THE CHROMATIC LAGRANGIAN: WAVEFUNCTIONS AND OPEN GROMOV-WITTEN CONJECTURES

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ABSTRACT. Inside a symplectic leaf of the cluster Poisson variety of Borel-decorated PGL_2 local systems on a punctured surface is an isotropic subvariety we will call the *chromatic Lagrangian*. Local charts for the quantized cluster variety are quantum tori defined by cubic planar graphs, and can be put in standard form after some additional markings giving the notion of a *framed seed*. The mutation structure is encoded as a groupoid. The local description of the chromatic Lagrangian defines a *wavefunction* which, we conjecture, encodes open Gromov-Witten invariants of a Lagrangian threefold in threespace defined by the cubic graph and the other data of the framed seed. We also find a relationship we call *framing duality*: for a family of "canoe" graphs, wavefunctions for different framings encode DT invariants of symmetric quivers.

CONTENTS

1. INTRODUCTION

This paper exploits cluster theory to compute wavefunctions for Lagrangian branes in threespace and to make explicit conjectures about their all-genus open Gromov-Witten invariants. For certain branes, these numbers also relate to the cohomologies of twisted character varieties and Donaldson-Thomas invariants of quivers. Two structural tools in the schema are the behavior under mutation and the dependence of quantities on phases and framings.

Let $\mathscr{P}$ be the symplectic cluster variety of Borel-decorated, PGL_2 local systems on a punctured sphere S with unipotent monodromy around the punctures. There is a Lagrangian subvariety $\mathscr{M} \subset \mathscr{P}$ of decorated local systems with trivial monodromy at the punctures. Cluster charts $\mathscr{P}_\Gamma$ of $\mathscr{P}$ are labeled by cubic graphs Γ on S, or dually ideal triangulations. They are algebraic tori, and can be identified with rank-one twisted local systems on a genus-g Legendrian surface S_Γ in the five-sphere. After choosing a spin structure, we can express the chart as $\mathscr{P}_\Gamma \cong H^1(S_\Gamma; \mathbb{C}^*) \cong (\mathbb{C}^*)^{2g}$.

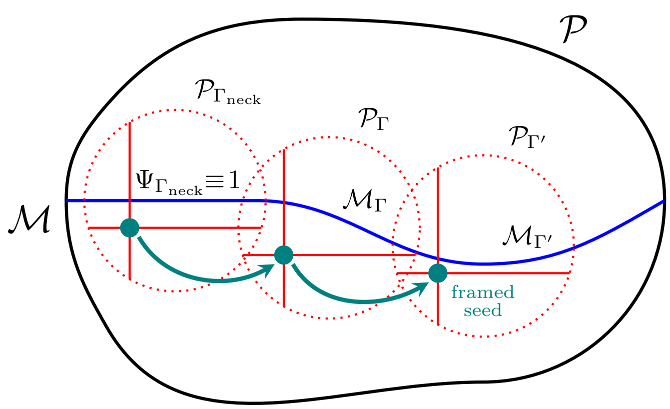

FIGURE 1.0.1. The cluster variety of decorated PGL_2-local systems $\mathscr{P}$ and the chromatic Lagrangian $\mathscr{M}$ (in blue). Each framed seed (teal dot) identifies the chart $\mathscr{P}_\Gamma$ with a quantum torus, in which the ideal $\mathscr{M}_\Gamma$ is described by a cyclic vector or *wavefunction,* Ψ_Γ. Arrows in the *framed seed groupoid* allow us to determine $\Psi_{\Gamma'}$ from Ψ_Γ. Any seed connected to the necklace graph Γ_{neck} by admissible mutations has a computable wavefunction, conjectured to be the generating function of all-genus open Gromov-Witten invariants of the corresponding Lagrangian.

Then $\mathscr{M}_\Gamma := \mathscr{M} \cap \mathscr{P}$ is a subspace of a torus closely related to the space of graph colorings of Γ, so we call $\mathscr{M}$ the *chromatic Lagrangian.* The explicit description of $\mathscr{M}_\Gamma$ will lead to enumerative predictions, but will also depend on further choices: a *phase*, a *framing* and a *cone*.

Central to the strategy for calculation is to understand the effects of a mutation $\Gamma \rightsquigarrow \Gamma'$, which is dual to a flop of a triangulation, and to understand its interaction with phases, framings and cones. The whole story has a quantization, conjecturally related to higher-genus open Gromov-Witten invariants. The entire structure is captured by the *framed seed groupoid*, an enhancement of the cluster groupoid, whose arrows are either mutations or changes of the various decorations — see Figure 1.0.1.

1.1. **Framed Seed Groupoid.** In a bit more detail, the edge lattice $\Lambda := \mathbb{Z}^{E_\Gamma}$ of a cubic graph on an oriented surface (for us, the sphere) has a natural skew form $(*,*)$ defined from the cyclic structure on edges meeting at a vertex. Quotienting by its kernel Λ_c defines a symplectic lattice $\underline{\Lambda}$. Roughly, a framed seed is an identification of this lattice with the standard symplectic lattice $\mathbb{Z}^g \oplus \mathbb{Z}^g$. More formally, it is a tuple $(\mathbf{i}, K, \lambda, \mathbf{f})$, where $\mathbf{i}$ is a skew form on a lattice Λ, $K \subset \underline{\Lambda}$ is a maximal isotropic sublattice, λ is a central character of Λ_c, and $\mathbf{f} = (\sigma, \{a_i\})$ is a pair of a splitting $\sigma : K^\vee \to \underline{\Lambda}$ of $0 \to K \to \underline{\Lambda} \to K^\vee \to 0$ and basis $\{a_i\}$ of $K^\vee$. Note we have $K^\vee \cong \underline{\Lambda}/K$ via the symplectic structure on $\underline{\Lambda}$, so the dual basis $\{b_i\}$. In total, the data of the framed seed provides an identification of $\underline{\Lambda}$ with the standard symplectic lattice $\mathbb{Z}^g \otimes \mathbb{Z}^g$ with symplectic form $\hbar \sum_i dx_i \wedge dy_i$.

We are interested in the set-up detailed in [TZ], i.e. the construction of a Legendrian surface $S_\Gamma \subset S^5$ from the data of $\Gamma \subset S$, and a singular exact Lagrangian filling L_0 of S_Γ as defined by an ideal *foam*, $\mathbf{F}$, the combinatorial dual of a tetrahedronization of a ball. A smoothing L can be defined by studying the local model of the Harvey-Lawson special Lagrangian smoothing of the singular Harvey-Lawson cone, and amounts to local choice of one of the three possible face-matchings at each tetrahedron. Then $H_1(S_\Gamma)$ is identified with $\underline{\Lambda}$, with its intersection form, and K is the kernel of the homology push-forward of inclusion of the boundary $S_\Gamma \hookrightarrow L$. Then $\{a_i\}$ is a basis for $H^1(L)$; the dual basis $\{b_i\}$ for $H_1(L)$ and these give coordinates x_i and y_j for $H^1(L, \mathbb{C}^*) \cong \mathscr{P}_\Gamma$, respectively. The quantization then leads to an isomorphism of $\mathscr{P}_\Gamma$ with the quantum torus $y_i x_j = q^{\delta_{i,j}} x_j y_i$. Each edge is then labeled by a monomial x_e in the x_i and y_j with q-dependent coefficient.

1.2. **Wavefunctions.** After quantization in each chart $\mathscr{P}_\Gamma$, the Lagrangian subvariety $\mathscr{M}_\Gamma \subset \mathscr{P}_\Gamma$ becomes a left ideal $\mathcal{I}_\Gamma$, and we can identify the left $\mathscr{P}_\Gamma$-module $\mathscr{P}_\Gamma/\mathcal{I}_\Gamma$ with the principal ideal defined by cyclic vector $\Psi_\Gamma \in \mathbb{C}[[\{x_i\}]]$, satisfying $\mathcal{I}_\Gamma \Psi_\Gamma = 0$ in the standard representation defined by exponentiating the Weyl representation: $(x_i \cdot f)(x) = x_i f(x)$, $(y_i \cdot f)(x) = f(qx_i)$, where $q = e^{\hbar}$. The generators for $\mathcal{I}_\Gamma$ are equations determined by the faces of Γ, giving us concrete equations for Ψ_Γ. For example, in the case where Γ is the tetrahedron graph, S_Γ is a genus-one surface and the quantum torus has generators x and y obeying $yx = q^2 xy$. For a certain choice of framed seed, the face equations are all equivalent to $(-1 + x + y)\Psi = 0$, and the unique power-series solution is $\Psi = \Phi(x)$, where $\Phi(x) = \prod_{n \geq 0}(1 - q^{2n}x)^{-1}$ is the quantum dilogarithm.

The equations for $\mathcal{I}_\Gamma$ are compatible with mutations $\Gamma \rightsquigarrow \Gamma'$, meaning generators of $\mathcal{I}_{\Gamma'}$ are related to generators of $\mathcal{I}_\Gamma$ by a cluster coordinate transformation, and these are effected (up to a known basis change) by conjugation by the quantum dilogarithm. The upshot is that graph mutations change the wavefunction by the action of the quantum dilogarithm, and as long as can make sense of this action on the ring of power series, we may compute the resulting wavefunction. We call such mutations *admissible.* What is more, we can effect changes of other aspects of a framed seed (phase, framing, basis) by known operators, as well. Moreover, the necklace graph Γ_{neck} (see Figure 1.5) is a distinguished base point for the framed seed groupoid, with known wavefunction $\Psi_{\Gamma_{\text{neck}}} \equiv 1$. So we can find any wavefunction for any point on the framed seed groupoid connected to this basepoint by an admissible path.

One must check that the resulting wavefunction is independent of path, and this amounts to checking that the cluster modular group (the automorphisms of the standard quantum torus determined by loops in the groupoid) acts trivially on the necklace wavefunction. This can be verified explicitly by observing that the necklace wavefunction is uniquely determined by the defining equations for the ideal.

In this way, the cluster structure of the cluster modular groupoid can be exploited to find wavefunctions. Some have conjectural interpretations.

1.3. **Open Gromov-Witten Conjectures.** The cubic planar graphs Γ that label cluster charts $\mathscr{P}_\Gamma$ also describe Legendrian surfaces S_Γ, which form asymptotic boundary conditions for categories of A-branes, by which we mean categories of constructible sheaves with singular support on S_Γ [N, NZ]. Non-exact Lagrangian fillings $L \subset \mathbb{C}^3$ asymptotic to S_Γ have open Gromov-Witten invariants which we conjecture, following the pioneering work of Aganagic-Vafa [AV], are predicted by the geometry of the brane moduli space $\mathscr{M}_\Gamma \subset \mathscr{P}_\Gamma$.

The classical geometry conjecturally leads to open Gromov-Witten invariants. The subvariety $\mathscr{M}_\Gamma \subset \mathscr{P}_\Gamma$ is Lagrangian. Choosing a framed seed A and lifting to the universal cover, we get $\widetilde{\mathscr{M}_\Gamma} \subset \mathbb{C}^{2g}$, and any connected component determines a prepotential W_Γ so that $\widetilde{\mathscr{M}_\Gamma}$ is the graph of dW_Γ. The instanton part of W_Γ is conjectured to be the open Gromov-Witten generating function.
Conjecture: $W_\Gamma^{(A)}$ is the generating function of disk invariants and obeys Ooguri-Vafa integrality: $W_\Gamma^{(A)}(x) = \sum_{d \in \mathbb{Z}_{\geq 0} \setminus \{0\}} n_d^{(A)} \mathrm{Li}_2(x^d)$, with $n_d^{(A)} \in \mathbb{Z}$. This conjecture appeared in essentially the same form in [TZ, Section 1.2].

The cluster variety $\mathscr{P}$ has a quantization, each chart $\mathscr{P}_\Gamma$ of which can be identified, through a splitting, with a quantum torus, $\mathcal{D}$: $V_i U_i = q^2 U_i V_i$, where $q = e^{i\pi\hbar^2}$. Then $\mathscr{M}_\Gamma$ quantizes as an ideal $\mathcal{I}$, and the left $\mathcal{D}$-module $\mathcal{D}/\mathcal{I}\mathcal{D}$ is cyclic for a vector Ψ_Γ, which we conjecture is the generating function of all-genus open Gromov-Witten invariants. As such, they should satisfy integrality conditions found by Ooguri and Vafa [OV].

Conjecture:

$\Psi_\Gamma^{(A)}$ is the generating function of all-genus open Gromov-Witten invariants and obeys Ooguri-Vafa integrality:

$$\Psi_\Gamma^{(A)} = \prod_{d \in \mathbb{Z}_{\geq 0}^g \setminus \{0\}} \prod_{s \geq 0} \Phi(x^d q^s)^{n_{v,s}^{(A)}},$$

with $n_{d,s}^{(A)} \in \mathbb{Z}$. (See Conjecture 7.7 for details.)

The conjecture implies the one above from [TZ] since $\Psi_\Gamma^{(A)} \sim e^{W_\Gamma^{(A)}/\hbar}$ and $\Phi(x) \sim e^{\mathrm{Li}_2(x)/\hbar}$ as $\hbar \to 0$, and then $n_d^{(A)} = n_{d,0}^{(A)}$.

Since all ideal triangulations are related by flips, every cubic planar graph of genus g (meaning it has $2g+2$ vertices) can be obtained from Γ_g^{neck} through a sequence of mutations. Our rubric therefore leads to conjectures for Lagrangian fillings for many Legendrian surfaces.

1.4. **Analytic Aspects.** A quantization in the physical sense would require that we construct, in addition to wavefunctions for each seed of the cluster modular groupoid, a Hilbert space with arrows acting by unitary isomorphisms. Fock and Goncharov constructed such a quantization depending on a parameter $\hbar \in \mathbb{R}$, a central character for the kernel of the skew form, with reality being crucial for each logarithmic cluster variable x to act in a unitary way, and for mutations to be effected by a unitary action of the Faddeev (noncompact quantum) dilogarithm $\varphi(x)$.

Such an approach cannot work for us, as the unipotency condition defining our cluster variety requires the central character to act as an *imaginary* number, ruling out self-adjointness in the naïve sense. Nevertheless, in Section 8 we present what we think of as good evidence for the existence of a quantization in the analytic sense, and for a well-defined wavefunction at each seed. Solutions are symmetric in $\hbar \leftrightarrow \hbar^{-1}$, reflecting the symmetry of the "squashed three-sphere" in the physical set-up (see, e.g., [CEHRV, Equation (2.16)]). In this set-up, all seed arrows would be admissible. For example, mutating with the same sign at all three strands of the genus-two necklace graph Γ^2_{neck} would not be admissible in the algebraic set-up of Section 1.2, but leads to an analytic wavefunction. Indeed, in Section 8 we show in this and several other cases that different paths to the same framed seed lead to the same wavefunction. The identities depend on the analytic properties of the Faddeev dilogarithm and its Fourier self-duality.

1.5. **Framing Duality.** Our starting point is the following observation about the dependence on the phase/framing parameter $A = (h)$, a 1×1 matrix, when Γ is the tetrahedron graph and S_Γ is the Legendrian Clifford torus:[1]

$$n_{d,0}^{(h)} = \dim H^{\text{middle}} \begin{pmatrix} \text{"twisted" character variety of rank } d \\ \text{local systems on a genus } h \text{ surface} \end{pmatrix}. \tag{1.5.1}$$

We are able to prove the equality assuming a conjecture of Häusel-Rodriguez Villegas [HRV]. Note that the $n_{d,0}^{(h)}$ are *disk* invariants — no higher genus surfaces were considered! The relevant HRV conjecture relates the right-hand side of (1.5.1) with the Donaldson-Thomas ("DT") series of the quiver with one node and h loops. Thus (h) is the adjacency matrix, a perspective which generalizes nicely.

