_1) = g - 1$$

which implies $l(p_1 + q_1) < 2$. If $\deg D \leq g$, we can continue this process until we obtain a non-special divisor $D = q_1 + \ldots + q_g$ where at each step q_i could be chosen arbitrarily close to p_i. Hence, a degree g divisor D in general position will be non-special, so $l(D) = 1$. We will exploit this fact in the proof of the trisecant identity.

Theta functions

We finally come to the definition of the theta function.

Definition 5. *(Theta function)* Let $\mathcal{B}$ be a symmetric $g \times g$ matrix with complex entries whose imaginary part is positive definite. The **theta function** associated to $\mathcal{B}$ is the holomorphic function $\theta : \mathbb{C}^g \to \mathbb{C}$ defined by the multidimensional Fourier series

$$\theta(z) = \sum_{n \in \mathbb{Z}^g} e^{\pi i n^t \mathcal{B} n} e^{2\pi i n^t z}$$

Because of the positive definiteness of $\text{Im}\mathcal{B}$, this series converges uniformly on compact sets and thus defines a holomorphic function on $\mathbb{C}^g$.

By Corollary (2) the matrix of $\mathcal{B}$-periods defines a theta function associated to a Torelli marked Riemann surface. Observe that although the theta function is not invariant with respect to translation by the period lattice $\Lambda = \mathbb{Z}^g + \mathcal{B}\mathbb{Z}^g$, in view of the symmetry of $\mathcal{B}$ it transforms by a simple multiplicative factor under such a translation: if $l \in \mathbb{Z}^g$, we have

$$\theta(z + l) = \theta(z) \tag{6}$$

$$\theta(z + \mathcal{B}l) = e^{-\pi i l^t \mathcal{B} l - 2\pi i l^t z} \theta(z) \tag{7}$$

One reason for the importance of theta functions in the theory of compact Riemann surfaces is that they can be combined with the Abel map to build multivalued meromorphic functions. Indeed, if U is a simply connected neighborhood of the basepoint of the Abel map, for any vector $c \in \mathbb{C}^g$ the formula

$$f(p) = \theta(\mathcal{A}(p) + c)$$

defines a holomorphic function on U. The problem is that because of the transformation rule (7), the function $f(p)$ transforms by a multiplicative factor when analytically continued around the cycle b_j of Σ, and so does not exted to a single valued holomorphic function on Σ. However, the function f is single valued on the cut Riemann surface Σ°. Riemann proved the following theorem characterizing the zeros of the function $f(p)$:

Theorem 2. *(Riemann's theorem) The function $f(p) = \theta(\mathcal{A}(p)+c)$ either vanishes identically in $p \in \Sigma$ or has exactly g zeroes $p_1, \ldots, p_g \in \Sigma$ such that the following equality holds in $J(\Sigma)$:*

$$\mathcal{A}(p_1) + \ldots + \mathcal{A}(p_g) + c + \mathcal{K} = 0 \tag{8}$$

where $\mathcal{K} \in J(\Sigma)$ is called the ***Riemann point****; this point depends on the curve Σ, its Torelli marking, and the basepoint P_0 of the Abel map, but not on the vector $c \in \mathbb{C}^g$.*

Proof. If the holomorphic function $f(p)$ is not identically zero on Σ°, by the argument principle its number of zeros can be computed as the integral

$$\frac{1}{2\pi i}\oint_{\partial\Sigma^\circ}\frac{df}{f}$$

If we write $\mathcal{A}_k$ for the k-th component of the Abel map, by definition we have $d\mathcal{A}_k = \omega_k$. Using the transformation rule (7), we see that the integrals of df/f along the cycles b_j, b_j^{-1} cancel. On the other hand, along the a-cycles we have

$$\oint_{a_j a_j^{-1}}\frac{df}{f} = \oint_{a_j} d\mathcal{A}_j = \oint_{a_j}\omega_j = 1$$

