Classroom notes

A measure blowup

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Last year Sam Chow and Gus Schrader were third-year students at the University of Melbourne enrolled in Linear Analysis, a subject involving — among other topics — Lebesgue measure and integration. This was based on parts of the recently published book *Metrics, Norms and Integrals* by the lecturer in the subject, Associate Professor Jerry Koliha. In addition to three lectures a week, the subject had a weekly practice class. The lecturer introduced an extra weekly practice class with a voluntary attendance, but most students chose to attend, showing a keen interest in the subject. The problems tackled in the extra practice class ranged from routine to very challenging. The present article arose in this environment, as Sam and Gus independently attacked one of the challenging problems.

Working in the Euclidean space $\mathbb{R}^d$ we can characterise Lebesgue measure very simply by relying on the Euclidean volume of the so-called *d-cells.* A d-cell C in $\mathbb{R}^d$ is the Cartesian product

$$C = I_1 \times \cdots \times I_d$$

of bounded nondegenerate one-dimensional intervals I_k; its Euclidean d-volume is the product of the lengths of the I_k. *Borel sets* in $\mathbb{R}^d$ are the members of the least family $\mathcal{B}^d$ of subsets of $\mathbb{R}^d$ containing the empty set, $\emptyset$, and all d-cells which are closed under countable unions and complements in $\mathbb{R}^d$. Since every open set in $\mathbb{R}^d$ can be expressed as the countable union of suitable d-cells, the open sets are Borel, as are all sets obtained from them by countable unions and intersections, and complements. The Lebesgue measure $m = m_d$ is a countably additive set function $m\colon \mathcal{B}^d \to [0,\infty]$ satisfying $m(\emptyset) = 0$ and coinciding with the Euclidean d-volume on d-cells; such a function is unique and monotonic with respect to the set inclusion. We do not need to know how to calculate the Lebesgue measure of a Borel set to get quite far on this characterisation alone.

The faces of d-cells are Borel sets and their d-dimensional Lebesgue measure is zero; that much can be obtained by embedding them in suitable d-cells and using the monotonicity of the Lebesgue measure. In other words, the boundaries of d-cells

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have Lebesgue measure zero. It is tempting to think that open sets in $\mathbb{R}^d$ (countable unions of d-cells) also have boundaries of Lebesgue measure zero. There do not seem to be obvious open set candidates to counter this.

However, Sam came up with this ingenious example.

Example 1. Let S be the 'fat Cantor set' obtained by removing the open middle quarter from the interval $[0,1]$, and then proceeding to remove the open subinterval of length $(\frac{1}{2})^{2n}$ from the middle of each of 2^{n-1} remaining intervals for $n = 2, 3, \ldots$. Then $G_1 = [0,1] \setminus S$ is open in $\mathbb{R}$, and the Lebesgue measure of G is $m(G_1) = \sum_{n=1}^{\infty} 2^{n-1}(\frac{1}{2})^{2n} = \frac{1}{2}$. Then $m(S) = \frac{1}{2}$, while S is the boundary of G_1. The set $G = G_1 \times (0,1) \times \cdots \times (0,1)$ (with $d-1$ factors $(0,1)$) is open in $\mathbb{R}^d$ with the boundary $\partial G = S \times (0,1) \times \cdots \times (0,1)$, where $m(G) = \frac{1}{2} = m(\partial(G))$.

Gus found an even more dramatic result.

Theorem 1. (i) *Given $\varepsilon > 0$ there exists an open set $G \subset \mathbb{R}^d$ with $m(G) \le \varepsilon$ and $m(\partial G) = \infty$.*

(ii) *Given $\varepsilon > 0$ and $0 < M < \infty$, there exists a bounded open set $G \subset \mathbb{R}^d$ with $m(G) \le \varepsilon$ and $m(\partial G) \ge M$.*

Proof. Let $\varepsilon > 0$, $0 < M \le \infty$, and let F be a closed subset of $\mathbb{R}^d$ with Lebesgue measure $m(F) \ge M + \varepsilon$; this is always possible ($\infty + \varepsilon = \infty$ by convention). Let A be the set of all points in F with rational coordinates; this set is countable and dense in F, that is, $\overline{A} = F$. We can order the points of A in a sequence, $A = \{q_n : n = 1, 2, \ldots\}$. Let B_n be an open d-cell in $\mathbb{R}^d$ containing q_n whose Euclidean volume is $(\frac{1}{2})^n \varepsilon$ (give an explicit construction of B_n — if you are so inclined). Then $G = \bigcup_{n=1}^{\infty} B_n$ is an open set whose Lebesgue measure satisfies

$$m(G) \le \sum_{n=1}^{\infty} m(B_n) = \sum_{n=1}^{\infty} \left(\tfrac{1}{2}\right)^n \varepsilon = \varepsilon.$$

Let $\overline{G}$ be the closure of G. Then $\overline{G} = G \cup \partial G$ is a disjoint union, and

$$m(\overline{G}) = m(G) + m(\partial G).$$

From $F = \overline{A} \subset \overline{G}$ it follows that $m(\overline{G}) \ge m(F) \ge M + \varepsilon$. Assertion (i) follows when we choose $M = \infty$: then $m(\overline{G}) = \infty = m(\partial G)$. To prove (ii) with finite M and bounded G, we choose F in the foregoing argument bounded and satisfying $m(F) \ge M + \varepsilon$. Then G, $\overline{G}$ and ∂G are also bounded and therefore of finite Lebesgue measure, so that $m(\partial G) = m(\overline{G}) - m(G) \ge (M + \varepsilon) - \varepsilon = M$.

Note: When requiring G in part (i) of the preceding theorem to be bounded we can no longer achieve $m(\partial G) = \infty$.

For more detail on the 'fat Cantor set' see Wikipedia
http://en.wikipedia.org/wiki/Smith-Volterra-Cantor_set.

Both Sam and Gus feel that although the Linear Analysis course was one of the most challenging they had taken, the enthusiasm and encouragement of the lecturer, and the opportunity to tackle challenging problems in the practice classes, made what would be an initimidating subject more accessible. Their experience of independent research in these practical classes led them to a greater appreciation of the subtleties of a subject as rich as analysis, and improved their understanding of mathematics dramatically.

Sam is a pure maths student at the University of Melbourne, hoping to study honours next year. He has always loved mathematical problem solving, and represented Australia in the 2005 IMO. Chess is his main hobby, often travelling overseas to play in tournaments. Other hobbies include soccer, squash, badminton and poker.

Gus Shrader is originally from Adelaide. At the time this note was written he was a third-year maths major at the University of Melbourne. He is now an Honours student in pure maths, and his Honours project is on algebraically completely integrable systems, in particular the so-called Hitchin systems.

Jerry Koliha is Associate Professor in the Mathematics and Statistics Department of the University of Melbourne. He specialises in Functional Analysis. He says that every year he learns from his students as much as they learn from him. At the end of this year, Jerry will be retiring from the University of Melbourne after 40 uninterrupted years. He can think of no better finish to his career than knowing he has inspired his students to appreciate mathematical analysis so much.