A special role is played by the Legendrian Clifford torus and its higher-genus generalizations. These Clifford surfaces of genus g arise from "canoe" graphs (see Figure 1.5.1). The Clifford surfaces arise from mutations of the higher-genus version of the Chekanov torus, a genus-g Legendrian surface corresponding to a "necklace" graph (see again Figure 1.5.1). Each Chekanov surface has a distinguished exact Lagrangian filling and therefore a distinguished phase and no holomorphic

[1]The twisted character variety of a genus-h surface is the set of isomorphism classes of irreducible representations of a central extension of the fundamental group, i.e. $n \times n$ matrices $A_1, ..., A_h, B_1, ..., B_h$ such that $[A_1, B_1][A_2, B_2] \cdots [A_h, B_h] = \zeta$, with ζ a central nth root of unity, modulo conjugation.

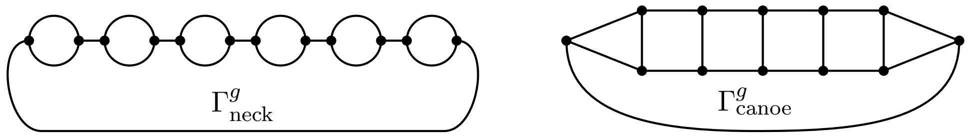

FIGURE 1.5.1. Mutating the necklace graph Γ^g_{neck} (left) along the g short strands results in the canoe graph Γ^g_{canoe} (right). Here $g = 5$. The Legendrian surfaces generalize the Chekanov and Clifford tori, respectively, which arise when $g = 1$.

disks: $\Psi_{\Gamma^g_{\text{neck}}} \equiv 1$ and $W_{\Gamma^g_{\text{neck}}} \equiv 0$. After mutation, we get a distinguised phase for the Clifford surface $S_{\Gamma^g_{\text{canoe}}}$, i.e. a Lagrangian filling L with $\partial L = S_{\Gamma^g_{\text{canoe}}}$ and $b_1(L) = g$, though the different framings in this phase are parametrized by a $g \times g$ symmetric integral matrix, A. The corresponding wavefunction $\Psi^{(A)}_{\Gamma^g_{\text{canoe}}}$ can be computed from cluster theory as in Section 1.1, and, as stated in Section 1.3 above, is conjecturally the partition function of the open topological string.

We can now state *framing duality* in the following way. Let Q_A be the symmetric quiver with g nodes and adjacency matrix A. Recall that the DT series is the generating function for cohomologies of quiver representation spaces M_d over different dimension vectors, d. Explicitly, $DT_{Q^A} = \sum_{k\in\mathbb{Z}_{\geq 0}\setminus\{0\}} \sum_{s>0} (-1)^s H^s(M_d) x^d q^{\frac{1}{2}s}$ Then we have:

$$\text{The wavefunction is the DT series of } Q_A: \qquad \Psi^{(A)}_{\Gamma^g_{\text{canoe}}} \;=\; DT_{Q^A}\ . \tag{1.5.2}$$

Remark 1.1. Many of the results which establish this equality were performed by Kontsevich-Soibelman in [KS]. For the genus-one case studied by Aganagic and Vafa, the connection between DT invariants and open GW invariants in different framings was observed also in [LZ]. As for other Legendrians, also in genus-one, wavefunctions for knot and link conormals were considered in [AENV]. Finding quiver duals for knot conormals is known as the Knot-Quiver Correspondence [KRSS]. The relationship (1.5.2) suggests that the quiver invariants arise from an effective quiver quantum-mechanical theory described by the capping data for the noncompact threefolds we construct from Harvey-Lawson components — see, e.g. [CEHRV, Section 5.1.1]. Framing duality is thus in the spirit as the knots-quiver correspondence of [KRSS], whose geometric and physical interpretations were proposed in [EKL]. It is however more general, in the following sense. The Legendrian surfaces considered here are higher genus and not tori, giving rise to all symmetric quivers and DT invariants depending on all g variables. In contrast, framings of a fixed knot are labeled by a single integer, corresponding to a one-parameter set of quivers, with DT invariants determined by specializing the g variables to a one-dimensional slice — see [KRSS, Equation (4.2)]. It would be interesting to pursue a geometric interpretation of framing duality along the lines of [EKL].

Remark 1.2. One wonders if the above relations extend to other cubic graphs and/or nonsymmetric quivers.

1.6. **Seminal Prior Works.** Very similar constructions were considered from related physical perspectives in prior works. In [CEHRV] and [DGGo] the authors consider an M5-brane on $S^3 \times L$, where L is a Lagrangian submanifold of a compactifying space. (Those authors call this Lagrangian M.) They describe the partition function of the effective 3d theory on S^3 as a quantum-mechanical state. The M-theory set-up expresses this partition function as an integer combination of dilogarithms. The partition function can also be computed by reduction to L. It is a general property of quantum field theory that the path integral on a manifold with boundary always defines a state in the Hilbert space defined by the boundary. In the present case, the boundary Hilbert space is the quantization of the space of flat U(1) connections on the genus-g Legendrian boundary surface

S_Γ (with monodromy -1 around $2g+2$ punctures – see Section 4). The wavefunction Ψ should be understood as the wavefunction of this quantum state.

On top of all this, many of the results of this paper have also appeared in important previous works, to which we owe a debt of gratitude. In [CCV] and [CEHRV], the authors studied the behavior of these wavefunctions under symplectic transformations, although not via cluster theory and without relating the results to Gromov-Witten invariants. The papers [DGGo] and [DGGu] overlap with the present paper, as well as [CEHRV], in considering Lagrangian double covers branched over tangles, and studied the corresponding Lagrangian moduli space. The paper [KS] studied quiver representations and preservation of integrality under changes of framings, providing many of the key formulas that we use. The idea of quantizing mirror curves goes back to [ADKMV] and has been integral to the spectral approach of [GHM] and others. Finally, open Gromov-Witten conjectures appeared previously in [TZ] and [Za]. Further citations are made in the text.

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2. Cluster Poisson Varieties and Quantizations

For the convenience of the reader, we briefly recall the needed background on cluster Poisson varieties and their quantizations. Within this paper, we focus on the cluster Poisson varieties that are skew-symmetric and without frozen variables. A more general definition of cluster Poisson varieties can be found in [FG2].

2.1. **Cluster Poisson varieties.**

Definition 2.1. A seed is a pair $\mathbf{i} = (\{x_1, \ldots, x_n\}, W)$, where $\{x_1, \ldots, x_n\}$ is a collection of commuting algebraically independent variables, and $W = \sum_{i,j} a_{ij} x_i \frac{\partial}{\partial x_i} \wedge x_j \frac{\partial}{\partial x_j}$ is a bi-vector encoded by an integer skew-symmetric matrix $A = (a_{ij})$.

Let $\mathbf{i}$ be a seed. Every $k \in \{1, \ldots, n\}$ creates a new seed $\mu_k(\mathbf{i}) = (\{x_1', \ldots, x_n'\}, W)$ such that

$$x_i' = \begin{cases} x_k^{-1} & \text{if } i = k, \\ x_i(1 + x_k^{-\operatorname{sgn}(a_{ik})})^{-a_{ik}} & \text{if } i \neq k. \end{cases}$$

In terms of $\{x_i'\}$, the bi-vector W can be presented as $\sum_{i,j} a_{ij}' x_i' \frac{\partial}{\partial x_i'} \wedge x_j' \frac{\partial}{\partial x_j'}$, where

$$a_{ij}' = \begin{cases} -a_{ij} & \text{if } i = k \text{ or } j = k; \\ a_{ij} + \frac{|a_{ik}|a_{kj} + a_{ik}|a_{kj}|}{2} & \text{otherwise.} \end{cases}$$

The process of obtaining the new seed $\mu_k(\mathbf{i})$ is called a *cluster mutation* in the direction k. The cluster mutation μ_k in the same direction is involutive: $\mu_k^2(\mathbf{i}) = \mathbf{i}$.

Let σ be a permutation of $\{1, \ldots, n\}$. It gives rise to a seed $\sigma(\mathbf{i}) = (\{x_1', \ldots, x_n'\}, W)$ such that

$$x_i' = x_{\sigma^{-1}(i)}, \qquad i \in \{1, \ldots, n\}.$$

A composition $\tau = \sigma \circ \mu_{i_1} \circ \cdots \circ \mu_{i_m}$ of cluster mutations and permutations taking a seed $\mathbf{i}$ to $\mathbf{i}'$ is called a cluster transformation.

Definition 2.2. Let $\mathscr{X}$ be a rational variety over $\mathbb{C}$ equipped with a rational bi-vector W. A cluster chart of $\mathscr{X}$ is a birational map

$$\pi = (x_1, \ldots, x_n) : \mathscr{X} \longrightarrow \mathbb{C}^n$$

such that $\mathbf{i}_\pi := (\{x_1, \ldots, x_n\},\ \pi_*(W))$ forms a seed. Two cluster charts are called equivalent if their corresponding seeds are related by a cluster transformation. The equivalence class of a cluster chart π is denoted by $|\pi|$.

Abusing notation[2], a variety $\mathscr{X}$ equipped with a pair $(|\pi|, W)$ is called a cluster Poisson variety.

Let $\mathbb{C}(\mathscr{X})$ be the field of rational functions on $\mathscr{X}$. For a cluster chart $\pi' = \{x'_1, \ldots, x'_n\}$, let $\mathcal{T}_{\pi'} = \mathbb{C}[x'^{\pm 1}_1, \ldots, x'^{\pm 1}_n] \subset \mathbb{C}(\mathscr{X})$ denote the ring of Laurent polynomials in $x'_1, \ldots, x'_n$. The *cluster Poisson algebra* is the intersection

$$\mathbb{L}_{\mathscr{X}} := \bigcap_{\pi' \in |\pi|} \mathcal{T}_{\pi'}. \tag{2.1.1}$$

Note that the bivector W induces a natural Poisson bracket on $\mathbb{L}_{\mathscr{X}}$:

$$\{\cdot, \cdot\} : \ \mathbb{L}_{\mathscr{X}} \times \mathbb{L}_{\mathscr{X}} \longrightarrow \mathbb{L}_{\mathscr{X}}, \qquad \{f, g\} := W(f, g).$$

Let p be a birational automorphism of $\mathscr{X}$. We say p is a cluster automorphism if

- p preserves the bi-vector: $p_*(W) = W$,
- p preserves the equivalence class of cluster charts: $\pi \circ p \in |\pi|$.

The set of cluster automorphisms forms a group. Denote it by $\mathcal{G}_{\mathscr{X}}$ and call it the *cluster modular group* of $\mathscr{X}$. The group $\mathcal{G}_{\mathscr{X}}$ acts by Poisson automorphisms on the algebra $\mathbb{L}_{\mathscr{X}}$.

2.2. **Quantization.** Let $\mathscr{X}$ be a cluster Poisson variety. Let $A = (a_{ij})$ be the $n \times n$ integer skewsymmetric matrix appearing in an initial seed defining $\mathscr{X}$ as in Definition 2.1. To A is associated a triple $(\Lambda, \Pi, (*, *))$, where Λ is a rank n lattice, $\Pi = \{e_1, \ldots, e_n\} \subset \Lambda$ is a basis, and $(*, *)$ is a bilinear form on Λ such that $(e_i, e_j) = a_{ij}$. We also set

$$\Lambda^+ = \bigoplus_{i=1}^n \mathbb{Z}_{\geq 0} e_i, \qquad \Lambda^- = \bigoplus_{i=1}^n \mathbb{Z}_{\leq 0} e_i.$$

Let $\mathbb{C}[q^{\pm 1}]$ be the ring of Laurent polynomials in q. Let $\mathcal{T}^q$ be the quantum torus algebra over $\mathbb{C}[q^{\pm 1}]$ with the generators X_v $(v \in \Lambda)$, subject to the relations

$$X_v X_w = q^{(v,w)} X_{v+w}. \tag{2.2.1}$$

Denote by $\mathbf{Frac}(\mathcal{T}^q)$ the non commutative field of fractions of $\mathcal{T}^q$ (cf. [BZ, Appendix]). The positive cone Λ^+ determines a formal completion of the algebra $\mathcal{T}^q$. We will consider the group of formal power series with leading term 1

$$\widehat{\mathcal{R}}_\Pi = \Big\{ \sum_{v \in \Lambda^+} a_v(q) X_v \ \Big| \ a_0(q) = 1,\ a_v(q) \in \mathbb{C}(q) \Big\}.$$

Now let us consider the mutations of the basis $\Pi = \{e_1, \ldots, e_n\}$. Let $\Pi^* = \{\alpha_1, \ldots, \alpha_n\} \subset \Lambda^*$ be the dual basis of Π. Let $k \in \{1, \ldots, n\}$. For an n-tuple $S = \{v_1, \ldots, v_n\}$ of elements in Λ, the mutated $\mu_k(S) = \{v'_1, \ldots, v'_n\}$ consists of elements

$$v'_i = \begin{cases} -v_k & \text{if } i = k, \\ v_i + \displaystyle\sum_{l=1}^n \max\{0, (v_i, v_k)\alpha_l(v_k)\} \cdot \operatorname{sgn}(\alpha_l(v_k)) e_l & \text{if } i \neq k. \end{cases} \tag{2.2.2}$$

Remark 2.3. There is a slightly more general version of mutations, which we will consider in Section 3.

[2] Within this paper, we only take into account the birational structure of $\mathscr{X}$.

Let $(k_1, \ldots, k_m)$ be a sequence of indices in $\{1, \ldots, n\}$. Let us start with the set $\Pi = S$. Applying the mutations (2.2.2) recursively, we obtain a sequence of bases of Λ

$$\Pi = \Pi_1 \xrightarrow{\mu_{k_1}} \Pi_2 \xrightarrow{\mu_{k_2}} \ldots \xrightarrow{\mu_{k_m}} \Pi_{m+1} = \Pi', \qquad \text{where } \Pi_j = \{e_1^{(j)}, \ldots, e_n^{(j)}\}. \tag{2.2.3}$$

A basis Π' obtained this way is said to be equivalent to Π. Let $|\Pi|$ consist of bases equivalent to Π.

The elements $e_i^{(j)}$ in (2.2.3) are called *c*-vectors by Fomin-Zelevinsky [FZ4]. The sign coherence of *c*-vectors asserts that each $e_i^{(j)}$ lies either in Λ^+ or Λ^- [DWZ]. Hence there is a unique sequence of signs $(\varepsilon_1, \ldots, \varepsilon_m)$ such that

$$f_j = \varepsilon_j e_{k_j}^{(j)} \in \Lambda^+, \qquad j = 1, \ldots, m. \tag{2.2.4}$$

We define the formal power series

$$\Psi_{\Pi'} = \Psi(X_{f_1})^{\varepsilon_1} \Psi(X_{f_2})^{\varepsilon_2} \cdots \Psi(X_{f_m})^{\varepsilon_m} \ \in \ \widehat{\mathcal{R}}_\Pi.$$

where

$$\Psi(X) = \prod_{n=0}^{\infty} (1 + q^{2n+1} X)^{-1}.$$

Theorem 2.4 ([K, Th.4.1])**.** *The $\Psi_{\Pi'}$ only depends on the set Π', not on the mutation sequences that take Π to Π'.*

Associated with each $\Pi' = \{e'_1, \ldots, e'_n\} \in |\Pi|$ is a quantum torus algebra $\mathcal{T}^q_{\Pi'}$ over $\mathbb{C}[q^{\pm 1}]$ with generators

$$X'_v = \mathrm{Ad}_{\Psi_{\Pi'}}(X_v) \ \in \mathbf{Frac}(\mathcal{T}^q), \qquad v \in \Lambda.$$

The generators X'_v satisfy the relations (2.2.1). In particular, the variables $X'_{e'_1}, \ldots, X'_{e'_n}$ are called *quantized cluster $\mathcal{X}$-variables.* The pair $(\Pi', \mathcal{T}^q_{\Pi'})$ is called a quantum cluster seed. The *quantum cluster algebra* is the intersection

$$\mathbb{L}^q_{\mathscr{X}} = \bigcap_{\Pi' \in |\Pi|} \mathcal{T}^q_{\Pi'} \subset \mathbf{Frac}(\mathcal{T}^q). \tag{2.2.5}$$

The quasiclassical limit $q \mapsto 1$ of (2.2.5) recovers the Poisson algebra (2.1.1).

