Towards a modular functor from quantum higher Teichmüller theory

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Talk based on joint work with Alexander Shapiro (UToronto).

Slides available online: `www.math.berkeley.edu/~guss`

Quantum group $\mathrm{U_q}(\mathfrak{g})$

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$.

The *quantum group* $\mathrm{U_q}(\mathfrak{g})$ is a Hopf algebra deformation of the enveloping algebra $\mathrm{U}(\mathfrak{g})$.

Coproduct in $\mathrm{U_q}(\mathfrak{g})$ is no longer co-commutative; instead $\mathrm{U_q}(\mathfrak{g})$ is *quasi-triangular* : it has an R-matrix $R \in \mathrm{U_q}(\mathfrak{g}) \otimes \mathrm{U_q}(\mathfrak{g})$ satisfying

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \in \mathrm{U_q}(\mathfrak{g})^{\otimes 3},$$
$$\Delta^{op} = \mathrm{Ad}_R \circ \Delta,$$

where Δ^{op} is the opposite coproduct.
This means $\mathrm{U_q}(\mathfrak{g})$ can be used to construct interesting braided tensor categories.

Jones-Reshetikhin-Turaev-Witten construction

One interesting class of such categories comes from finite dimensional representations of $\mathrm{U_q}(\mathfrak{g})$ when q is a root of unity.

Reshetikin and Turaev used this category to define a 3d-TQFT, which yields invariants of 3-manifolds and framed links. When $\mathfrak{g} = \mathfrak{sl}_2$, they recover Jones polynomial.

Witten: construct these invariants from geometric quantization of Chern-Simons theory with compact gauge group K. If $K = SU_2$, again obtain Jones polynomials.

Question: Is there an analagous construction for Chern-Simons theory with split real gauge group, e.g. $G = SL_n(\mathbb{R})$?

Moduli spaces of framed local systems

A *marked surface* $\widehat{S}$ is a compact oriented surface S with a finite set $\{x_1, \ldots, x_k\} \subset \partial S$ of marked boundary points.

Its punctured boundary is $\partial \widehat{S} := \partial S \setminus \{x_1, \ldots, x_k\}$.

Fix $G = PGL_n\mathbb{C}$ and $B \subset G$ a Borel. A *framed G-local system* on $\widehat{S}$ is:

1. a G-local system $\mathcal{L}$ on S, together with
2. a flat section β of the restriction to the punctured boundary of the associated flag bundle $(\mathcal{L} \times_G G/B)|_{\partial \widehat{S}}$.

Definition

$$\mathcal{X}_{G,\widehat{S}} := \text{moduli of framed } G\text{-local systems on } \widehat{S}$$

$\mathcal{X}_{G,\widehat{S}}$ is a cluster variety

Fock-Goncharov: $\mathcal{X}_{G,\widehat{S}}$ is a *cluster Poisson variety:* it is covered up to codimension 2 by an atlas of toric charts

$$\mathcal{T}_{\Sigma} : (\mathbb{C}^*)^d \longrightarrow \mathcal{X}_{G,\widehat{S}},$$

labelled by quivers Q_{Σ}. Poisson brackets determined by signed adjacency matrix ϵ_{jk} of Q_{Σ}:

$$\{x_j, x_k\} = \epsilon_{jk} x_j x_k.$$

Different charts are related by subtraction-free birational transformations called *cluster mutations*.

Higher Teichmüller spaces

Subtraction-free gluing maps means there is well-defined notion of totally positive points $\mathcal{X}^+_{G,\widehat{S}} \subset \mathcal{X}_{G,\widehat{S}}$. (It just means points where all toric coordinates are positive.)

When $G = PGL_2\mathbb{C}$, $\mathcal{X}^+_{G,\widehat{S}}$ is identified with a component in moduli space $\mathcal{M}_{flat}(\widehat{S}, PSL_2\mathbb{R})$ isomorphic with Teichmüller space.

So $\mathcal{X}^+_{G,\widehat{S}}$ is higher rank analog of Teichmüller space.

Canonical quantization of cluster varieties

Promote chartwise Poisson structure to collection of quantum torus algebras

$$\mathcal{T}_\Sigma^q \quad : \quad x_j x_k = q^{2\epsilon_{jk}} x_k x_j.$$

Quantum mutation in direction k is now defined in terms of conjugation by *quantum dilogarithm*

$$\Gamma_q(x) = \prod_{n=1}^{\infty} \frac{1}{1+q^{2n+1}x}.$$

Cluster realization of $\mathrm{U}_q(\mathfrak{sl}_n)$

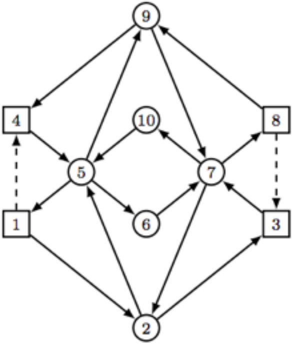
Theorem (S.-Shapiro '16)

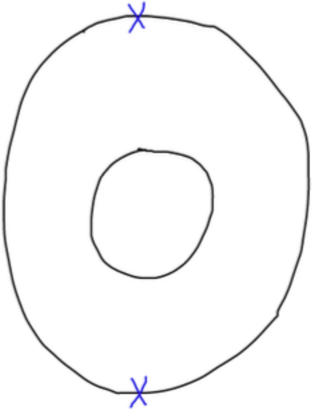
Let $\widehat{S}$ be an annulus with two marked points on one of its boundary components, and let $\mathcal{X}^q_{PGL_n,\widehat{S}}$ be the corresponding quantum cluster algebra. Then there is an embedding of algebras

$$\mathrm{U}_q(\mathfrak{sl}_n) \hookrightarrow \mathcal{X}^q_{PGL_n,\widehat{S}},$$

with the property that for each Chevalley generator $X_i^{\pm}$ of the quantum group, there is a cluster in which $X_i^{\pm}$ is a cluster monomial.