Summing over all g of the a-cycles yields

$$\frac{1}{2\pi i}\oint_{\partial\Sigma^\circ}\frac{df}{f} = g$$

as claimed. To prove the remaining assertion, consider the integrals

$$\frac{1}{2\pi i}\oint_{\partial\Sigma^\circ}\mathcal{A}_j\frac{df}{f} = \sum_{k=1}^{g}\mathcal{A}_j(p_g)$$

where $p_1, \ldots, p_g$ are the zeroes of f. But once again explicitly computing the integral as a sum of integrals over the cycles $\{a_i, b_i\}$ yields

$$\frac{1}{2\pi i}\oint_{\partial\Sigma^\circ}\mathcal{A}_j\frac{df}{f} = -c_j - \mathcal{A}_j(Q) + \sum_{k=1}^{g}\oint_{a_k}\mathcal{A}_j\omega_k$$

modulo Λ, where Q is the common basepoint of the loops $\{a_i, b_i\}$. Equating the two expressions for the integral, we obtain Riemann's formula with the coordinates of the point $\mathcal{K}$ given (up to periods) by

$$\mathcal{K}_i = \mathcal{A}_i(q) + \frac{1}{2}\mathcal{B}_{ii} - \sum_{k=1}^{g}\oint_{a_k}\mathcal{A}_i(p)\omega_k \tag{9}$$

which is manifestly independent of $c \in \mathbb{C}^g$. □

The vectors c for which the function $f(p) = \theta(\mathcal{A}(p) + c)$ is identically zero can be characterized as follows:

Proposition 2. *The function $f(p) = \theta(\mathcal{A}(p) + c)$ is identically zero on Σ° if and only if there exists a* ***special*** *divisor $D = q_1 + \ldots + q_g$ such that*

$$\mathcal{A}(q_1) + \cdots + \mathcal{A}(q_g) + c + \mathcal{K} = 0$$

in $J(\Sigma)$.

For the proof we point the reader to Farkas and Kra's book [6]. Combining this proposition, the previous theorem and Abel's theorem, we have

Theorem 3. *Let $c \in \mathbb{C}^g$ be a vector such that the function $F(p) = \theta(\mathcal{A}(p) + c)$ is not identically zero on Σ°. Then*

(a) The divisor $(F) = p_1 + \cdots + p_g$ is non-special.
(b) The points $p_1, \ldots, p_g$ are uniquely determined from equation (8) up to reordering.

Proof. Item (a) follows immediately from the Proposition. For (b), given two divisors $D = p_1 + \ldots + p_g$, $D' = p_1' + \ldots + p_g'$ satisfying (8), we would have $\mathcal{A}(D - D') = 0$ in $J(\Sigma)$. Hence by Abel's theorem there exists a meromorphic function h on Σ with $(h) + D \geq 0$. If D and D' are not identical, this function is nonconstant (it has poles), which contradicts the fact that D is non-special. □

By the transformation rule (7), the zero locus

$$\Theta = \{c \in J(\Sigma) | \theta(c) = 0\}$$

of the theta theta is a well-defined subvariety of $J(\Sigma)$ of dimension $g-1$ which admits the following parameteriztion via the Abel map:

Theorem 4. *A vector $c \in \mathbb{C}^g$ satisfies $\theta(c) = 0$ if and only if there exist $g-1$ points $p_1, \ldots, p_{g-1}$ on Σ such that*

$$c = -\mathcal{K} - \mathcal{A}(p_1) - \ldots - \mathcal{A}(p_{g-1})$$

in $J(\Sigma)$.

Proof. Suppose first that $\theta(c) = 0$. Set $f(q) = \theta(\mathcal{A}(q) + c)$. If f is identically zero, there exists a special divisor $D = r_1 + \ldots + r_g$ with

$$c + \mathcal{A}(D) + \mathcal{K} = 0$$

Since D is special, we can construct a meromorphic function h whose only poles are $s_1, \ldots, s_g$, and which has a zero at the basepoint Q of the Abel map. If $D' = Q + r_1 + \ldots + r_{g-1}$ is the divisor of zeros of h, then by Abel's theorem we have $\mathcal{A}(D') = \mathcal{A}(D)$. Since $\mathcal{A}(Q) = 0$, we see that

$$c + \mathcal{K} + \mathcal{A}(r_1) + \cdots + \mathcal{A}(r_{g-1}) = 0$$

On the other hand, if $f(q)$ does not vanish identically it has g zeroes $p_1, \ldots, p_g$ which are the unique solution to

$$c + \mathcal{A}(p_1) + \cdots + \mathcal{A}(p_g) + \mathcal{K} = 0$$

Now since $\theta(c) = 0$, the basepoint of the Abel map must be one of these zeroes, and the 'forwards' direction follows.