The cluster modular group $\mathcal{G}_{\mathscr{X}}$ acts on $\mathbb{L}^q_{\mathscr{X}}$ via *quantum cluster automorphism*, constructed as follows. Every element in $\mathcal{G}_{\mathscr{X}}$ one-to-one corresponds to a linear automorphism τ of the lattice Λ such that τ preserves the bilinear form on Λ and maps the initial basis set Π to $\Pi' := \tau(\Pi) \in |\Pi|$. Each τ gives rise to an algebra isomorphism

$$g_\tau : \mathcal{T}^q_{\Pi'} \xrightarrow{\sim} \mathcal{T}^q_{\Pi}, \qquad X'_v \longmapsto X_{\tau^{-1}(v)}.$$

The restriction of g_τ on $\mathbb{L}^q_{\mathscr{X}}$ induces an algebra automorphism of $\mathbb{L}^q_{\mathscr{X}}$, called a quantum cluster automorphism.

2.3. Casimirs.

The bilinear form $(*, *)$ on Λ gives rise to a linear map c from Λ to its dual Λ^*

$$\forall v \in \Lambda, \qquad c(v)(*) = (v, *).$$

The kernel of c forms a sub-lattice Λ_c of Λ. The quotient Λ/Λ_c is a symplectic lattice.

If $v \in \Lambda_c$, then X_v commutes with every generator X_w by (2.2.1). For every $\Pi' \in |\Pi|$, we have

$$X'_v = \mathrm{Ad}_{\Psi_{\Pi'}}(X_v) = X_v.$$

Therefore X_v $(v \in \Lambda_c)$ are contained in $\mathbb{L}^q_{\mathscr{X}}$ and are called *Casimirs.* It is easy to see that the center $Z(\mathbb{L}^q_{\mathscr{X}})$ of $\mathbb{L}^q_{\mathscr{X}}$ is the torus algebra generated by Casimirs.

Definition 2.5. Let $\mathbf{t}$ be a homomorphism from $Z(\mathbb{L}^q_{\mathscr{X}})$ to $\mathbb{C}[q^{\pm 1}]$. The quotient algebra $\mathbb{L}^q_{\mathscr{X},\mathbf{t}}$ of $Z(\mathbb{L}^q_{\mathscr{X}})$ is obtained by modulo the relations

$$X_v = \mathbf{t}(X_v),$$

where v goes through Λ_c.

3. Groupoids of polarized and framed seeds

3.1. **Polarizations and framings for seeds.** Suppose the rank of the skew-form $(*,*)$ of the seed $\mathbf{i}$ is $2g$, and that we fix a central character λ of its kernel $\Lambda_c \subset \Lambda$. In what follows, we will write $\underline{\Lambda} := \Lambda/\Lambda_c$ for the corresponding symplectic lattice. A *polarization* for $\mathbf{i}$ is the choice of an isotropic sublattice $K \subset \underline{\Lambda}$ of maximal rank g, such that the skew form induces a short exact sequence of lattices

$$0 \to K \to \underline{\Lambda} \to K^\vee \to 0.$$

We consider two polarized seeds $(\mathbf{i}, K)$ and $(\mathbf{i}', K')$ to be equivalent if the canonical map $\underline{\Lambda}_{\mathbf{i}} \to \underline{\Lambda}_{\mathbf{i}'}$ is an isometry which sends K to K'. If $(\mathbf{i}, L)$ is a polarized seed and $\mathbf{i}'$ is a seed related to $\mathbf{i}$ by a signed mutation or central-character preserving permutation, then the induced isomorphism of symplectic lattices $\iota : \underline{\Lambda}_{\mathbf{i}} \simeq \underline{\Lambda}_{\mathbf{i}'}$ determines a polarization $K' = \iota(K)$ for $\mathbf{i}'$.

Our reason for introducing the additional data of polarizations is that they define representations of the symplectic torus $\mathcal{T}^q_{\underline{\Lambda}}$ associated to the seed $\mathbf{i}$ and a central character λ. Indeed, a polarization K for $\mathbf{i}$ determines a commutative subalgebra $\mathcal{T}^q_K \subset \mathcal{T}^q_{\underline{\Lambda}}$. The subalgebra $\mathcal{T}_K$ is identified with the coordinate ring of a split algebraic torus of rank g, and let us write $\mathbf{1}_K$ for its 1-dimensional representation given by evaluation at the identity element. From the latter we may construct an induced representation of $\mathcal{T}^q_{\underline{\Lambda}}$

$$\begin{aligned}\mathcal{V}_K :&= \mathrm{Ind}^{\mathcal{T}^q_{\underline{\Lambda}}}_{\mathcal{T}^q_K}(\mathbf{1}_K)\\ &= \mathcal{T}^q_{\underline{\Lambda}} \otimes_{\mathcal{T}^q_K} \mathbf{1}_K.\end{aligned}$$

The representation $\mathcal{V}_K$ is a $\mathbb{Z}$-module of infinite rank. In order to give a concrete model for it, it is necessary to equip the polarized seed $(\mathbf{i}, K)$ with another piece of additional data, which we now describe. We define a *framing* for $(\mathbf{i}, K)$ to be a splitting $\sigma : K^\vee \to \underline{\Lambda}$ of the short exact sequence (3.1), together with a basis $\{a_i\}$ for $K^\vee$. Now consider the standard quantum torus

$$\mathcal{D}_{2g} := \mathbb{Z}[q^{\pm 1}]\langle U_1^{\pm}, \dots, U_n^{\pm}, V_1^{\pm}, \dots, V_n^{\pm}\rangle$$

with the relations

$$U_iU_j = U_jU_i, \qquad V_iV_j = V_jV_i, \qquad V_iU_j = q^{2\delta_{ij}}U_jV_i$$

The choice of a framing $\mathbf{f} = (\sigma, \{a_i\})$ for $(\mathbf{i}, K)$ canonically determines an isomorphism

$$\iota_{\mathbf{f}} \ : \ \mathcal{T}^q_{\underline{\Lambda}} \longrightarrow \mathcal{D}_{2g}.$$

The isomorphism $\iota_{\mathbf{f}}$ is uniquely characterized by the requirement that the element $X_{\sigma(a_i)}$ of $\mathcal{T}^q_{\underline{\Lambda}}$ is mapped to the generator U_i of $\mathcal{D}_{2g}$. The generators V_i correspond under the inverse isomorphism to elements of the basis $\{b_i\}$ of K dual to the basis $\{a_i\}$ for $K^\vee$.

Let $\mathcal{R} := \mathbb{Z}[q^{\pm}][X_1^{\pm}, \dots, X_g^{\pm}]$ be the ring of Laurent polynomials in g variables. There is natural action of $\mathcal{D}_{2g}$ on $\mathcal{R}$ such that

$$\forall F \in \mathcal{R}, \qquad U_i \cdot F = X_iF, \qquad V_i \cdot F = F(X_1, \dots, q^2X_i, \dots, X_n), \tag{3.1.1}$$

and we obtain an isomorphism of $\mathcal{T}^q_{\underline{\Lambda}}$-modules

$$\mathcal{V}_K \simeq \iota^*_{\mathbf{f}}\mathcal{R},$$

thus providing the promised model for the induced representation $\mathcal{V}_K$.

A *framed seed* $\underline{\mathbf{i}}$ is the data $(\mathbf{i}, K, \lambda, \mathbf{f})$ of a seed $\mathbf{i}$ together with a polarization K, central character λ, and framing $\mathbf{f}$. We consider two framed seeds to be equivalent if the canonical map $\underline{\Lambda}_{\mathbf{i}} \to \underline{\Lambda}_{\mathbf{i}'}$ is a central-character-preserving isometry which intertwines the isomorphisms $\iota_{\mathbf{f}}, \iota_{\mathbf{f}'}$.

A framed seed $\underline{\mathbf{i}}$ gives rise to a symplectic basis $\{\sigma(a_i), b_i\}$ for $\underline{\Lambda}$, where we again write b_i for the elements of the basis for K dual to the basis $\{a_i\}$ for $K^\vee$. Indeed, the framed seed is completely determined by the symplectic basis $\{\sigma(a_i), b_i\}$, since the polarization K is recovered as the span of the $\{b_i\}$. We say that two framed seeds are related by a *framing change morphism* if their underlying polarized feeds are identical, and they have identical bases $\{a_i\}$ for $K^\vee$. The space of framing change morphisms based at a given framed seed is naturally identified with the space of $g \times g$ symmetric integer matrices $\Omega = (\omega_{ij})$, where the new framing σ' is related to the original by

$$\sigma'(a_i) = \sigma(a_i) + \sum_{j=1}^{g} \omega_{ij} b_j, \quad i = 1, \dots, g.$$

Remark 3.1. We recall that if $a_i' = \sum C_{ij} a_j$ is another basis of $K^\vee$, then the corresponding dual basis is given by $b_i' = \sum (C^{-1})_{ji} a_j$. Hence the symmetric matrix Ω transforms under such a change of basis C as

$$\Omega \longmapsto C \Omega C^T.$$

The *framed seed groupoid* is a category whose objects are equivalence classes of framed seeds. The arrows are generated by those of three elementary kinds: signed mutations, permutations and framing change morphisms. An arrow $a : (\mathbf{i}, K, \lambda, \mathbf{f}) \to (\mathbf{i}', K', \lambda, \mathbf{f}')$ induces a birational automorphism of $\mathcal{D}_{2g}$ by pre- and post-composing the quantum cluster transformation associated to $\mathbf{i} \to \mathbf{i}'$ with the isomorphisms $\iota_{\mathbf{f}}^{-1}$ and $\iota_{\mathbf{f}'}$ respectively. We put a relation on the arrows in the framed seeds groupoid by identifying arrows with the same source and target which induce identical birational automorphisms of $\mathcal{D}_{2g}$.

3.2. Suppose that $\underline{\mathbf{i}}$ is a framed seed, and recall the corresponding representation

$$\mathcal{V}_{\underline{\mathbf{i}}} \simeq \mathbb{Z}[q^{\pm}][X_1^{\pm}, \dots, X_g^{\pm}]$$

of the quantum torus $\mathcal{T}^q_{\underline{\Lambda}}$. The embedding of the Laurent series ring into the ring

$$\mathcal{K} := \mathbb{Z}((q))((X_1, \dots, X_g))$$

of formal Laurent series also gives rise to a representation of $\mathcal{T}^q_{\underline{\Lambda}}$ which we denote by $\widehat{\mathcal{V}}_{\underline{\mathbf{i}}}$.

For the purposes of constructing wavefunctions, it will be necessary to consider the action of a somewhat larger algebra on the representation $\widehat{\mathcal{V}}_{\underline{\mathbf{i}}}$. Write $\widehat{\mathcal{D}}_{2g}$ for the 'complete quantum torus' associated to $\mathcal{D}_{2g}$, which may be regarded as the ring of non-commutative formal Laurent series in U_i, V_i. Inside $\widehat{\mathcal{D}}_{2g}$, consider the subalgebra

$$\mathcal{A}_{2g} := \mathbb{Z}((q))((U_1, \dots, U_g))\langle V_1^{\pm 1}, \dots V_g^{\pm 1}\rangle$$

consisting of formal Laurent series in the U_i whose coefficients are Laurent polynomials in the V_i. Unlike in the case of $\widehat{\mathcal{D}}_{2g}$, there is a well-defined action of the algebra $\mathcal{A}_{2g}$ on $\widehat{\mathcal{V}}_{\underline{\mathbf{i}}}$. Indeed, under (3.1.1) each V_i acts on the 'vacuum vector' $1 \in \widehat{\mathcal{V}}_{\underline{\mathbf{i}}}$ by $V_i \cdot 1 = 1$, and so the action of a arbitrary Laurent polynomial in the V_i, being a finite $\mathbb{Z}((q))$-linear combination of such, is also well-defined.

Recall that the space of change of framing morphisms based at a given framed seed can be identified with the additive group $\mathfrak{p}_g$ of $g \times g$ symmetric matrices $\Omega = (\omega_{ij})$ with $\omega_{ij} \in \mathbb{Z}$. Its group algebra $\mathbb{Z}\mathfrak{p}_g$ is generated by symbols $T_\Omega, \Omega \in \mathfrak{p}_g$ satisfying $T_\Omega T_{\Omega'} = T_{\Omega + \Omega'}$. The group $\mathfrak{p}_g$ acts on $\mathcal{A}_{2g}$ by automorphisms called *changes of framing*:

$$T_\Omega : \mathcal{A}_{2g} \xrightarrow{\sim} \mathcal{A}_{2g}, \qquad V_j \longmapsto V_j, \qquad U_j \longmapsto q^{\omega_{jj}} U_j \prod_{k=1}^{g} V_k^{\omega_{jk}}, \tag{3.2.1}$$

and we may form the semi-direct product algebra

$$\widehat{\mathcal{A}}_{2g} = \mathcal{A}_{2g} \otimes_{\mathbb{Z}} \mathbb{Z}\mathfrak{p}_g.$$

Given $U_{\mathbf{w}} = \prod_j U_j^{w_j}$, it follows from (3.2.1) that we have

$$T_\Omega(U_{\mathbf{w}}) = q^{\mathbf{w}^t\Omega\mathbf{w}} U_{\mathbf{w}} V_{\Omega\mathbf{w}}.$$

Remark 3.2. The reader may find the following interpretation of the framing shift automorphisms useful. Consider the topological Heisenberg algebra $\mathcal{H}_g$ over $\mathbb{C}[[\hbar]]$ generated by $\{u_j, v_j\}$ subject to the relations

$$[u_j, v_k] = \frac{\delta_{j,k}}{2\pi i},$$

and set $q = e^{\pi i \hbar^2}$. The algebra $\mathcal{A}_{2g}$ embeds into this Heisenberg algebra via $U_k \mapsto e^{2\pi\hbar u_k}$, $V_k \mapsto e^{2\pi\hbar v_k}$. Now given a $g \times g$ symmetric matrix $\Omega \in \mathfrak{p}_g$, consider the associated quadratic form

$$Q(\mathbf{v}) = \sum_{j,k=1}^{g} \omega_{jk} v_j v_k,$$

and write $e^{-\pi i Q(\mathbf{v})}$ for the corresponding element of the group algebra $\mathbb{Z}\mathfrak{p}_g$. Note that the $e^{-\pi i Q(\mathbf{v})}$ are not elements of the Heisenberg algebra $\mathcal{H}_g$, but one can nonetheless formally compute the result of conjugating the generators of $\mathcal{H}_g$ by them using the Baker-Campbell-Hausdorff formula:

$$\begin{aligned}\mathrm{Ad}_{e^{-\pi i Q(\mathbf{v})}}(u_j) &= u_j - \pi i [Q(\mathbf{v}), u_j] \\ &= u_j + \sum_k \omega_{jk} v_k,\end{aligned}$$

so that

$$\begin{aligned}\mathrm{Ad}_{e^{-\pi i Q(\mathbf{v})}}(U_j) &= \mathrm{Ad}_{e^{-\pi i Q(\mathbf{v})}}(e^{2\pi\hbar u_j}) \\ &= e^{2\pi\hbar(u_j + \sum_k \omega_{jk} v_k)} \\ &= q^{\omega_{jj}} e^{2\pi\hbar u_j} e^{2\pi\hbar \sum_k \omega_{jk} v_k} \\ &= q^{\omega_{jj}} U_j \prod_{k=1}^{g} V_k^{\omega_{jk}},\end{aligned}$$

recovering (3.2.1).

The extended algebra $\widehat{\mathcal{A}}_{2g}$ also acts in the representation $\widehat{\mathcal{V}}_{\underline{\mathbf{i}}} \simeq \mathcal{K}$: given $F = \sum_{\mathbf{w}} C_{\mathbf{w}}(q) X^{\mathbf{w}} \in \mathcal{K}$, we define

$$T_\Omega \cdot F := \sum_{\mathbf{w}} q^{\mathbf{w}^t\Omega\mathbf{w}} C_{\mathbf{w}}(q) X^{\mathbf{w}}. \tag{3.2.2}$$

That (3.2.2) indeed defines a representation of the extended algebra $\mathcal{A}_{2g}$ follows easily from the considerations of Remark (3.2), or can be readily verified directly.