Example: $U_q(\mathfrak{sl}_3)$





Positive representations of $\mathrm{U_q}(\mathfrak{sl}_n)$

Now suppose

$$q = e^{2\pi i \hbar^2}, \quad \hbar^2 \in \mathbb{R}_+ \setminus \mathbb{Q}_+.$$

One can construct infinite dimensional representations of $\mathrm{U_q}(\mathfrak{sl}_n)$ by pulling back representations of the quantum torus algebra $\mathcal{T}^q$.

These representations $\mathcal{P}_\lambda$ are labelled by positive real points λ of a Weyl chamber, and have the property that the quantum group generators act by positive, essentially selfadjoint operators.

Studied when $\mathfrak{g} = \mathfrak{sl}_2$ by Faddeev, Bytsko-Ponsot-Teschner.

For general $\mathfrak{g}$, they have also been constructed by Frenkel-Ip.

Continuous tensor categories

Modular duality: $\mathcal{P}_\lambda$ admits commuting action of $\mathrm{U}_{\mathrm{q}^\vee}(\mathfrak{sl}_n)$, where

$$q = e^{2\pi i \hbar^2}, \quad q^\vee = e^{2\pi i/\hbar^2}.$$

Theorem (Ponsot-Teschner)

For $\mathfrak{g} = \mathfrak{sl}_2$, positive representations form a 'continuous tensor category':

$$\mathcal{P}_{s_1} \otimes \mathcal{P}_{s_2} = \int_{\mathbb{R}_{>0}}^{\oplus} \mathcal{P}_s d\mu(s).$$

The measure is $d\mu(s) = 4\sinh(2\pi\hbar s)\sinh\left(2\pi\hbar^{-1}s\right) ds$.

Continuous tensor categories in higher rank?

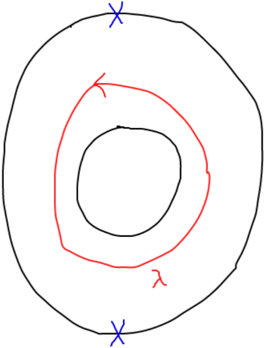
Conjecture (Frenkel-Ip)

The category of positive representations $\mathcal{P}_\lambda$ of $\mathrm{U_q}(\mathfrak{sl}_n)$ is also closed under tensor product.

Geometric approach: cutting and gluing isomorphisms

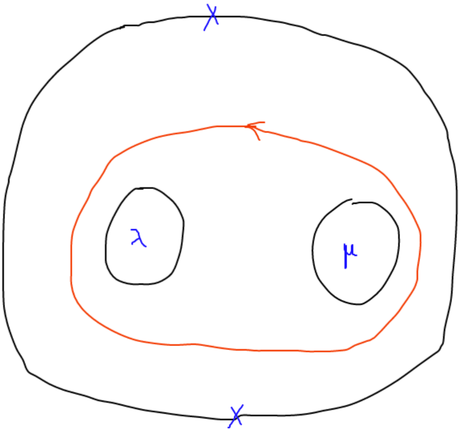
With A. Shapiro, we have a geometric approach to the conjecture using quantum higher Teichmüller theory.

First observation: central character of $\mathcal{P}_\lambda$ determined by monodromy around puncture.



Geometric approach: cutting and gluing isomorphisms

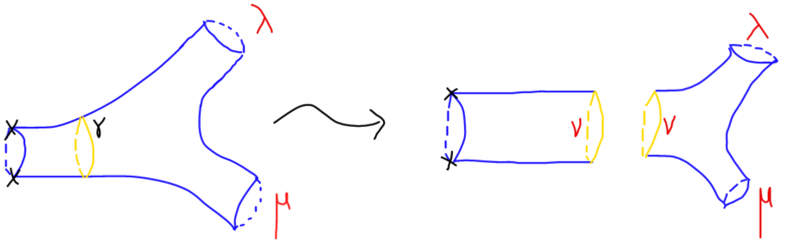
Picture for $P_\lambda \otimes P_\mu$:



Casimirs of $\Delta U_q(\mathfrak{sl}_\mathfrak{n})$ correspond to holonomies around red loop.

Geometric approach: cutting and gluing isomorphisms

Teichmüller theory interpretation of the decomposition of the tensor product of positive representations $P_\lambda \otimes P_\mu$ into positive representations P_ν:



Reduction to q-difference Toda spectral problem

Algebraically: the picture let us read off a natural sequence of cluster transformations, which identifies the traces of holonomies with the Hamiltonians of the q-difference *open Toda lattice.*

The eigenfunctions of these operators are the *q-Whittaker functions.* Their orthogonality and completeness yields the desired the direct integral decomposition.

Towards a modular functor?

We see this as first step towards constructing an infinite dimensional analog of a modular functor from the quantization of higher Teichmüller spaces.

When $\mathfrak{g} = \mathfrak{sl}_2$, this has been investigated by Teschner. On the loop group side, closely related to modular functor coming from Virasoro conformal blocks.

Suggests connection with 4D gauge theory (AGT...)

Will be interesting to complete the construction in higher ranks and compare with the $\mathcal{W}$-algebras construction.

Thanks for listening!