Towards a modular functor from quantum higher Teichmüller theory: cutting and gluing

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Talk based on joint work with Alexander Shapiro (UToronto).

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Quantum group $\mathrm{U_q}(\mathfrak{g})$

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$.

The *quantum group* $\mathrm{U_q}(\mathfrak{g})$ is a Hopf algebra deformation of the enveloping algebra $\mathrm{U}(\mathfrak{g})$.

Coproduct in $\mathrm{U_q}(\mathfrak{g})$ is no longer co-commutative; instead $\mathrm{U_q}(\mathfrak{g})$ is *quasi-triangular* : it has an R-matrix $R \in \mathrm{U_q}(\mathfrak{g}) \otimes \mathrm{U_q}(\mathfrak{g})$ satisfying

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \in \mathrm{U_q}(\mathfrak{g})^{\ 3},$$
$$\Delta^{op} = \mathrm{Ad}_R \circ \Delta,$$

where Δ^{op} is the opposite coproduct.
This means $\mathrm{U_q}(\mathfrak{g})$ can be used to construct interesting braided tensor categories.

Jones-Reshetikhin-Turaev-Witten construction

One interesting class of such categories comes from finite dimensional representations of $\mathrm{U_q}(\mathfrak{g})$ when q is a root of unity.

Reshetikin and Turaev used this category to define a 3d-TQFT, which yields invariants of 3-manifolds and framed links. When $\mathfrak{g} = \mathfrak{sl}_2$, they recover Jones polynomial.

Witten: construct these invariants from geometric quantization of Chern-Simons theory with compact gauge group K. If $K = SU_2$, again obtain Jones polynomials.

Question: Is there an analagous construction for Chern-Simons theory with split real gauge group, e.g. $G = SL_n(\mathbb{R})$?

Moduli spaces of framed local systems

Recall the moduli space $\mathcal{X}_{G,\widehat{S}}$ of framed G-local systems on a marked surface $\widehat{S}$.

When $G = PGL_{n+1}$, $\mathcal{X}_{G,\widehat{S}}$ is a cluster Poisson variety (Fock-Goncharov 2003).

So there is a well-defined set of totally positive points $\mathcal{X}^{+}_{G,\widehat{S}} \subset \mathcal{X}_{G,\widehat{S}}$.

When $G = PGL_2\mathbb{C}$, $\mathcal{X}^{+}_{G,\widehat{S}}$ is identified with a component in moduli space $\mathcal{M}_{flat}(\widehat{S}, PSL_2\mathbb{R})$ isomorphic with Teichmüller space.

So $\mathcal{X}^{+}_{G,\widehat{S}}$ is a higher rank analog of Teichmüller space.

Cluster realization of $\mathrm{U}_q(\mathfrak{sl}_n)$

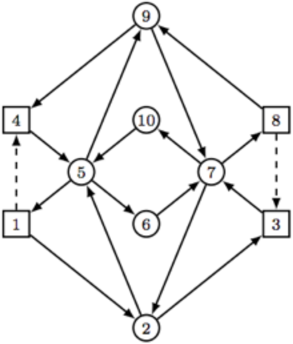
Theorem (S.-Shapiro '16)

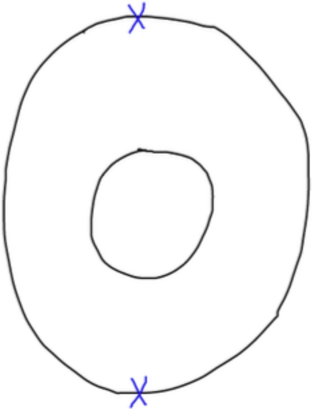
Let $\widehat{S}$ be an annulus with two marked points on one of its boundary components, and let $\mathcal{X}^q_{PGL_n,\widehat{S}}$ be the corresponding quantum cluster algebra. Then there is an embedding of algebras

$$\mathrm{U}_q(\mathfrak{sl}_n) \hookrightarrow \mathcal{X}^q_{PGL_n,\widehat{S}},$$

with the property that for each Chevalley generator E_i, F_i of the quantum group, there is a cluster in which that generator is a cluster monomial.

Example: $U_q(\mathfrak{sl}_3)$





Application to quantum group representations

Question: What does this theorem tell us about representation theory of the quantum group $U_q(\mathfrak{sl}_n)$?

Positive representations of quantum cluster varieties

Idea: The quantum cluster algebra $\mathcal{X}^q_{PGL_n,\widehat{S}} \supset U_q(\mathfrak{sl}_n)$ is a subalgebra in a quantum torus algebra $\mathcal{T}^q$.

Suppose that

$$q = e^{\pi i \hbar^2}, \quad \hbar^2 \in \mathbb{R}_+ \setminus \mathbb{Q}_+.$$

One can construct infinite dimensional representations of $\mathrm{U_q}(\mathfrak{sl}_n)$ by pulling back representations of the quantum torus algebra $\mathcal{T}^q$ in which the cluster variables X_k act by positive, essentially self-adjoint unbounded operators on a Hilbert space.