3.3. Suppose that $\underline{\mathbf{i}}$ is a framed seed, and e_k is an element of the basis Π for Λ associated to the underlying seed $\mathbf{i}$. Recall that the data of the framing $\mathbf{f}$ allows us to associate to e_k a monomial $M_{\mathbf{f}}(e_k) \in \mathcal{D}_{2g}$ of the form

$$M_{\mathbf{f}}(e_k) = (-q)^r \prod_{j=1}^{g} U_j^{m_j} V_j^{n_j}, \qquad m_j, n_j, r \in \mathbb{Z}.$$

We say that a mutation of the framed seed $\underline{\mathbf{i}}$ in direction e_k with sign ϵ is *admissible* if in the monomial $M_{\mathbf{f}}(e_k)$ we have $m_j \geq 0$ for all $j = 1, \ldots, g$, and in addition there is at least one j for which $m_j \neq 0$. Let us make a few simple remarks about this definition.

Remark 3.3. If two framed seeds $\underline{\mathbf{i}}, \underline{\mathbf{i}}'$ are related by a change of framing, then evidently a signed mutation is admissible with respect to $\underline{\mathbf{i}}$ if and only if it is admissible with respect to $\underline{\mathbf{i}}'$.

Remark 3.4. Let a be an admissible mutation of framed seed $\underline{\mathbf{i}}$ in direction k with sign ϵ, and let $\underline{\mathbf{i}}' = a(\underline{\mathbf{i}})$ be the resulting framed seed. Then the mutation of $\underline{\mathbf{i}}'$ in direction k with sign $-\epsilon$, which is the inverse of a in the framed seed groupoid, is also an admissible mutation.

It follows from these remarks there is a sub-groupoid of the framed seeds groupoid whose morphisms are generated by framing shifts and admissible mutations.

Our reason for introducing the notion of admissibility of mutations is the following: a mutation of a framed seed in direction e_k with sign ϵ is admissible (if and) only if the quantum dilogarithm formal power series $\Phi\left(M_{\mathbf{f}}(e_k)\right)^{\epsilon}$ is an element of the algebra $\mathcal{A}_{2g}$.

Suppose that $\vec{a} = (a_1, \ldots, a_l)$ is a morphism in the framed seed groupoid, i.e. a sequence of admissible mutations and framing shifts. Let us say that such a morphism is admissible if each signed mutation in the corresponding sequence is. Then to each admissible morphism we may associate an invertible element $\Phi_{\vec{a}}$ of the extended algebra $\widehat{\mathcal{A}}_{2g}$. This element $\Phi_{\vec{a}}$ determines a birational automorphism of $\mathcal{D}_{2g}$ (by conjugation), along with an automorphism of $\mathcal{K}$ (via the representation (3.1.1), (3.2.2).)

Lemma 3.5. *Suppose that two chains of $\vec{a}_1$, $\vec{a}_2$ of admissible mutations and framing shifts induce the same birational automorphism of $\mathcal{D}_{2g}$. Then $\Phi_{\vec{a}_1} = \Phi_{\vec{a}_2}$.*

Proof. The Lemma is proved by the following standard argument, cf [KN]. If the $\Phi_{\vec{a}_i}$ induce the same birational automorphism of $\mathcal{D}_{2g}$, then the element $\Phi_{\vec{a}_1}^{-1}\Phi_{\vec{a}_2} \in \widehat{\mathcal{A}}_{2g}$ commutes with all generators U_i, V_i. An easy calculation shows that this implies that $\Phi_{\vec{a}_1}^{-1}\Phi_{\vec{a}_2}$ must be an element of the ground ring $\mathbb{Z}((q))$. But since each quantum dilogarithm corresponding to an admissible mutation is a formal power series in U_i starting from 1, we see that $\Phi_{\vec{a}_1}^{-1}\Phi_{\vec{a}_2} = 1$, and the Lemma is proved. $\square$

4. Cubic Planar Graphs and Fukaya Moduli

Let $\Gamma \subset S^2$ be a cubic planar graph. There is an integer g such that Γ has $\mathsf{v} = 2g+2$ vertices, $\mathsf{e} = 3g+3$ edges, and $\mathsf{f} = g+3$ faces. In [TZ] we associated the following objects to Γ.

(1) A Legendrian surface $S_\Gamma \subset T^\infty\mathbb{R}^3 \subset S^5$ of genus g [TZ, Def.2.1]. The surface S_Γ is a branched double cover of S^2, branched over the vertices of Γ. It is defined by its front projection, which is taken to be a two-sheeted cover of S^2 with crossing locus over the edges of Γ and looking like the following near vertices:

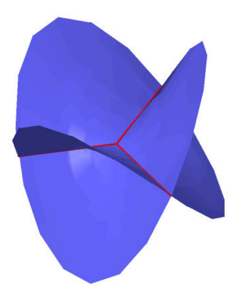
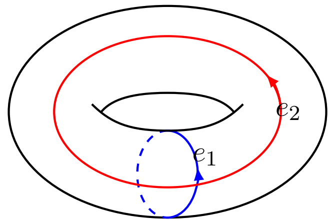
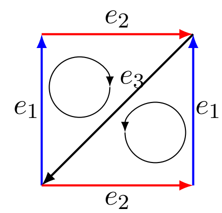


Figure 4.0.1. The front projection of S_Γ near a vertex.

(2) A period domain $\mathscr{P}_\Gamma := H^1(S_\Gamma, \mathbb{C}^*)$, which is an algebraic torus equipped with an algebraic symplectic form coming from the intersection pairing on $H^1(S_\Gamma)$ [TZ, §4.6].

Example 4.1. Let $g = 1$. Then $H_1(S_\Gamma)$ has two generators, denoted by e_1, e_2 respectively. Let $e_1 + e_2 + e_3 = 0$.





As illustrated by the graph, we get

$$X_{e_1}X_{e_2}X_{e_3} = q, \qquad X_{e_2}X_{e_1}X_{e_3} = q^{-1}.$$

Therefore

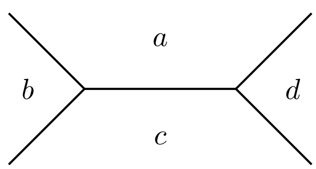
$$X_{e_1}X_{e_2} = q^2 X_{e_2}X_{e_1}.$$

(3) A moduli space $\mathscr{M}_\Gamma$ of microlocal-rank-one constructible sheaves on $\mathbb{R}^3$, whose singular support lies in S_Γ [TZ, §4.3]. More concretely, $\mathscr{M}_\Gamma$ is the space of PGL_2-equivalence classes of $\mathbb{P}^1$-colorings of the faces of Γ.

(4) A Lagrangian map $\mathscr{M}_\Gamma \to \mathscr{P}_\Gamma$ [TZ, §4.7]. It can be described as follows. Every edge e of Γ connects branch points and therefore defines an element of $H_1(S_\Gamma)$. It gives rise to a character $x_e : \mathscr{P}_\Gamma \to \mathbb{C}^*$ by the canonical pairing between H_1 and H^1. The sum of edges surrounding a face f is a trivial cycle in H_1, so $\prod_{e\in\partial f} x_e = 1$. The map $\mathscr{M}_\Gamma \to \mathscr{P}_\Gamma$ is defined by setting x_e to be the cross ratio

$$x_e = -\frac{a-b}{b-c}\cdot\frac{c-d}{d-a} \tag{4.0.1}$$

where $a, b, c, d \in \mathbb{P}^1 = \mathbb{C}\cup\{\infty\}$ are the colors of faces surrounding an edge e in the following pattern:



One easily verifies the relations $\prod_{e\in\partial f} x_e = 1$.

We exhibit defining equations for $\mathscr{M}_\Gamma$. The characters x_e generate the coordinate ring of $\mathscr{P}_\Gamma$, obeying the relation $\prod_e x_e = (-1)^{g+1}$ and further the equation

$$x_{e_1}x_{e_2}\cdots x_{e_n} = 1. \tag{4.0.2}$$

whenever $e_1, \dots, e_n$ label the edges of a face of Γ. In these coordinates, the map $\mathscr{M}_\Gamma \to \mathscr{P}_\Gamma$ is given parametrically by the cross ratio (4.0.1). But it is also given by equations, as a complete intersection, in the following way. Let F be the set of faces of Γ. If $e_1, \dots, e_n$ are the edges around a face $f \in F$ taken counterclockwise, then the expression

$$V_f := 1 + x_{e_1} + x_{e_1}x_{e_2} + \cdots + x_{e_1}\cdots x_{e_{n-1}} \tag{4.0.3}$$

is independent of which edge is called e_1. $\mathscr{M}_\Gamma$ is cut out by the equations $V_f = 0, f \in F$.

Now let $\hat\Gamma$ denote the dual planar graph, with vertex set $V(\hat\Gamma)$. Since Γ is cubic, $\hat\Gamma$ is a triangulation of S^2. Fock-Goncharov associate (in [FG1, §9], and even more specifically in [FG3, §4.1]) to the pair $(S^2, V(\hat\Gamma))$ a cluster Poisson variety, with the triangulation $\hat\Gamma$ determining a cluster chart of this variety. We will make a precise comparison in the next section, where we prove the following as a special case of Theorem 5.2

Theorem 4.2. *Let $\mathscr{P}$ be the symplectic leaf of the cluster Poisson variety $\mathscr{X}$ defined by $(S^2, V(\hat{\Gamma}))$ corresponding to local systems with unipotent monodromy, with its natural symplectic structure. There is a canonical algebraic Lagrangian subvariety $\mathscr{M} \subset \mathscr{P}$ with the following property: for every cubic planar graph Γ with $2g+2$ vertices, there is a cluster chart $\mathscr{P}_\Gamma \subset \mathscr{P}$ such that the embedding $\mathscr{M}_\Gamma \to \mathscr{P}_\Gamma$ is isomorphic to $\mathscr{M} \cap \mathscr{P}_\Gamma \to \mathscr{P}_\Gamma$.*

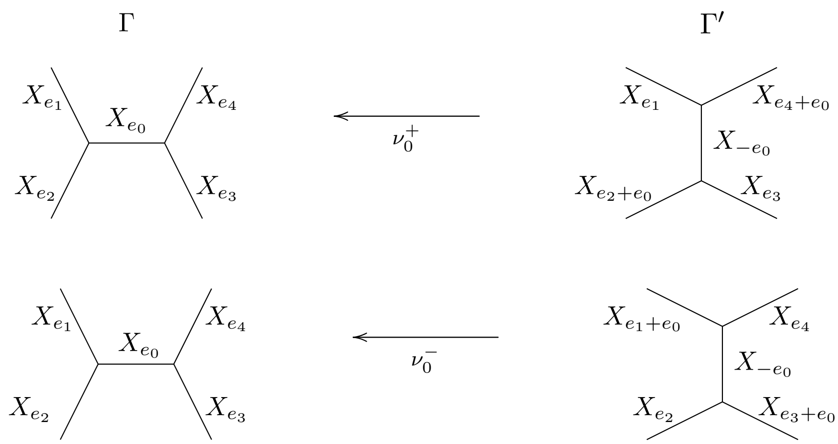
Proof. This is a special case of Theorem 5.2 — see Remark 5.3. □

Because of item (3) above, we call $\mathscr{M}$ the *chromatic Lagrangian*. It was defined from a dual perspective in earlier work of Dimofte-Gabella-Goncharov [DGGo].

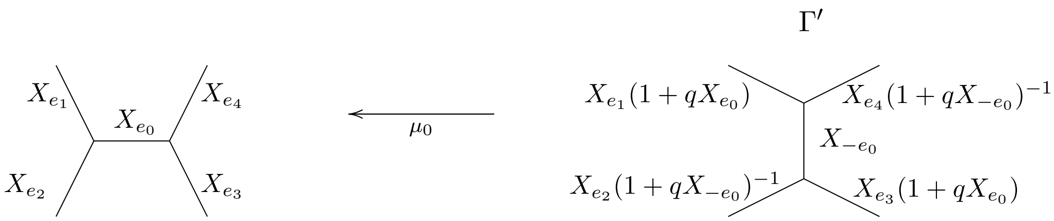
4.1. **Mutation and quantization.** We define $\mathcal{T}_\Gamma := \Lambda_\Gamma \otimes_{\mathbb{Z}} \mathbb{C}^*$, a Poisson torus. It has a canonical quantization $\mathcal{T}_\Gamma^q$, generated by coordinates X_v, $v \in \Lambda$, with relations

$$X_v X_w = q^{(v,w)} X_{v+w}.$$

Let $\underline{\mathbf{i}}$ be a framed seed with underlying cubic graph Γ, and let Γ' be the graph obtained from Γ by flipping a single edge e_0. Then the positive and negative lattive mutation maps $\nu^\pm : \Lambda_{\Gamma'} \to \Lambda_\Gamma$ deliver isometries of edge lattices $\Lambda_{\Gamma'} \cong \Lambda_\Gamma$, and so define framed seeds $\nu_0^\pm(\underline{\mathbf{i}})$. The corresponding isometries of lattices are illustrated below:

(4.1.1)


Also associated to each flip of triangulation is a cluster transformation, i.e. a birational map of tori $\mathcal{T}_{\Gamma'} \dashrightarrow \mathcal{T}_\Gamma$. As explained in Section **??** these maps admit quantizations $\mathcal{T}_\Gamma^q \dashrightarrow \mathcal{T}_{\Gamma'}^q$, which in our case take the form



The map μ can be factored in one of two ways, corresponding to the choice of sign in the lattice isomorphism $\nu^\pm$. Indeed, one easily verifies that the quantum cluster transformation μ_k corresponding to the flip at edge k can be written as

$$\begin{aligned}\mu_k &= \mathrm{Ad}_{\Phi(X_{e_k})} \circ \nu_k^+ \\ &= \mathrm{Ad}_{\Phi(X_{-e_k})^{-1}} \circ \nu_k^-.\end{aligned}$$

Now consider a morphism in the framed seed groupoid represented by a sequence of n signed edge mutations $a(\mathbf{k}) : \underline{\mathbf{i}} \to \underline{\mathbf{i}}'$:

$$\underline{\mathbf{i}} = \underline{\mathbf{i}}_0 \to^{k_1} \underline{\mathbf{i}}_1 \to^{k_2} \cdots \to^{k_n} \underline{\mathbf{i}}_n = \underline{\mathbf{i}}',$$

where the jth mutation takes place at edge k_j and has sign ϵ_j. It gives rise to an isomorphism of quantum tori $\nu_{\mathbf{k}} : \mathcal{T}^q_{\underline{\mathbf{i}}'} \to \mathcal{T}^q_{\underline{\mathbf{i}}}$ given by

$$\nu_{\mathbf{k}} = \nu_{k_1}^{\epsilon_1} \circ \cdots \circ \nu_{k_n}^{\epsilon_n}.$$

Moreover, if we write M_j for the image in $\mathcal{T}_{\underline{\mathbf{i}}}$ of the quantum torus element $X_{e_{k_j}}^{\epsilon_j} \in \mathcal{T}_{\underline{\mathbf{i}}_{j-1}}$ under the isomorphism

$$\nu_{k_1}^{\epsilon_1} \circ \cdots \circ \nu_{k_{j-1}}^{\epsilon_{j-1}} \ : \ \mathcal{T}^q_{\underline{\mathbf{i}}_{j-1}} \to \mathcal{T}^q_{\underline{\mathbf{i}}_0},$$

then we have

$$\begin{aligned} \mu_{\mathbf{k}} &:= \mu_{k_1} \circ \cdots \circ \mu_{k_n} \\ &= \mathrm{Ad}_{\Phi(M_1)^{\epsilon_1}} \circ \cdots \circ \mathrm{Ad}_{\Phi(M_n)^{\epsilon_n}} \circ \nu_{\mathbf{k}}. \end{aligned}$$

Such a sequence of mutations of framed seeds gives rise to a birational automorphism $\mu^{\mathcal{D}}_{\mathbf{k}} := \iota_{\mathbf{i}} \circ \mu_{\mathbf{k}} \circ \iota_{\mathbf{i}'}^{-1}$ of $\mathcal{D}_{2g}$, which evidently factors as

$$\mu^{\mathcal{D}}_{\mathbf{k}} = \mathrm{Ad}_{\Phi(\iota_1(M_1))^{\epsilon_1}} \circ \cdots \circ \mathrm{Ad}_{\Phi(\iota_n(M_n))^{\epsilon_n}} \circ \iota^* \nu_{\mathbf{k}},$$

where we have set

$$\iota^*\nu_{\mathbf{k}} := \iota_0 \circ \nu_{\mathbf{k}} \circ \iota_n^{-1}.$$

The reader may find it convenient to visualize the automorphism $\iota^*\nu_{\mathbf{k}}$ as follows. Recall that the data of a framing for a seed gives rise to a decoration of the edges of its cubic graph by monomials $\mathcal{D}_{2g}$. Then the automorphism $\iota^*\nu_{\mathbf{k}}$ is characterized by the property that is maps the monomial sitting on edge e of Γ' in framed seed $\underline{\mathbf{i}}_n$ to the monomial sitting on the corresponding edge of Γ in framed seed $\underline{\mathbf{i}}_0$.