For the 'backwards' direction, if $c = -\mathcal{K} - \mathcal{A}(p_1) - \ldots - \mathcal{A}(p_{g-1})$, again consider $f(q) = \theta(\mathcal{A}(q) + c)$. Then if f is identically zero, evaluation at the basepoint of the Abel map shows $\theta(c) = 0$. If f does not vanish identically, it has g zeroes $q_1, \ldots, q_g$ such that

$$\mathcal{A}(q_1) + \ldots + \mathcal{A}(q_g) - \mathcal{A}(p_1) - \ldots \mathcal{A}(p_{g-1}) - \mathcal{A}(Q) = 0$$

where again Q is the basepoint of Abel's map. But then Abel's theorem implies the divisor $q_1 + \ldots + q_g$ is special, which by the Proposition implies $f(q) \equiv 0$. $\square$

Example: What do these theorems say when $\Sigma = \mathbb{C}/\{1, \tau\}$ has genus 1? Define a Torelli marking of Σ with the a-cycle being $[0,1] \subset \mathbb{C}$, and the b-cycle being $\{t \cdot \tau | t \in [0,1]\} \subset \mathbb{C}$. Then $\{dz\}$ is a normalized basis for the space of holomorphic differentials on Σ, so the Abel map with basepoint $z = x$ is the map $z \mapsto z - x$. By formula (9), the Riemann point is

$$\mathcal{K} = \frac{1+\tau}{2} - x$$

Since any degree 1 positive divisor on Σ is non-special, the function $\theta(z+c)$ is never identically zero; it always has one zero z_0 with the property that

$$z_0 + c + \mathcal{K} = 0$$

This equation allows us to interpret the zeros of the genus 1 theta function geometrically in terms of the plane cubic model $C \subset \mathbb{P}^2$ of Σ. Let $\pi : \Sigma \to C$ be the isomorphism

$$z \longmapsto [\wp(z) : \wp'(z) : 1]$$

Let $E = [0 : 0 : 1] = \pi(0)$ be the identity element of C and let $R = \pi(\mathcal{K})$ be the image of the Riemann point. Let $\overline{ER}$ be the line spanned by E and R. Then the zero of the theta function $\theta(z)$ is the third point of the intersection of the line $\overline{ER}$ with C. More generally, for any point $P \in C$, the zero of $\theta(z + \pi^{-1}(P))$ is given by the third intersection point of the line $\overline{PR}$ with C.

The cross-ratio function

Let p_1, p_2, p_3, p_4 be any four distinct points on the projective line $\mathbb{P}^1$. Recall that there is a unique projective (Mobius) transformation $g \in PGL_2$ such that $g(p_2) = [1 : 1], g(p_3) = [0 : 1]$ and $g(p_4) = [1 : 0]$. Then since g is a bijection, we have $g(p_1) = [a : 1]$ for some a. The **cross-ratio** of these four points is defined to be

$$\rho(p_1, p_2; p_3, p_4) = a$$

Hence the cross-ratio function is a meromorphic function on an open subset of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, which we can extend to a meromorphic function on the whole space. If we choose a local coordinate $[z : 1]$ on $\mathbb{P}^1$, then it is easy to check that ρ is given by the formula

$$\rho(z_1, z_2;_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$$

The cross-ratio function ρ has the following symmetries:

(i) $\rho(p_1, p_2; q_1, q_2) = \rho(q_1, q_2; p_1, p_2) = \rho(p_2, p_1; q_2, q_1)$
(ii) $\rho(p_2, p_1; q_1, q_2) = \rho(p_1, p_2; q_1, q_2)^{-1}$
(iii) $\rho(p_1, p_2; q_1, q_2)\rho(p_1, q_2; p_2, q_1)\rho(p_1, q_1; q_2, p_2) = 1$

Let us also observe that the cross-ratio ρ also has the following analytic properties as a meromorphic function of $p_1 \in \mathbb{P}^1$: it has a simple pole at q_2 and a simple zero at q_1, and no other zeros and poles; moreover $\rho \equiv 1$ when $q_1 = q_2$. Since the ratio of two functions with these analytic properties would be a holomorphic function on $\mathbb{P}^1$ and thus the constant function 1, *these properties completely determine the cross-ratio function ρ.*

A natural question to ask is whether there exists an analog of the cross-ratio function on a compact Riemann surface of positive genus. It turns out that there does exist such a function, and it can be constructed as follows. Let Q be the basepoint of the Abel map, and choose any $g - 1$ points $y_1, \dots, y_{g-1}$ such that the divisor $D = y_1 + \dots + y_{g-1} + Q$ is non-special. Then $\alpha = -K - \mathcal{A}(D)$ is a point of the theta divisor. Moreover, the function

$$e(p, q) = \theta(\mathcal{A}(p) - \mathcal{A}(q) + \alpha) \tag{10}$$

is not identically zero, and considered as a function of p has divisor $q + y_1 + \ldots + y_{g-1}$. Such an α is called a **non-singular**[3] point of the theta divisor.