Positive representations of quantum cluster varieties

We can embed a quantum cluster chart $\mathcal{T}^q$ into a Heisenberg algebra $\mathcal{H}$ generated by $x_1, \ldots, x_d$ with relations

$$[x_j, x_k] = \frac{1}{2\pi i}\epsilon_{jk},$$

by the homomorphism

$$X_j \mapsto e^{2\pi\hbar x_j}.$$

The algebra $\mathcal{H}$ has a family of irreducible Hilbert space representations V_χ parameterized by central characters $\chi \in \mathrm{Hom}(\ker\epsilon, \mathbb{R})$, in which the generators x_j act by unbounded self-adjoint operators.

Modular duality and mutation for $|q| = 1$

Now consider $q^{\vee} = e^{\pi i/\hbar^2}$, obtained from $q = e^{\pi i \hbar^2}$ by the modular S-transformation $\hbar \mapsto 1/\hbar$.
We also have an embedding of $\mathcal{T}^{q^{\vee}}$ into the Heisenberg algebra $\mathcal{H}$ given by

$$\tilde{X}_j = e^{2\pi\hbar^{-1}x_j},$$

so

$$\tilde{X}_k\tilde{X}_j = (q^{\vee})^{2\epsilon_{kj}}\tilde{X}_j\tilde{X}_k.$$

Note that the generators $\tilde{X}_j$ commute with the original ones $X_j = e^{2\pi\hbar x_k}$:

$$\begin{aligned} X_j\tilde{X}_k &= e^{2\pi i\epsilon_{jk}}\tilde{X}_kX_j \\ &= \tilde{X}_kX_j, \end{aligned}$$

since $\epsilon_{jk} \in \mathbb{Z}$.

Quantum total positivity

Cluster mutation in direction k is now realized by conjugation by *non-compact quantum dilogarithm*

$$\Phi^{\hbar}(z) = \frac{\Gamma_q(e^{2\pi\hbar z})}{\Gamma_{q^\vee}(e^{2\pi\hbar^{-1} z})},$$

where $q^\vee = e^{\pi i/\hbar^2}$ is modular S-dual to $q = e^{\pi i \hbar^2}$.

We have

$$\overline{\Phi^{\hbar}(\overline{z})} = \frac{1}{\Phi^{\hbar}(z)}.$$

So since x_k is self-adjoint, the operator $\Phi^{\hbar}(x_k)$ is a unitary operator on V_χ.

This means the mutated operators $\mu_k(X_j)$ are also positive self-adjoint, and we get a unitary representation of the groupoid of cluster transformations.

Example: positive representations of $U_q(\mathfrak{sl}_2)$

Consider the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$, and the self-adjoint, unbounded operators

$$\hat{p} = \frac{i}{2\pi}\frac{\partial}{\partial x}, \quad \hat{x} = x.$$

Then for all $s \in \mathbb{R}$, we have positive self-adjoint operators

$$X_1 = e^{2\pi\hbar(\hat{p} \quad s)}, \quad X_2 = e^{2\pi\hbar(\hat{x}+2s)}$$

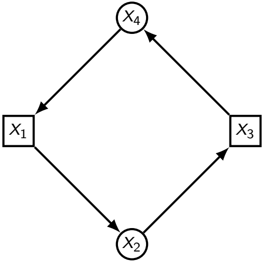
$$X_3 = e^{2\pi\hbar(\quad \hat{p} \quad s)}, \quad X_4 = e^{2\pi\hbar(\quad \hat{x}+2s)}$$

satisfying the cyclic quiver relations

$$q^2 X_k X_{k+1} = X_{k+1} X_k, \quad k \in \mathbb{Z}/4\mathbb{Z},$$

where $q = e^{\pi i \hbar^2}$.

Example: positive representations of $U_q(\mathfrak{sl}_2)$



Chevalley generators E, F of $U_q(\mathfrak{sl}_2)$ act by positive, self-adjoint operators

$$E \mapsto X_1 + qX_1X_2, \qquad F \mapsto X_3 + qX_3X_4$$

The $U_q(\mathfrak{sl}_2)$ Casimir element Ω acts by

$$\Omega \mapsto e^{2\pi\hbar s} + e^{\;\;2\pi\hbar s}.$$

Positive representations of $\mathrm{U_q}(\mathfrak{sl}_n)$

Positive representations of $\mathfrak{g} = \mathfrak{sl}_2$ studied by Faddeev, Bytsko-Ponsot-Teschner (1999).

Positive representations $\mathcal{P}_s$ for $U_q(\mathfrak{sl}_\mathfrak{n})$ have also been considered by Frenkel-Ip (2011).

These representations $\mathcal{P}_\lambda$ are labelled by positive real points λ of a Weyl chamber, and have the property that the quantum group generators act by positive, essentially selfadjoint operators.