Now let us suppose that each signed mutation in the sequence $\mathbf{k}$ is admissible, so that under the framing isomorphism ι_j from $\underline{\mathbf{i}}_j$ the monomial M_j is mapped to an element of the algebra $\mathcal{A}_{2g}$. Then we may form the product

$$\Phi_{a(\mathbf{k})} := \Phi(\iota_n(M_n))^{-\epsilon_n} \circ \cdots \circ \Phi(\iota_1(M_1))^{-\epsilon_1} \in \mathcal{A}_{2g}.$$

Recall the representation $\mathcal{K} \simeq \mathbb{Z}((q))((X_1, \ldots, X_g))$ of the algebra $\mathcal{A}_{2g}$. The action of $\Phi_{a(\mathbf{k})}$ defines an automorphism

$$a(\mathbf{k}) \ : \ \mathcal{K} \to \mathcal{K}, \qquad f \longmapsto \Phi_{a(\mathbf{k})} \cdot f,$$

and for all $A \in \mathcal{D}_{2g}$, we have the following identity of operators on $\mathcal{K}$:

$$\Phi_{a(\mathbf{k})} \circ \mu^{\mathcal{D}}_{\mathbf{k}}(A) = \iota^*\nu_{\mathbf{k}}(A) \circ \Phi_{a(\mathbf{k})}. \tag{4.1.2}$$

In particular, if $L \in \mathcal{T}^q_{\underline{\mathbf{i}}_0}$ and $L' \in \underline{\mathbf{i}}_n$ are related by $L = \mu_{\mathbf{k}}(L')$, then we have

$$\iota_0(L) \circ \Phi_a = \iota_n(L') \circ \Phi_a \tag{4.1.3}$$

as operators on $\mathbf{K}$.

The torus $\mathcal{T}_\Gamma$ associated to a cubic graph Γ, or its quantization $\mathcal{T}^q_\Gamma$, is the cluster chart $\mathscr{P}_\Gamma$ of $\mathscr{P}$, described in Sections 4 and 5. In the next section we show that the global Lagrangian submanifold $\mathscr{M} \subset \mathscr{P}$ is compatible with this chart-wise quantization.

4.2. **Quantizing the Chromatic Lagrangian.** We begin by discussing the quantization of the relevant connected component of the moduli space of framed PGL_2 local systems with unipotent monodromy on the punctured sphere. Fix a cubic graph Γ of genus g, and as in the previous section let $\mathcal{T}^q_\Gamma$ be the associated quantum torus. Suppose that $e_1, \ldots, e_n$ are the edges around a face f of Γ, listed in counterclockwise cyclic order around the face; note that this means that each e_{i+1} precedes e_i in the counterclockwise order with respect to their common vertex, so that we have

$$X_{e_i} X_{e_{i+1}} = q^{-2} X_{e_{i+1}} X_{e_i}.$$

Then the relation (4.0.2), which imposes unipotence of the monodromy around the puncture dual to corresponding f, is quantized as

$$X_{e_1+\ldots+e_n} = q^{-2}. \tag{4.2.1}$$

In order to pick out the required component, let $s = \sum_{e_i \in E} e_i \in \widetilde{\Lambda}_\Gamma$ be the sum of the edges. We then further impose the relation that

$$X_s = (-q)^{g+3}. \tag{4.2.2}$$

After quotienting by these relations, we obtain a symplectic quantum torus algebra $\mathcal{T}^q_{\underline{\Gamma}}$.

We now proceed to the quantization of the additive face relations that are equivalent to the triviality of the underlying unipotent local system at a point of $\mathcal{P}$. To this end, set

$$R_f = q^{-1} + X_{e_1} + X_{e_1+e_2} + \ldots + X_{e_1+e_2+\cdots e_{n-1}}. \tag{4.2.3}$$

Remark 4.3. It follows from the multiplicative face relation (4.2.1) that multiplying (4.2.3) by qX_{e_n} yields

$$X_{e_n} + X_{e_n+e_1} + X_{e_n+e_1+e_2} + \ldots + q^{-1},$$

so we see that the ideal in the quantum torus $\mathcal{T}^q_\Gamma$ generated by R_f is independent of our arbitrary linearization of the cyclic order on the edges around the face f implicit in (4.2.3).

Let $\mathcal{I}_\Gamma$ be the left ideal in $\mathcal{T}^q_\Gamma$ generated by all (4.2.1) along the global relation (4.2.2) and the relations R_f for all faces f.

Theorem 4.4. *The above system is compatible with the quantum cluster mutation.*

Proof. Reading in counterclockwise order for the left face of the right graph, let us set

$$\tilde{Z} = Z\phi_2(X), \qquad \tilde{X} = X^{-1}, \qquad \tilde{S} = S\phi_1(X).$$

The relations (**??**) yield

$$SX = q^{-2}XS$$

Therefore we get

$$\tilde{Z}\tilde{X}\tilde{S} = Z\phi_2(X)\cdot X^{-1}\cdot S\phi_1(X) = Z\cdot\phi_2(X)X^{-1}\phi_1(q^{-2}X)\cdot S = q^{-1}ZS.$$

Therefore (**??**) is preserved (and likewise for other faces). Similarly, we have

$$\begin{aligned} &1 + q\tilde{Z} + q^2\tilde{Z}\tilde{X} + q^3\tilde{Z}\tilde{X}\tilde{S} \\ =&1 + q\tilde{Z}(1+q\tilde{X}) + q^2 ZS \\ =&1 + qZ + q^2 ZS \end{aligned}$$

Hence (4.2.3) is preserved. ∎

We now illustrate the constructions of this section in the following simple but fundamental example.

Example 4.5. Consider the framed seed $\underline{\mathbf{i}}_0$ for the $g=1$ necklace graph Γ_0 shown in Figure 4.2.1. The additive face relation

$$R = q^{-1} + X_{e_2}$$

corresponding to its left bead is mapped under the corresponding framing isomorphism $\iota_0 : \mathcal{T}^q_{\Gamma_0} \to \mathcal{D}_2$ to the element

$$\iota_0(R) = q^{-1}(1-V).$$

Let us now perform a positive mutation at the edge e_3 of Γ_0 to obtain the framed seed for the canoe graph Γ_1 shown in Figure 4.2.2. Then we see that

$$R = \mu_3\left(R'\right), \qquad R' = q^{-1} + X_{e'_2} + X_{e'_2+e'_3},$$

where R' is the additive face relation associated to the face of Γ' bounded by (e'_1, e'_2, e'_3). Under the new framing isomorphism $\iota_1 : \mathcal{T}^q_{\Gamma_1} \to \mathcal{D}_2$, the element R' is mapped to

$$\iota_1(R') = q^{-1} + q^{-1}UV - q^{-1}V. \tag{4.2.4}$$

The element $\Phi_{\mu_3^+} \in \mathcal{A}_2$ is given by

$$\begin{aligned}\Phi_{\mu_3^+} &= \Phi\left(-q^{-1}U\right)^{-1}\\ &= (U, q^2)_\infty,\end{aligned}$$

and hence the operators associated to the face relations R, R' are indeed intertwined under by the action of Φ_a: we have

$$\iota_0(R) \circ \Phi_{\mu_3^+} = \iota_1(R') \circ \Phi_{\mu_3^+}.$$

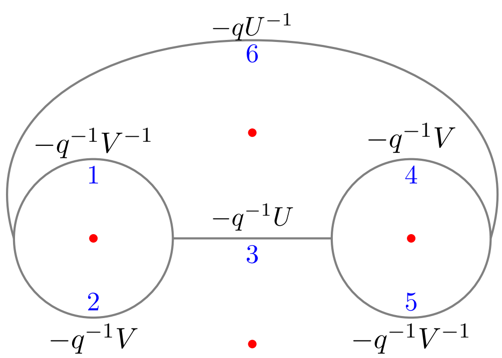


FIGURE 4.2.1. The standard necklace framed seed $\underline{\mathbf{i}}_0$ for $g = 1$

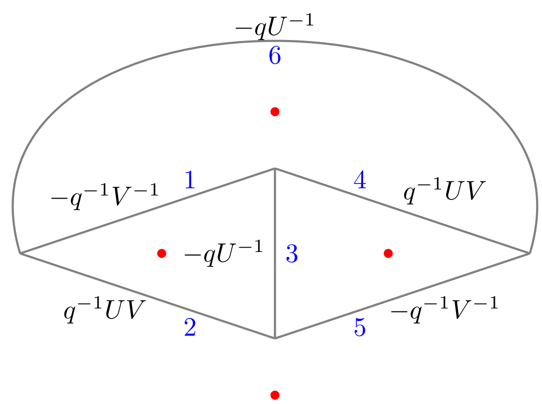


FIGURE 4.2.2. The framed seed $\underline{\mathbf{i}}_1 = \mu_3^+(\underline{\mathbf{i}}_0)$ for the canoe graph.

5. GLOBALIZATION

We now show that Γ labels a cluster chart $\mathscr{P}_\Gamma$ of a global moduli space $\mathscr{P}$ with a global Lagrangian $\mathscr{M}$ such that in each chart $\mathscr{M} \cap \mathscr{P}_\Gamma = \mathscr{M}_\Gamma$. We begin by considering a more general space.

5.1. **Moduli space of decorated local systems.** Let G be a split semisimple algebraic group over $\mathbb{Q}$ with trivial center. Let S be an oriented compact topological surface equipped with a non-empty subset $I = \{p_1, \ldots, p_n\}$ called punctures. A framed G-local system on S consists of data $(\mathcal{L}, \{B_1, \ldots, B_n\})$ where

- $\mathcal{L} \in \mathrm{Hom}(\pi_1(S - I), \mathrm{G})$ is a local system.
- for each puncture p_i, we assign a a flag (Borel subgroup) B_i of G invariant under the monodromy of $\mathcal{L}$ around p_i.

The moduli space $\mathscr{X}_{\mathrm{G},S}$ parametrizes G-orbits of framed G-local systems.

Theorem 5.1. *The space $\mathscr{X}_{\mathrm{G},S}$ is a cluster Poisson variety.*

Proof. For $\mathrm{G} = \mathrm{PGL}_n$, this is due to Fock-Goncharov [FG1, §9]. When G is of type B, C, D, this is [Le]. For a general G, this is [GS2]. □

For each puncture, the flag B_i chosen gives rise to a map from $\mathscr{X}_{\mathrm{G},S}$ to the Cartan subgroup H (by taking "eigenvalues" of monodromies). Therefore we get a map

$$\pi : \mathscr{X}_{\mathrm{G},S} \longrightarrow \mathrm{H}^n \tag{5.1.1}$$

The fibers of π are symplectic varieties. In particular, let us take $\mathscr{X}^{\mathrm{un}}_{\mathrm{G},S} := \pi^{-1}(1)$. This corresponds to the equations (4.0.2).

Now let i be a reflection of S that fixes the punctures. For example, if S is a sphere, then one can put all the punctures on the equator, and i exchanges the two hemispheres. Note that i changes the orientation of S. Therefore i induces an anti-Poisson involution of $\mathcal{X}_{G,S}$. Let s be the inverse map of H^n which takes $(h_1, ..., h_n)$ to $(h_1^{-1}, \ldots, h_n^{-1})$. By definition, the following maps commute

$$\begin{tikzcd}
\mathscr{X}_{\mathrm{G},S} \arrow[r, "\pi"] \arrow[d, "i"'] & \mathrm{H}^n \arrow[d, "s"] \\
\mathscr{X}_{\mathrm{G},S} \arrow[r, "\pi"] & \mathrm{H}^n
\end{tikzcd}$$

Therefore i maps $\mathscr{X}^{\mathrm{un}}_{\mathrm{G},S}$ to $\mathscr{X}^{\mathrm{un}}_{\mathrm{G},S}$.

Taking all the fixed points of the map i, we get a subvariety $\mathscr{M}_i$ of $\mathscr{X}^{\mathrm{un}}_{\mathrm{G},S}$.

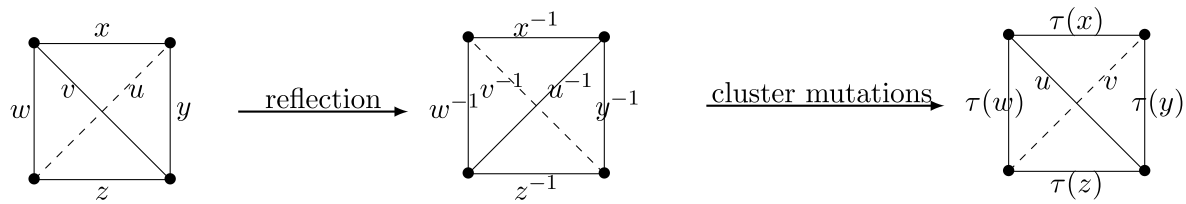
Theorem 5.2. *$\mathscr{M}_i$ is a Lagrangian subvariety of $\mathscr{X}^{\mathrm{un}}_{\mathrm{G},S}$.*

Proof. Let w be the symplectic form on $\mathscr{X}^{\mathrm{un}}_{\mathrm{G},S}$. Note that $i^*(w) = -w$. Since i is the identity map on $\mathscr{M}_i$, the restriction of w to $\mathscr{M}$ is trivial. It remains to check the dimension : $\dim \mathscr{M} = \frac{1}{2} \dim \mathscr{X}^{\mathrm{un}}_{\mathrm{G},S}$. □

5.2. **Example: the moduli space $\mathscr{X}_{PGL_2, S_{g+3}}$.** Now let S_{g+3} be a sphere with $g+3$ many punctures. Let $\mathcal{T}$ be an ideal triangulation of S_{g+3}, i.e., a triangulation with vertices at the punctures.

Remark 5.3. If S is a sphere, and $G = PGL_2$, then $\mathscr{M}$ in Theorem 4.2 is a connected component contained in each $\mathscr{M}$. Therefore Theorem 4.2 is a special case of Theorem 5.2.

Example 5.4. The following triangulations show an example of involution for a sphere with 4 punctures.



Here

$$\tau(u) = v; \qquad \tau(v) = u.$$

$$\tau(w) = w^{-1}(1+v^{-1})(1+u^{-1}); \qquad \tau(y) = y^{-1}(1+v^{-1})(1+u^{-1});$$

$$\tau(x) = x^{-1}(1+v)^{-1}(1+u)^{-1}; \qquad \tau(z) = z^{-1}(1+v)^{-1}(1+u)^{-1};$$

Note that the mapping class group of punctured sphere acts on $\mathcal{P}$ by symplectormorphisms. The mapping class group preserves $\mathcal{M}$, but it interchanges the other components of $\mathcal{M}_i$.

For general surfaces, we get many Lagrangians which depend on the reflection i chosen.

6. Foams, Phases and Framings

We have shown that the moduli space of constructible sheaves with singular support on a Legendrian surface Λ is a (quantum) Lagrangian subvariety (ideal) of a symplectic leaf in a (quantized) cluster Poisson variety. This ideal is defined by a "wavefunction." The purpose of this section is to describe the combinatorics of non-exact Lagrangian fillings $L \subset \mathbb{C}^3$ of the Legendrian. The geometric/combinatoric set-up will allow us to make conjectures about open Gromov-Witten invariants of the pair $(\mathbb{C}^3, L)$.