Definition 6. (*Cross-ratio function*) The **cross-ratio function** of Σ is the meromorphic function on $(\Sigma^\circ)^4$ defined by

$$\rho(p_1, p_2; q_1, q_2) = \frac{e(p_1, q_1)e(p_2, q_2)}{e(p_1, q_2)e(p_2, q_1)}$$

It is not immediately obvious that $\rho(p_1, p_2; q_1, q_2)$ has all of the symmetry properties (i-iii) of the genus 0 cross-ratio, although it obviously shares some of them. But note that the cross-ratio has the following analytic properties as a function of p_1:

(1) As a function of p_1, ρ has divisor $q_1 - q_2$, and $\rho = 1$ when $q_1 = q_2$.

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[3]For the reason behind this terminology, see Lemma VI.3.5 in [6]

(2) ρ is invariant under after passing around any a-cycle, but after passing around the cycle b_j transforms via

$$\rho \longmapsto e^{2\pi i \mathcal{A}_j(q_1-q_2)}\rho$$

where $\mathcal{A}_j$ denotes the j-th component of the Abel map.

By the same argument as in the genus zero case, *the cross-ratio function is uniquely determined by these analytical properties*. Hence we have the remarkable fact that definition of the cross-ratio is *independent* [2] of the choice of non-singular point $\alpha \in \Theta$ used to define the function $e(p, q)$. In particular, we can choose α to be a so-called **odd, non-singular characteristic**: such a α has the property that the function $\theta_\alpha(z) = \theta(z + \alpha)$ is an odd function of z, so we have $e(p, q) = -e(q, p)$. In what follows, we will always assume $e(p, q)$ has been defined using a non-singular odd characteristic, and it thus antisymmetric. With this in mind, is it is straightforward to see that ρ in fact has all of the symmetries (i-iii) of the genus zero cross-ratio function.

Fay's trisecant identity

Recall that the genus zero cross-ratio function ρ_0 satisfies the following simple addition formula:

$$\rho_0(p_1, p_2; q_1, q_2) + \rho_0(p_1, q_1; p_2; q_2) = 1 \tag{11}$$

If we replace ρ_0 by the cross-ratio function of a Riemann surface of genus $g > 0$, this simple identity fails to hold. There is, however, another identity satisfied by the higher genus cross-ratio function which can be viewed as a generalization of formula (11):

Theorem 5. *(Fay's trisecant identity [1]) Let $c \in \mathbb{C}^g$ be such that $\theta(c) \neq 0$. Then the following identity holds:*

$$\begin{aligned} &\theta\left(\int_{q_1}^{p_1} \omega + c\right)\theta\left(\int_{q_2}^{p_2} \omega + c\right)\rho(p_1, q_1; q_2, p_2) \\ &+ \theta\left(\int_{q_2}^{p_1} \omega + c\right)\theta\left(\int_{q_1}^{p_2} \omega + c\right)\rho(p_1, q_2; q_1, p_2) \\ &\qquad = \theta(c)\theta\left(\int_{q_1+q_2}^{p_1+p_2} \omega + c\right) \end{aligned} \tag{12}$$

where we use the compact notation

$$\int_q^p \omega = \mathcal{A}(p) - \mathcal{A}(q)$$

for the Abel map.