Continuous tensor categories

Theorem (Ponsot-Teschner)

For $\mathfrak{g} = \mathfrak{sl}_2$*, positive representations form a 'continuous tensor category':*

$$\mathcal{P}_{s_1} \otimes \mathcal{P}_{s_2} = \int_{\mathbb{R}_{>0}} \mathcal{P}_s d\mu(s).$$

The measure is $d\mu(s) = 4\sinh(2\pi\hbar s)\sinh\left(2\pi\hbar^{-1}s\right) ds$.

Continuous tensor categories in higher rank?

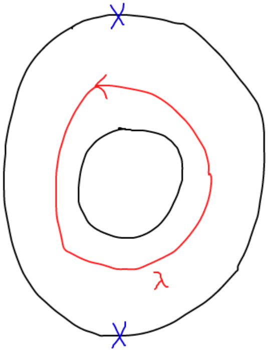
Conjecture (Frenkel-Ip)

The category of positive representations $\mathcal{P}_\lambda$ of $\mathrm{U_q}(\mathfrak{sl}_n)$ is also closed under tensor product.

Geometric approach: cutting and gluing isomorphisms

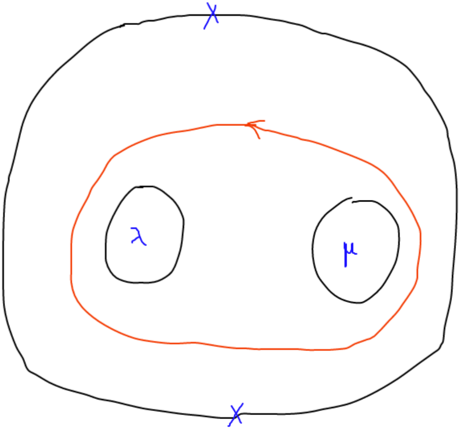
With A. Shapiro, we have a geometric approach to the conjecture using quantum higher Teichmüller theory.

First observation: central character of $\mathcal{P}_\lambda$ determined by eigenvalues of holonomy around the puncture.



Geometric approach: cutting and gluing isomorphisms

Picture for $P_\lambda \otimes P_\mu$:

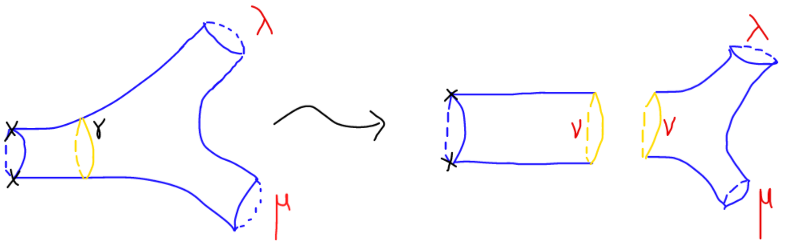


Casimirs of $\Delta U_q(\mathfrak{sl}_\mathfrak{n})$ determined by holonomies around red loop.

Geometric approach: cutting and gluing isomorphisms

Teichmüller theory interpretation of the decomposition of the tensor product of positive representations $P_\lambda \otimes P_\mu$ into positive representations P_ν:

$$P_\lambda \otimes P_\mu = \int_\nu P_\nu \otimes M^\nu_{\lambda,\mu} d\nu.$$



Reduction to q-difference Toda spectral problem

Algebraically: the picture let us read off a natural sequence of cluster transformations, which identifies the traces of holonomies with the Hamiltonians of the q-difference *open Toda lattice.*

Classical limit of q-difference Toda: let c be an Coxeter element of the symmetric group S_{n+1}, e.g. $c = s_1 s_2 s_3 s_4$ for $n = 4$.

Let $H \subset SL_{n+1}(\mathbb{C})$ be a Cartan subgroup, B a pair of opposite Borels, and consider the double Bruhat cell

$$SL_{n+1}^{c,c} = B_+ c B_+ \cap B \ \ c B \ \ .$$

Then $\dim SL_{n+1}^{c,c}/\mathrm{Ad}_H = 2n$, and the conjugation-invariant functions define an integrable system.

Reduction to q-difference Toda spectral problem

The eigenfunctions of the quantum Toda Hamiltonians have been determined by Kharchev-Lebedev-Semenov-Tian-Shansky: they are the *q-Whittaker functions.*

Their orthogonality and completeness yields the desired the direct integral decomposition.

Towards a modular functor?

This cutting and gluing result is the first step towards constructing an infinite dimensional analog of a modular functor from the quantization of higher Teichmüller spaces.

When $\mathfrak{g} = \mathfrak{sl}_2$, this has been investigated by Teschner. On the loop group side, closely related to modular functor coming from Virasoro conformal blocks.

Suggests connection with 4D gauge theory (AGT...)

Will be interesting to complete the construction in higher ranks and compare with the $\mathcal{W}$-algebras construction.

Thanks for listening!