Here are the constructions we describe. We begin with a Legendrian surface S_Γ defined by a cubic planar graph $\Gamma \subset S^2$, as described in previous sections.

- A singular exact Lagrangian filling L_0 is constructed from an ideal foam, $\mathbf{F}$.
- A deformed foam $\mathbf{F}'$ determines a non-exact Lagrangian filling, L.
- L is a branched double cover of the three-ball, branched over a tangle, also defined by $\mathbf{F}'$.
- A deformation is described by a short *arc* between strands of the tangle at each vertex.
- The map $\tau : H_1(\Lambda) \twoheadrightarrow H_1(L)$ is determined combinatorially from $\mathbf{F}'$ and the arcs.
- A *splitting* of the map τ gives a *phase* and *framing.*
- We further require a maximal cone of $H_1(L)$.
- These constructions allow us to make open Gromov-Witten predictions about $(\mathbb{C}^3, L)$.
- All these notions can be carried through allowed mutations of the deformed foam $\mathbf{F}'$.

The upshot is that we get open Gromov-Witten predictions from the wavefunction at all points of the framed seed groupoid accessed by allowed mutations from the necklace foam. This is a large class of Lagrangian fillings and framings.

We now proceed as outlined above.

6.1. **Foams.** A cubic graph Γ on the sphere S is dual to a triangulation of S. If Γ is three-connected, then by Steinitz's theorem it is the edge graph of a polyhedron. A *foam* $\mathbf{F}$ is the dual structure to a tetrahedronization of the polyhedron: it is a polyhedral decompsition of the three-ball B with $\partial B = S$. The data of $\mathbf{F}$ includes the quadruple (R, F, E, V) of regions, faces, edges and vertices. A face or edge is called *external* if it intersects the boundary, and *internal* if it does not. The foam is *ideal* if it is dual to an ideal tetrahedronization of B, i.e one with no internal vertices. Even if Γ is not dual to a polyhedron, the notion of ideal foam makes sense — see [TZ, Definition 3.1]. For example, if there is a bigon between two vertices, then there is a single edge of the foam whose boundary is those vertices — see Example 6.1.

Example 6.1 (Foam filling for Γ_g^{neck})**.** The necklace graph has a distinguished foam filling, that we in fact believe to be unique. This foam has no vertices: $\mathbf{F}^1$ is already smooth — in other words there is a unique phase. See Figure 6.1.1 below. In fact, using the local construction at the left of Figure 6.1.1, we can construct similar foam fillings of any iterated sequence of bigon additions (handle attachments for the Legendrian surface), starting from the genus-zero necklace (theta graph). We refer to these as necklace-type graphs, and equip them with these canonical foam fillings. Note that while generic foams are dual to tetrahedronizations, these foams are dual to somewhat degenerate tetrahedronizations. For that reason, we will mainly focus on foams and not their duals.

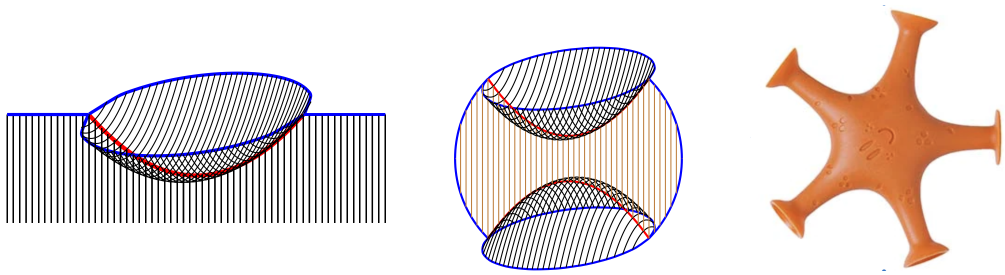


FIGURE 6.1.1. The necklace graph Γ_g^{neck}, pictured in blue, and its foam filling. At left is a local model near a bead, with tangle strand in red. In the middle is the foam filling for $g = 1$. The Ubbi toy at right is something close to a foam for Γ_4^{neck}.

6.1.1. *The Harvey-Lawson Foam.* The foam $\mathbf{F}_{\text{HL}}$ of the Harvey-Lawson Lagrangian $L_0 \subset \mathbb{C}^3$ has a single vertex at the origin in $\mathbb{R}^3$, four edges E_i equal to the rays $\mathbb{R}_{\geq 0} \cdot v_i$ where $v_0 = (1, 1, 1)$ and $v_i = -v_0 + 2e_i$, with e_i the standard basis vectors. There are $6 = \binom{4}{2}$ faces F_{ij} equal to the cones spanned by unordered pairs of edges, and $4 = \binom{4}{3}$ regions equal to the cones spanned by triples of edges. (It can be succinctly described as the toric fan of $\mathbf{P}^3$.)

The singular, exact Harvey-Lawson Lagrangian L_0 in $(\mathbb{C}^3, \omega_{\text{std}} = d\theta_{\text{std}})$ is a branched $2 : 1$ cover of $\mathbb{R}^3$, branched over the over edges. L_0 is a cone over $S^1 \times S^1$ with parametrization $(r, s, t) \mapsto (re^{is}, re^{it}, re^{-i(s+t)}) \in \mathbb{C}^3$, where $r \in \mathbb{R}_{\geq 0}$ and $(s, t) \in S^1 \times S^1$. The covering map is the restriction to L_0 of $\mathbb{C}^3 \to \mathbb{R}^3$ sending a complex triple to its real part: explicitly $(r, s, t) \mapsto (r\cos(s), r\cos(t), r\cos(s + t)) \in \mathbb{C}^3$. The map is $1 : 1$ over the four rays with $(s, t) = (0, 0), (\pi, 0), (0, \pi), (\pi, \pi)$, which we think of as a singular tangle. The six sheets of the foam are defined by $s = 0, s = \pi, t = 0, t = \pi, s + t = 0, s + t = \pi$.

The primitive function f obeying $df = \theta_{\text{std}}|_L$ is $f = \frac{1}{4}r^2 (\sin(2s) + \sin(2t) - \sin(2s + 2t))$. Note that f is odd under the hyperelliptic-type involution $(s, t) \leftrightarrow (-s, -t)$ and $f = 0$ along the preimages of the sheets of the foam. Thus f allows us to label the branches of L_0 on the regions R.

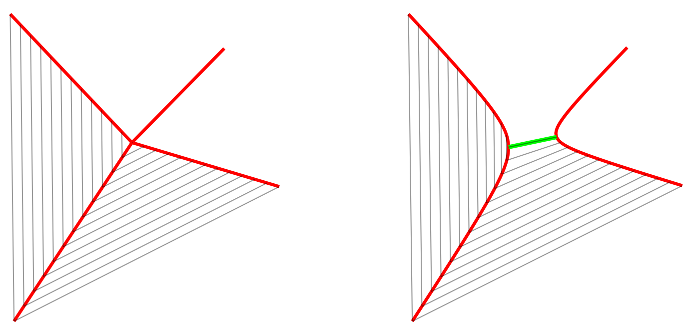


FIGURE 6.1.2. Left: the Harvey-Lawson foam, with its four edges but just two of the $\binom{4}{2} = 6$ faces drawn. Right: the deformed foam, with arc in green, and the two deformed faces drawn.

6.1.2. *Foams and singular exact Lagrangians.* From a foam $\mathbf{F}$ we can define a singular exact (not necessarily special) Lagrangian L_0 locally modeled on the Harvey-Lawson cone and foam — see [TZ, Section 3.2]. As with the Harvey-Lawson cone and foam, we can define a multi-valued function f whose sign labels the branches of L_0 in the regions of the foam.

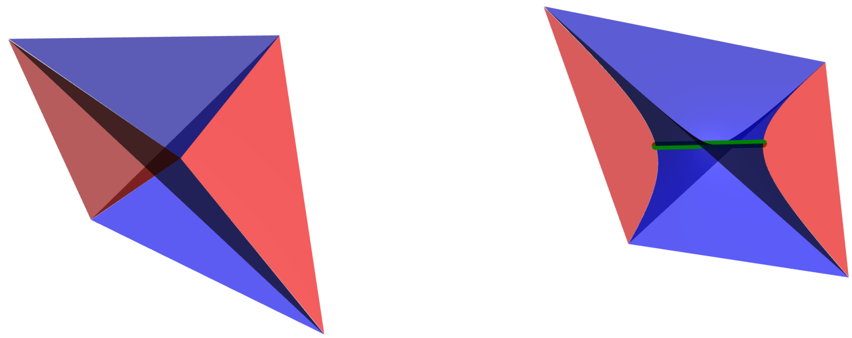


FIGURE 6.1.3. At left is the foam of the Harvey-Lawson Lagrangian, with all $6 = \binom{4}{2}$ sheets drawn. Warning: diagonally opposite vertices lie in different half-spaces, so despite appearances the two corresponding triangular sheets do *not* intersect outside the origin. At right is the deformed foam of its smoothing. Two sheets (pink) are smoothed to have hyperbolas as boundaries, while the boundaries of the other four consist of two halves of different hyperbolas, as well as the arc (green).

6.1.3. *Deformation of the Harvey-Lawson foam.* There are three distinct families of smoothings of L_0 corresponding to the three matchings of the four edges. We will describe the one for the matching $0 \leftrightarrow 1, 2 \leftrightarrow 3$; the others are similar and are related by a permutation of coordinates. The smoothing L_ϵ has the topology of $\mathbb{R}^2 \times S^1$ and has a parametrization in polar coordinates $(r, s, t) \mapsto (\sqrt{r^2+\epsilon^2}e^{is}, re^{it}, re^{-is-it}) \in \mathbb{C}^3$, which maps to $(\sqrt{r^2+\epsilon^2}\cos(s), r\cos(t), r\cos(s+t)) \in \mathbb{R}^3$. These are all diffeomorphic for $\epsilon \neq 0$, so when we are interested in topological questions, we can restrict to L_1 without loss of generality. The branched cover is $1:1$ over the points with $(s,t) = (0,0), (\pi, 0), (0, \pi), (\pi, \pi)$, and these parametrize four rays E_i which trace out *two* hyperbola components ($E_0 \cup E_1 = \{x^2 - y^2 = 1, y = z, x > 0\}$ and $E_2 \cup E_3 = \{x^2 - y^2 = 1, y = -z, x < 0\}$), a smoothing of the singular tangle of L_0.)

There is also the line segment $a \subset \mathbb{R}^3$ between $(-1, 0, 0)$ and $(1, 0, 0)$ which we call an *arc* — it is the image of $r = 0$. Note that $L_1 \to \mathbb{R}^3$ is $2:1$ over the arc. The six sheets F_{ij} now bound either a smooth edge $E_i \cup E_j$ if $(ij) = (01)$ or (23), or otherwise the union $E_i \cup E_j \cup a$. This will be our local model of a deformed foam. More generally, let s_i be the matching of edges of $\mathbf{F}_{\mathrm{HL}}$ which pairs v_0 and v_i. We write $\mathbf{F}_{\mathrm{HL},s_i}$ for the local deformed foam of L_1 Its arc is the line segment between $-e_i$ and e_i. We write $\mathbf{F}_{\mathrm{HL},\epsilon,s_i}$ for the deformed foam of L_ϵ.

Away from the origin and the preimage of the arc, the Harvey-Lawson cone and its smoothing are homeomorphic: $L_0|_{r\neq 0} \cong L_1|_{r \neq 0}$. As a result, the same primitive function f can be used to label regions of the foam and of its deformation, at least away from the arc. The local geometry of L and the deformed foam near an arc is shown in Figure 6.1.4.

6.1.4. *Deformed Foams.* Given a foam $\mathbf{F}$, we will define a deformed foam $\mathbf{F}'$ to be a structure locally modeled near each vertex on a Harvey-Lawson deformed foam.

Definition 6.2. Let $\mathbf{F}$ be a foam with vertex set V consisting of $n := \#V$ vertices. Write $\mathcal{S}$ for the set of matchings of half edges at each vertex, so $\#\mathcal{S} = 3^n$. Let $s \in \mathcal{S}$. We define a *deformed foam* $\mathbf{F}_s$ to be any set of vertices, edges, arcs, faces and regions which agrees with $\mathbf{F}$ outside some 3ϵ-neighborhood of V, is homeomorphic to $\mathbf{F}$ outside of a 2ϵ-neighborhood of V, and which is linearly equivalent to the local deformed foam of $\mathbf{F}_{\mathrm{HL},\epsilon,s}$ of Section 6.1.3 within a 2ϵ-ball of each vertex.

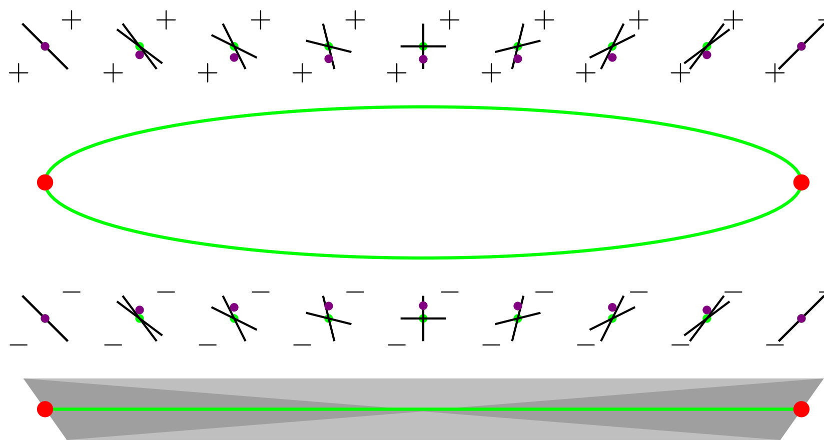
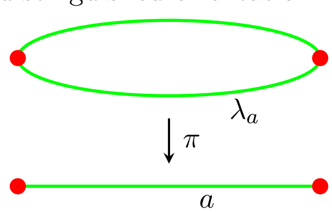


FIGURE 6.1.4. The neighborhood of an arc (the green line segment) and its lift to the Lagrangian (green oval). Four sheets, forming two surfaces (gray) meet at the arc. The cross-sectional planes are shown, along with the sign of the primitive function f on L. The red dots are where the sheets $s = 0, \pi$ meet the arc, so the sign of f changes as they are crossed. The purple dot is the cross section of the oriented loop γ_a — see Definition 6.9.

The smoothed Lagrangian L is branched over a tangle $T \subset B$, i.e. T is a one-manifold. The construction of L from a deformed foam identities a particular set of branch cuts we call *arcs*.

Definition 6.3. Let s_i be the smoothing of the Harvey-Lawson foam which matches the ray $\mathbb{R}_{\geq 0} \cdot v_0$ with $\mathbb{R}_{\geq 0} \cdot v_i$, where v_0 and v_i are as in Section 6.1.1. The *arc* of the deformed Harvey-Lawson foam $\mathbf{F}_{\mathrm{HL},s_i}$ is the line segment from $-e_i$ to e_i. An *arc* of a deformed foam $\mathbf{F}'$ is the locus in B corresponding to the arc of the Harvey-Lawson foam under the local identification of $\mathbf{F}'$ with $\mathbf{F}_{\mathrm{HL},s_i}$. We write A for the set of arcs of a deformed foam $\mathbf{F}'$.

Given an arc a of a deformed foam $\mathbf{F}'$ and its associated branched double cover $\pi : L \to B$, define $\lambda_a = \pi^{-1}(a)$. Since π is $2 : 1$ over the interior of the arc and $1 : 1$ at its edges, λ_a is a circle. Note that λ_a does not (yet) have a distinguished orientation.



Remark 6.4. If $\mathbf{F}$ is a foam, a deformed foam $\mathbf{F}'$ is defined by choosing a matching of the four internal edges meeting at each vertex. In the case of the necklace graph Γ^g_{neck}, the foam $\mathbf{F}$ has no internal vertices, and therefore $\mathbf{F}' = \mathbf{F}$. In particular, $\mathbf{F}$ is already deformed. In fact, we will learn that the $g + 1$ strands of Γ^g_{neck} can be thought of as the arcs of $\mathbf{F}' = \mathbf{F}$, and the face of $\mathbf{F}'$ which they bound gives rise to a single relation among them – see Definition 6.12 and Proposition 6.13. Similar considerations apply whenever Γ has a bigon.