Proof. We consider both sides as functions of $p_1 \in \Sigma$. Inserting the explicit formula for the cross-ratio function and rearranging some terms, Fay's identity becomes

$$\begin{aligned} &\theta(c)\theta\left(\int_{q_1+q_2}^{p_1+p_2} \omega + c\right)\frac{e(p_1, p_2)e(q_2, q_1)}{\prod_{i,j=1}^2 e(p_i, q_j)} \\ &\qquad = \frac{\theta\left(\int_{q_1}^{p_1} \omega + c\right)\theta\left(\int_{q_2}^{p_2} \omega + c\right)}{e(p_1, q_1)e(p_2, q_2)} - \frac{\theta\left(\int_{q_2}^{p_1} \omega + c\right)\theta\left(\int_{q_1}^{p_2} \omega + c\right)}{e(p_1, q_2)e(p_2, q_1)} \end{aligned} \tag{13}$$

One observes that both sides of this equation have the same monodromy around a and b-cycles. Thus the quotient $h(p_1) = LHS/RHS$ is a *meromorphic function* on Σ. Moreover, the divisor of the LHS as a function of p_1 is $(LHS) = D + p_2 - q_1 - q_2$, where D is a positive degree g divisor coming from the nonconstant theta on the

LHS. On the other hand, we can see that the only poles of the RHS are when $p_1 = q_1, q_2$, so it must have divisor $(RHS) = E + p_2 - q_1 - q_2$ where E is another positive degree g divisor. Hence the quotient h has divisor $(h) = D - E$. But since

$$\mathcal{A}(D) + \mathcal{A}(p_2 - q_1 - q_2) + c + \mathcal{K} = 0$$

we see that for p_2, q_1, q_2 in general position $\mathcal{A}(D) \notin \Theta$, which means D is non-special so h is constant. It remains only to show this constant is in fact 1; this can be done straightforwardly by considering the behavior of both sides as $p_1 \to q_1$ while $q_1 \neq q_2$. □

Determinantal forms and degenerations

Observe that equation (13) can be written in the following compact determinantal form

$$\theta(c)\theta\left(\int_{q_1+q_2}^{p_1+p_2} \omega + c\right) \frac{e(p_1,p_2)e(q_2,q_1)}{\prod_{i,j=1}^{2} e(p_i,q_j)} = \begin{vmatrix} \frac{\theta\left(\int_{q_1}^{p_1} \omega+c\right)}{e(p_1,q_1)} & \frac{\theta\left(\int_{q_2}^{p_1} \omega+c\right)}{e(p_1,q_2)} \\ \frac{\theta\left(\int_{q_1}^{p_2} \omega+c\right)}{e(p_2,q_1)} & \frac{\theta\left(\int_{q_2}^{p_2} \omega+c\right)}{e(p_2,q_2)} \end{vmatrix} \tag{14}$$

In fact, by an identical argument to the one used to prove the trisecant identity one can show that the following "$2N$-point formula" holds:

Theorem 6. *(Multisecant identity [1]) Let $c \in \mathbb{C}^g$ be such that $\theta(c) \neq 0$. Then the following identity holds:*

(15)

$$\theta(c)^{N-1}\theta\left(\int_{q_1+\ldots+q_N}^{p_1+\ldots+p_N} \omega + c\right) \frac{\prod_{i<j} e(p_i,p_j)e(q_j,q_i)}{\prod_{i,j=1}^{N} e(p_i,q_j)} = \det\left[\frac{\theta\left(\int_{q_j}^{p_i} \omega + c\right)}{e(p_i,q_j)}\right]_{i,j=1,\ldots,N}$$

On the Riemann sphere, the function $p - q$ can be thought of playing the role of $e(p,q)$. In this spirit, this formula can be thought of as a higher genus generalization of the Cauchy determinant evaluation

$$\frac{\prod_{i<j}(p_i-p_j)(q_j-q_i)}{\prod_{i,j=1}^{N}(p_i-q_j)} = \det\left[\frac{1}{p_i-q_j}\right]_{i,j=1,\ldots,N}$$

We can also consider a degeneration of formula (15) as the points q_i merge to the basepoint Q of the Abel map. In order to avoid dealing with derivatives of theta functions, we will state this formula in a slightly different fashion. Let D be a non-special positive divisor of degree g. This means that the vector space $\mathcal{L}(-D-(N-1)Q)$ is N-dimensional, and that we can choose a basis $\{h_0, \ldots, h_{N-1}\}$ normalized so that

$$h_k(z) = z^{-k} + O(z^{-k+1})$$

Then the following formula holds:

$$\frac{\theta(c)^{N-1}\theta\left(\sum_{k=1}^{N}\mathcal{A}(p_k)+c\right)}{\prod_{i=1}^{N}\theta\left(\mathcal{A}(p_i)+c\right)} \prod_{i<j}\frac{e(p_i,Q)}{e(p_i,p_j)} = \begin{vmatrix} 1 & h_1(p_1) & \cdots & h_{N-1}(p_1) \\ 1 & h_1(p_2) & \cdots & h_{N-1}(p_2) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & h_1(p_N) & \cdots & h_{N-1}(p_N) \end{vmatrix} \tag{16}$$