6.2. **Phases and Framings.**

Definition 6.5. Let $H \cong \mathbb{Z}^{2g}$ be a rank-$2g$ lattice with a non-degenerate, antisymmetric pairing ω. A *phase* is a rank-g isotropic subgroup $K \subset H$. A *framing* of K is a transverse isotropic subspace. We call the combination of phase and framing an *isotropic splitting*, or sometimes just *splitting*.

We will be studying phases and framings when $H = H_1(\Lambda)$ is the homology of a genus-g surface, Λ and ω is the intersection pairing. So let L be an orientable three-manifold with boundary a genus-g surface $\Lambda = \partial L$, with $H_2(L) = 0$. Then it follows from the long exact sequence in homology together with the Poincaré-Lefschetz duality isomorphisms $H^1(L) \simeq H_2(L, \Lambda), H_1(L, \Lambda) \simeq H^2(L)$ that $b_1(L) = g$,[3] so that we obtain the short exact sequence

$$0 \longrightarrow H^1(L) \longrightarrow H_1(\Lambda) \xrightarrow{\tau} H_1(L) \longrightarrow 0 \tag{6.2.1}$$

The notion of phases and framings will apply to above geometic setting.

Definition 6.6. Suppose $H = H_1(\Lambda)$ is the first homology of a genus-g oriented surface Λ, and L is an orientable three-fold with $\partial L = \Lambda$. We have the short exact sequence of Equation 6.2.1. We say that a phase $K \subset H$ is *geometric* if $K = Ker(\tau) \cong H^1(L)$. An accompanying framing is an ω-isotropic splitting $\tau : H_1(L) \hookrightarrow H_1(\Lambda)$ of the short exact sequence (6.2.1).

In the context of open Gromov-Witten theory, Λ is Legendrian in a contact manifold and L is Lagrangian in a symplectic filling.

Remark 6.7. In [TZ], the above geometric phases were called "OGW framings" to connote open Gromov-Witten theory. The definition was generalized from [AKV], where mirror symmetry was used to make conjectures in open Gromov-Witten theory.[4] The terminology stems from the connection to Chern-Simons theory through large-N duality, where Lagrangians are knot conormals and framing relates to the framing of knots. We describe the connection to open Gromov-Witten theory later in this section.

Remark 6.8. Here is why we need phases and framings. Let $\mathscr{X} = \mathscr{X}_{PGL_2,S^2}$ be the cluster variety of framed local systems on a sphere. Let $\mathscr{P}$ be the symplectic leaf of unipotents, and let $\mathscr{M}$ be the Lagrangian subvariety defined by trivial monodromy. Let Γ be a cubic graph on the sphere $S = S^2$ and $S_\Gamma \subset J^1(S)$ the associated a Legendrian surface up to isotopy. We write $\mathscr{P}_\Gamma = H^1(S_\Gamma; \mathbb{C}^*)$ for the corresponding cluster chart.[5] A splitting allows us to write $H^1(S_\Gamma; \mathbb{C}^*)$ as $T^*(H^1(L; \mathbb{C}^*))/H_1(L)$. When we lift $\mathscr{M}$ to $T^*H^1(L; \mathbb{C}^*)$ we can write it locally as the graph of the differential of a function W_Γ on $H^1(L; \mathbb{C}^*)$, from which we will extract enumerative information.

Recall from [TZ] the combinatorial model of the first homology of a Legendrian $\Lambda := S_\Gamma$ defined from a cubic planar graph Γ on a sphere, S. The faces of Γ define a relation $\sim_\Gamma$ on the edge lattice $\mathbb{Z}^{E_\Gamma}$, namely $\sum_{e\in\partial f} e \sim_\Gamma 0$. We then have $H_1(S_\Gamma) \cong \mathbb{Z}^{E_\Gamma}/\sim_\Gamma$. We have an antisymmetric pairing $\overline{\omega}$ on $\mathbb{Z}^{E_\Gamma}$, depending only on the orientation of S, defined by $\overline{\omega}(e, e') = \pm 1$ if e and e' are adjacent to a vertex v with e preceding/following e' in the cyclic ordering at the vertex , and zero otherwise. Since $\sum_{e\in\partial f} e$ generates the kernel of this pairing, $\overline{\omega}$ descends to a nondegenerate, antisymmetric intersection pairing ω on

$$H_1(S_\Gamma) \cong \mathbb{Z}^{E_\Gamma}/\sim_\Gamma . \tag{6.2.2}$$

We now have a combinatorial model of $H_1(S_\Gamma)$. We next build combinatorial models of $H_1(L)$ for L arising from a deformed foam, and of the map $H_1(S_\Gamma) \to H_1(L)$.

[3] In this section, homology will be taken with $\mathbb{Z}$ coefficients unless otherwise stated.

[4] In [I] a similar definition of framing is made, but without the isotropic condition.

[5] As explained in Section 4, the cluster charts $\mathscr{P}_\Gamma$ of $\mathscr{P}$ are spaces of rank-one local systems with fixed monodromy -1 at the critical points of the branched double cover $S_\Gamma \to S^2$. This space is non-canonically isomoprhic to the torus $Loc_1(S_\Gamma) \cong H^1(S_\Gamma, \mathbb{C}^*)$. In fact, it is a torsor modeled on the latter space. For given any base point $\mathcal{L}_0 \in \mathscr{P}_\Gamma$ and any point $\mathcal{L} \in Loc_1(S_\Gamma)$, we have $\mathcal{L} \otimes \mathcal{L}_1 \in \mathscr{P}_\Gamma$. As such, the tangent space of $\mathscr{P}_\Gamma$ at any point is canonically $H^1(S_\Gamma; \mathbb{C})$ and its Poisson structure is determined by the intersection form on S_Γ, independent of choice of base point. Hereafter, we often omit the distinction and refer to cluster charts as the tori $H^1(S_\Gamma, \mathbb{C}^*)$.

6.3. **Combinatorics of Tangles from Deformed Foams.** We continue our study of smooth Lagrangians arising from deformed foams. Cutting to the chase, the loops defined by the edge set E and arc set A will generate $H_1(S_\Gamma)$ and $H_1(L)$, with relations determined by faces. In total, we find

$$\begin{tikzcd} \mathbb{Z}^{E_\Gamma} \arrow[r, "\sim_\Gamma"] \arrow[d, "\iota"] & H_1(S_\Gamma) \arrow[d, "\tau"] \\ \mathbb{Z}^{E_\Gamma \cup A} \arrow[r, "\sim_{\mathbf{F}'}"] & H_1(L) \end{tikzcd} .$$

Here ι is induced by the inclusion $E_\Gamma \to E_\Gamma \cup A$, and the top line was defined in the previous section. The bottom line will be defined in this section.

Let $\Gamma \subset S$ be a cubic graph on the sphere, let $\mathbf{F}$ be an ideal foam on the three-ball B, whose regions, faces and edges respectively bound the faces, edges and vertices of Γ. Let $\mathcal{S}$ be the discrete set of smoothings of $\mathbf{F}$, i.e. the set of matchings of edges incident to each vertex of $\mathbf{F}$ — so $\#\mathcal{S} = 3^{\#V}$. Let s be a smoothing whose resulting tangle T has no circle components. Let L be a smooth Lagrangian corresponding to the deformed foam $\mathbf{F}_s$.

Recall that for an arc a we werite $\lambda_a := \pi^{-1}(a)$. We now define an orientation on λ_a, thus definining an element $\gamma_a \in H_1(L)$.

Since the construction of the smoothing is local, we need only look at the Harvey-Lawson smoothing L_1 and its unique arc a, which we can lift to the *parametrized* curve $(e^{it}, 0, 0)$ and take the induced orientation. This is the orientation induced from the unique holomorphic disk in $\mathbb{C}^3$ bounding L, i.e. $|z_1| \leq 1$. We can also give a more combinatorial construction that does not require an explicit local model, as follows.

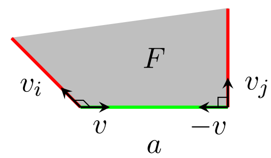
Definition 6.9. We choose a canonical orientation for λ_a by orienting the arc arbitrarily and taking a push-off $\tilde{\lambda}_a$ of the path along the arc that has some combinatorial properties, using the primitive function, f. We require that near the start of the push-off, in the chosen orientation, that f has a negative value and lies in one of the two fat regions (see Figure 6.1.4) — in particular, outside of two sheets which meet at the arc's origin — then crosses once at the midpoint of the arc in a counterclockwise direction (in the induced orientation of the transverse plane). The remainder of $\tilde{\lambda}_a$ traverses the arc backwards after crossing the origin of the transverse plane at the arc's endpoint, and has the same combinatorial recipe as the first half of $\tilde{\lambda}_a$. This completes the description of the push-off, $\tilde{\lambda}_a$. There are actually two such push-offs, but the resulting paths are homotopic. Likewise, the opposite orientation of the arc leads to a homotopic path (just shifted). For an arc a, write γ_a for the resulting element of $H_1(L)$.

Remark 6.10. The prescription in Definition 6.9 is similar to how unoriented edges of a cubic planar graph lead to an oriented loop of the associated branched cover — see [TZ, Section 4.6] — albeit somewhat more intricate.

For each face of the deformed foam $\mathbf{F}'$, we will define a relation among the edges e and arc loops γ_a along its boundary. Together these relations will characterize $H_1(L)$ as $\mathbb{Z}^{E_\Gamma \cup A}/\sim_{\mathbf{F}'}$. To define the relation, we need a careful discussion of the sign of an arc relative to a face.

Definition 6.11. Let F be a face of a deformed foam $\mathbf{F}'$ bounding an arc $a \in A$. Then we have a homeomorphism of a neighborhood of a with a neighborhood of the lone arc of the Harvey-Lawson deformed foam $\mathbf{F}_{\mathrm{HL},s_i}$ defined by some smoothing s_i which pairs the edges containing vectors v_0 and v_i — see Section 6.1.3. Let F'_{ij} be the face of $\mathbf{F}_{\mathrm{HL},s_i}$ corresponding to F, which deforms the face F_{ij} of $\mathbf{F}_{\mathrm{HL}}$ containing v_i and v_j (note $i, j \in \{0, 1, 2, 3\}$). Let us orient the arc from the end bounding the strand of the tangle deforming the edge of $\mathbf{F}_{\mathrm{HL}}$ with v_i to the end bounding the tangle strand deforming the edge with v_j. Call a vector along the arc in this orientation v. We define the sign of the arc relative to the face by

$$\sigma(F,a) := \operatorname{sgn}\det(v, v_i, v_j) = \operatorname{sgn}\det(-v, v_j, v_i)$$



Note that the opposite orientation on the arc leads to $\operatorname{sgn}\det(-v, v_j, v_i)$, which is the same. The definition therefore only depends on the orientation of B.

Definition 6.12. Let $\mathbf{F}$ be foam with boundary Γ, and let $\mathbf{F}'$ be a deformed foam with arc set A. We define a relation $\sim_{\mathbf{F}'}$ on $\mathbb{Z}^{E_\Gamma \cup A}$ by setting

$$\sum_{e\in\partial F} e + \sum_{a\in\partial F} \sigma(F,a)\cdot a \;\sim_{\mathbf{F}'}\; 0, \tag{6.3.1}$$

for each face F of $\mathbf{F}'$.

6.4. **Face relations for foams.** On general grounds, L with $\partial L = S_\Gamma$ and $b_1(L) = g = \frac{1}{2}b_1(S_\Gamma)$ defines a phase as the kernel of the surjection $\tau : H_1(S_\Gamma) \twoheadrightarrow H_1(L)$. Here we want to understand this combinatorially when L arises from a deformed foam $\mathbf{F}'$, in terms of its arcs and the edges of Γ.

Proposition 6.13. *Let $\mathbf{F}$ be an ideal foam filling a cubic graph Γ with edge set E_Γ, and let L be a smoothing associated to a deformed foam $\mathbf{F}'$ with arc set A, such that the corresponding tangle has no circle components. Let $\sim_{\mathbf{F}'}$ be as in Definition 6.12. Then $H_2(L) = 0$, and we have an isomorphism*

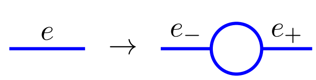
$$H_1(L) \cong \mathbb{Z}^{E_\Gamma \cup A}/\sim_{\mathbf{F}'}$$

such that the homology pushforward $H_1(\Lambda) \to H_1(L)$ is identified with the map induced by the inclusion $\mathbb{Z}^{E_\Gamma} \hookrightarrow \mathbb{Z}^{E_\Gamma \cup A}$.

Before the proof, a remark.

Remark 6.14. If Γ has no bigons, then each edge e is equivalent to a sum of arcs under Equation 6.3.1 by the external face of $\mathbf{F}'$ containing e in its boundary. Then after taking the partial quotient of $\mathbb{Z}^{E_\Gamma \cup A} \to \mathbb{Z}^A$ by the external faces of $\mathbf{F}'$, we may think of $H_1(L)$ as $\mathbb{Z}^A/\sim_{\mathbf{F}'}$.

Proof. We prove the Proposition by induction on the number of internal vertices of foams $\mathbf{F}$ with genus-g boundary surface. The base case then consists of any foam filling of a necklace-type graph of genus-g, as in Example 6.1. In fact, each such graph is itself obtained from the genus-0 necklace (theta graph) by bigon addition, or Legendrian one-handle attachment of the corresponding Legendrian surface (see [CZ, Theorem 4.10(1)]). So we can induct on the genus of the base case. The genus-zero foam consists of the three filled semicircles in the unit balls at azimuthal angles $0, 2\pi/3, 4\pi/3$. Now we add bigons. Each bigon addition adds three edges and one face to the boundary, as seen here,



thereby increasing H_1 of the Legendrian by 2 and the genus by 1. Two faces are added to the foam, which end in the two edges of the bigon. The bigon edges sum to zero in homology of the Legendrian, by the relation from the bigon face. The foam face relations then show that these edges are trivial in H_1 of the filling, thus in the kernel of the homology map corresponding to inclusion of the boundary — see Figure 6.1.1. The difference $e_+ - e_-$ is in no boundary and therefore is an additional nontrivial class in H_1 of the Lagrangian filling the new Legendrian. This establishes the base case of no internal vertices, for every genus.

We now induct on the number of internal vertices by attaching a Harvey-Lawson foam. We can attach at a single vertex or along an edge. (What about along a face?) To verify the inductive step in the former case, let $\mathbf{F}'$ be a smoothed ideal foam, whose boundary is a cubic graph $\Gamma_{\mathbf{F}}$ of genus g.

Now suppose Δ is a single tetrahedron together with a smoothed Harvey-Lawson foam $\mathbf{F}'_\Delta$ in it. Let us choose a vertex v of $\Gamma_{\mathbf{F}}$ along with a vertex w of the cubic graph Γ_Δ on the boundary of the tetrahedron. Let us write e_1, e_2, e_3 for the three edges of $\Gamma_{\mathbf{F}}$ incident to the vertex v listed in cyclic order determined by the orientation, and similarly write $\epsilon_1, \epsilon_2, \epsilon_3$ for the edges of Γ_Δ incident to w, but listed in opposite cyclic order. Each of these edges determines an external face of the corresponding foam, which we denote by f_{e_i} or f_{ϵ_i}. We glue a neighborhood of the vertex w to $\mathbf{F}'$ by identifying the tetrahedron (dual) face corresponding to w with the boundary (dual) face of $\mathbf{F}$ corresponding to v as indicated in Figure 6.4.1, so that each edge e_i is glued to the corresponding ϵ_i to form a new edge $\bar{e}_i$. As a result of this gluing, we obtain a new ideal foam $\widetilde{\mathbf{F}}'$.