We can think of this formula as the higher genus analog of the Vandermonde determinant evaluation

$$\prod_{i<j}(p_i-p_j) = \det\left[p_i^{j-1}\right]_{i,j=1,\ldots,N}$$

The case of genus 1

We make a few remarks about the determinantal formulae when $g = 1$. In this case, there is a function closely related[4] to the theta function called the **Weierstrass sigma function**. The sigma function is related to the $\wp$ function by

$$\wp(z) = -\frac{d^2}{dz^2}\log\sigma(z)$$

If we rewrite equation (15) in terms of the sigma function, we obtain the classical Cauchy formula [4]

$$\sigma(c)^{N-1}\sigma\left(\sum_{k=1}^{N}(x_k - y_k) + c\right)\frac{\prod_{i<j}\sigma(x_i - x_j)\sigma(y_j - y_i)}{\prod_{i,j=1}^{N}\sigma(x_i - y_j)} = \det\left[\frac{\sigma\left(x_i - y_j + c\right)}{\sigma(x_i - y_j)}\right]_{i,j=1,\ldots,N}$$

This formula remains true if we replace everywhere the function $\sigma(z)$ by the functions $f(z) = \sin(\pi z)$ or $h(z) = z$; these cases correspond to *singular* genus 1 curves. The formula (16) becomes the classical Frobenius-Stickelberger addition formula [1]

$$C_N\sigma\left(\sum_{k=1}^{N} z_k\right)\frac{\prod_{i<j}\sigma(z_i - z_j)}{\prod_{i=1}^{N}\sigma^{N-1}(z_i)} = \begin{vmatrix} 1 & \wp(z_1) & \wp'(z_1) & \cdots & \wp^{(N-1)}(z_1) \\ 1 & \wp(z_2) & \wp'(z_2) & \cdots & \wp^{(N-1)}(z_2) \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & \wp(z_N) & \wp'(z_N) & \cdots & \wp^{(N-1)}(z_N) \end{vmatrix} \tag{17}$$

where the constant $C_N = (-1)^{\frac{1}{2}N(N-1)}1!2!\cdots N!$. Using the fact that the sigma function is odd, so $\sigma(0) = 0$, we see that this formula is really a generalization of problem 3 (addition formula for $\wp$) on Math 255 Homework 3.

The Schottky problem and the meaning of 'trisecant'

It would be remiss to conclude without giving some explanation of why the name 'trisecant' is attached to Fay's identity. Recall that *any* complex symmetric matrix Ω with positive definite imaginary part defines a lattice $\Lambda = \mathbb{Z}^g + \Omega\mathbb{Z}^g$ of rank $2g$ in $\mathbb{C}^g$, a complex torus $X = \mathbb{C}^g/\Lambda$ and a theta function $\theta(z)$. Such a complex torus is called an **abelian variety**. If the matrix Ω is unimodular, the abelian variety X is called **principally polarized**. A natural question to ask is which abelian varieties arise as Jacobians of some Riemann surface; this is, roughly speaking, what is known as the **Schottky problem** in classical algebraic geometry. Given a principally polarized abelian variety X, one can use the so-called second-order theta functions to embed the quotient $X/\{\pm 1\}$ into the projective space $\mathbb{P}^{2^g-1}$; the image of X under this embedding is called the **Kummer variety** of X. In this context, Fay's identity (12) has a remarkable geometric interpretation: it says that given any four points p, p_1, p_2, p_3 on Σ the images of the following points of $J(\Sigma)$

$$\mathcal{A}(p_1 + p_2 - q_1 - q_2),\ \mathcal{A}(p_1 + q_1 - p_2 - q_2),\ \mathcal{A}(p_1 + q_2 - p_2 - q_1)$$

on the Kummer variety are **collinear** in $\mathbb{P}^{2^g-1}$. Hence the Kummer variety of a Jacobian admits a four-dimensional family of trisecant lines parameterized by the four points of Σ [7]. Moreover, it was proved by Gunning [3] that this property of Jacobians essentially characterizes them among all abelian varieties, thereby providing a solution to (one version of) the Schottky problem. But this is the subject for another term paper...

[4] For the precise relation between the two, see [5]

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