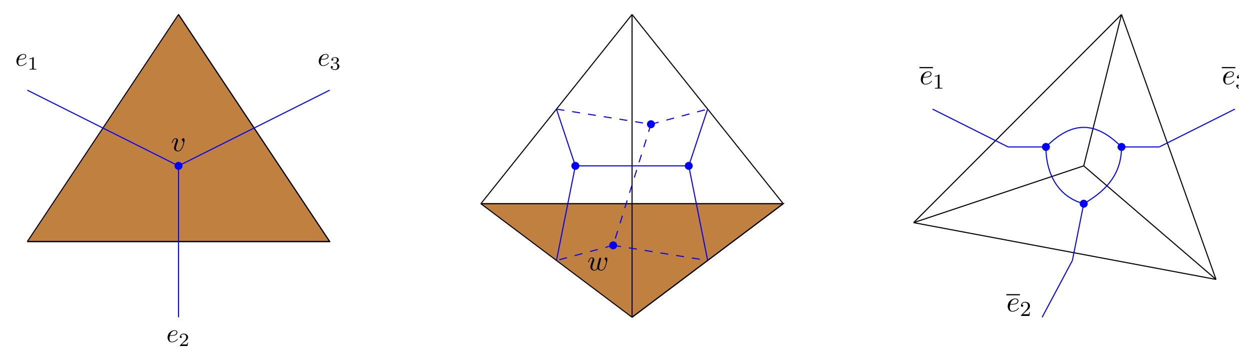


FIGURE 6.4.1. Attaching a Harvey-Lawson foam to $\mathbf{F}$ in a neighborhood $\widehat{f}$ of a vertex, shaded in brown. (Dually, $\widehat{f}$ is a face of the dual triangulation to $\Gamma_{\mathbf{F}}$.)

The set of faces of the new deformed foam $\widetilde{\mathbf{F}}'$ may be described as follows. The internal faces of $\widetilde{\mathbf{F}}'$ are the same as those of $\mathbf{F}'$. The set of external faces of $\widetilde{\mathbf{F}}'$ consists of all those external faces of $\mathbf{F}'$ and $\mathbf{F}'_\Delta$ that correspond to edges of $\Gamma_{\mathbf{F}}$ or Γ_Δ not incident to v, w, along with three faces f_1, f_2, f_3 obtained by gluing each f_{e_i} to the corresponding f_{ϵ_i}.

We now turn our attention to the effect of this gluing at the level of the double covers of the ball. Let us write $\pi : L \to B$, $\pi_\epsilon : \mathrm{HL}_\epsilon \to B$ for the branched double covers corresponding to $\mathbf{F}'$ and $\mathbf{F}'_\Delta$ respectively, so that we have $\widetilde{L} = L \cup_{\pi^{-1}(\hat{f})} \mathrm{HL}_\epsilon$, where $\hat{f}$ is the neighborhood of the vertex v where we attached $\mathbf{F}'_\Delta$, i.e. the face of along which the dual tetrahedron Δ was glued. This gluing is illustrated in Figure 6.4.2.

Now since the space $\pi^{-1}(\hat{f})$ is homeomorphic to a disk, the Mayer-Vietoris long exact sequence shows that $H_2(\widetilde{L}) \simeq H_2(L) \oplus H_2(\mathrm{HL}_\epsilon) = 0$. Similarly, it delivers an isomorphism

$$i_* + \iota_* : H_1(L) \oplus H_1(\mathrm{HL}_\epsilon) \to H_1(L'), \tag{6.4.1}$$

such that $[\bar{e}_i] = i_*([e_i]) + \iota_*([\epsilon_i])$. Hence all that remains is to verify the face relations for the faces f_1, f_2, f_3 of $\widetilde{\mathbf{F}}'$ obtained by gluing faces of $\mathbf{F}'$ to those of $\mathbf{F}'_\Delta$. But recall from 6.11 that the definition

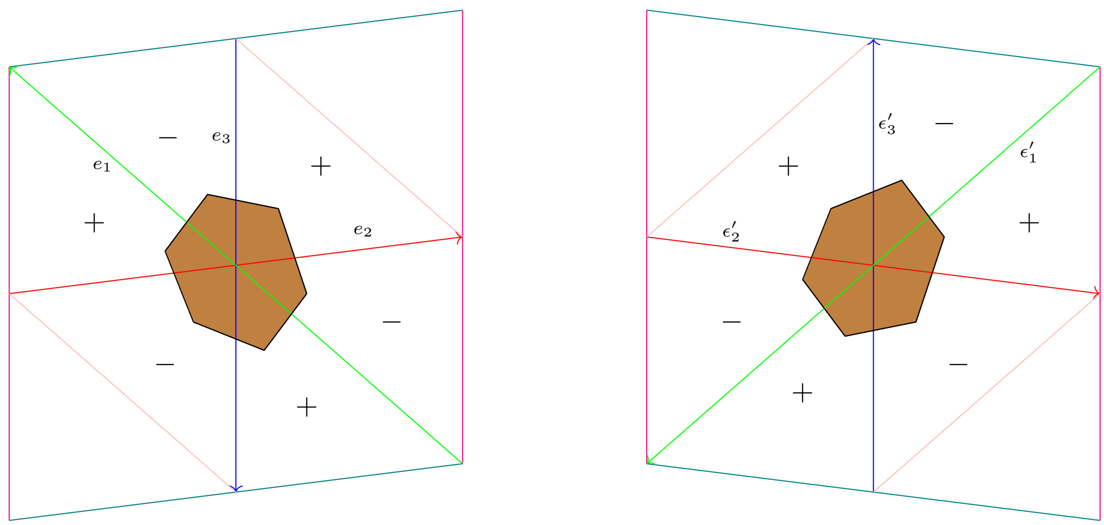


FIGURE 6.4.2. Gluing two copies of HL_ϵ along the disk $\pi^{-1}(\hat{f}) \subset \partial \mathrm{HL}_\epsilon \simeq T^2$. The set $\pi^{-1}(\hat{f})$ is shaded brown, and opposite pairs of boundary edges of the square are identified in the figure. The arrows on edges indicate the canonical lifts of the edges of the cubic graphs on S^2 to cycles in $H_1(\Lambda)$. External faces of the foam are labelled by the corresponding sign of the primitive function.

of the sign of an arc a relative to a face f is entirely local, depending only on the tangent vectors v_i, v_j to the two edges of the deformed foam that meet a and bound f. So if a_0 is the unique arc at the vertex of $\mathbf{F}'$ that is connected to v by an edge, and a_1 the corresponding arc in $\mathbf{F}'_\Delta$ (connected to w), the sign of a_0 with respect to face f_{e_i} in $\mathbf{F}$ is identical to its sign with respect to face f_i of the glued foam $\mathbf{F}'$. Similarly, the sign of a_1 with respect to f_{ϵ_i} coincides with its sign with respect to f_i. The face relation for f_i is therefore obtained as the sum of those for f_{ϵ_i} and f_{e_i} under the isomorphism (6.4.1).

It remains to consider the case of gluing in a Harvey-Lawson cone along an edge. We have a foam $\mathbf{F}$ with boundary $\Gamma_{\mathbf{F}}$ and a Harvey-Lawson foam $\mathbf{F}_\Delta$ with boundary a tetrahedron graph Γ_Δ. Suppose that we fix an edge e_0 of $\Gamma_{\mathbf{F}}$ connecting two vertices v_1, v_2, and correspondingly fix an edge ϵ_0 of Γ_Δ connecting vertices w_1, w_2 of Δ. Let us denote the edges of $\Gamma_{\mathbf{F}}$ incident to v_1 by e_0, e_1, e_2, cyclically ordered in accordance with the orientation of ∂B, and similarly write e_0, e_3, e_4 for the edges incident to v_2. We denote by $\epsilon_0, \epsilon_1, \epsilon_2$ the edges incident to w_1 but ordered with respect to the opposite of the orientation on Δ, and similarly write $\epsilon_0, \epsilon_3, \epsilon_4$. We write ϵ_5 for the remaining edge of Γ_Δ which is incident to neither w_1 nor w_2. We now glue the foam $\mathbf{F}_\Delta$ to $\mathbf{F}$ by identifying the edges so that each edge e_i is glued to the corresponding ϵ_i. We denote by $\widetilde{\mathbf{F}}$ the ideal foam produced as a result of this gluing.

Note that the cubic graphs $\Gamma_{\widetilde{\mathbf{F}}}$ and $\Gamma_{\mathbf{F}}$ have the same genus: indeed, the two are related by a single diagonal exchange/edge mutation, as illustrated in Figure 6.4.3. (We will return to this point in Proposition 6.22.) The set of external faces of $\widetilde{\mathbf{F}}$ is thus in natural bijection with that of $\mathbf{F}$: the latter contains the external faces $\{f_i, i = 1, \dots 4\}$ obtained by gluing each face f_{e_i} to the corresponding f_{ϵ_i}, along with the external face with boundary f_{ϵ_5}. On the other hand, we now have a new *internal* face f_0 created by gluing f_{e_0} to f_{ϵ_0}. By assumption, the smoothing of the foam in

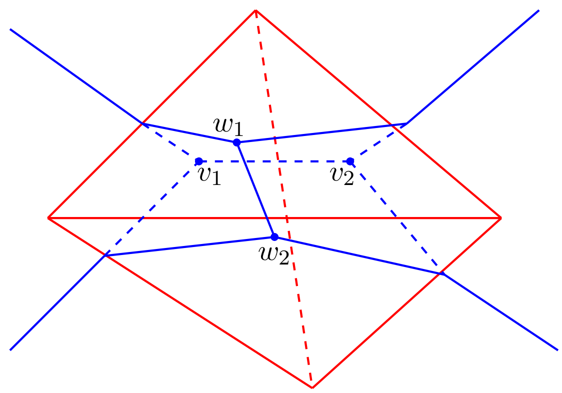


FIGURE 6.4.3. The cubic graph (shown solid in blue) produced by gluing a Harvey-Lawson foam an edge is related to the orignal (blue, dotted) by a diagonal exchange. The Harvey-Lawson dual tetrahedron is shown in red.

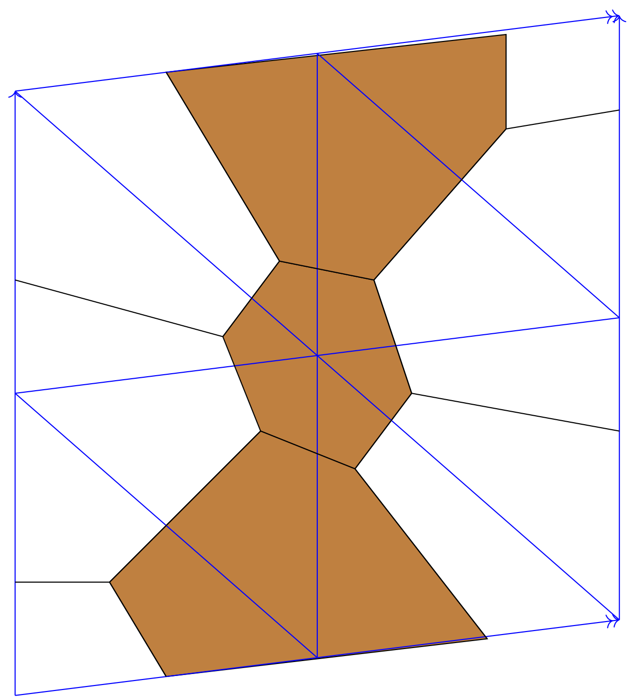


FIGURE 6.4.4. Shaded in brown is the subset $\pi^{-1}(\hat{\sigma})$ of $\partial \mathrm{HL}_\epsilon = T^2$. Edges of the cubic graph are shown in blue, and those of the dual triangulation in black. Since the opposite pairs of blue boundary edges are identified, the space $\pi^{-1}(\hat{\sigma})$ is homeomorphic to a cylinder.

Δ is chosen such that the tangle in the glued deformed foam $\widetilde{\mathbf{F}}'$ has no circle components; this is equivalent to requiring that at least one of the faces f_{e_0}, f_{ϵ_0} contains an arc as part of its boundary.

We now consider the gluing of double covers $\pi : L \to B$ and $\pi_\epsilon : \mathrm{HL}_\epsilon \to B$. We have $\widetilde{L} = L \cup_{\pi^{-1}(\hat{\sigma})} \mathrm{HL}_\epsilon$, where $\hat{\sigma}$ is a neighborhood in B of the edge ϵ_0, or dually the quadrilateral along which the dual tetrahedron Δ is glued to B. As shown in Figure 6.4.4 the space $\pi^{-1}(\hat{\sigma})$ is homeomorphic to a cylinder C. We fix the isomorphism $H_1(C) \simeq \mathbb{Z}\{\gamma\}$, where we take the generator γ to be the oriented loop on $\partial \mathrm{HL}_\epsilon$ given by canonical lift of the edge ϵ_0 of Γ_Δ. The relevant part of

Mayer-Vietoris sequence then reads

$$0 \to H_2(\widetilde{L}) \to \mathbb{Z}\{\gamma\} \to H_1(L) \oplus H_1(\mathrm{HL}_\epsilon) \to H_1(\widetilde{L}) \to 0. \tag{6.4.2}$$

By our assumption that at least one of the faces f_{e_0}, f_{ϵ_0} contains an arc as part of its boundary, we see that the map

$$i_* \oplus (-\iota_*) : \mathbb{Z}\{\gamma\} \to H_1(L) \oplus H_1(\mathrm{HL}_\epsilon)$$

is injective. Hence $H_2(L') = 0$, and

$$H_1(\widetilde{L}) \simeq \frac{H_1(L) \oplus H_1(\mathrm{HL}_\epsilon)}{\mathbb{Z}\{\gamma\}}.$$

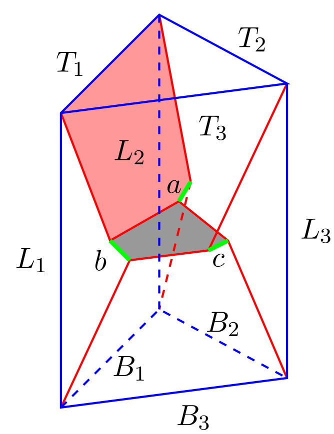
The face relations for all external faces of $\widetilde{\mathbf{F}}'$ now follow from this description of $H_1(\widetilde{L})$ exactly as in the case of the single-triangle gluing. Finally, since

$$(i_* \oplus (-\iota)_*)\,(\gamma) = ([e_0], [\epsilon_0]),$$

we see from the isomorphism (6.4.2) that the relation in $H_1(\widetilde{L})$ corresponding to the new internal face f_0 is also obtained as the sum of the relation corresponding to f_{e_0} in $H_1(L)$ with that corresponding to f_{ϵ_0} in $H_1(\mathrm{HL}_\epsilon)$. This completes the proof of the Proposition.

□

6.5. **Example – triangular prism.** Let Γ be the edge graph of a triangular prism and let $\mathbf{F}'$ be the deformed foam pictured here:



Write G for the gray face and P for the pink face. Then $\sigma(P, a) = 1$, and from Equation 6.12, P gives the relation

$$\tau(T_1) + \gamma_a = 0.$$

In total, the external face relations give

$$\begin{array}{rclrclrcl} \tau(T_1) + \gamma_a & = & 0 & \tau(L_1) + \gamma_b & = & 0 & \tau(B_1) - \gamma_b - \gamma_a & = & 0 \\ \tau(T_2) - \gamma_a - \gamma_c & = & 0 & \tau(L_2) & = & 0 & \tau(B_2) + \gamma_a & = & 0 \\ \tau(T_3) - \gamma_b & = & 0 & \tau(L_3) + \gamma_c & = & 0 & \tau(B_3) - \gamma_c & = & 0 \end{array}$$

We also have the internal (gray) face relation, and since $\sigma(G, b) = \sigma(G, c) = 1$, we see

$$\gamma_b + \gamma_c = 0.$$

The relations are consistent with the face relations from Γ. For example, the sum $T_1 + T_2 + T_3 \sim_\Gamma 0$, and this implies $\tau(T_1 + T_2 + T_3) = 0$, or $\gamma_b + \gamma_c = 0$, and this is true by the internal gray face relation of $\mathbf{F}'$. The other face relations are consistent, as well.

So τ indeed descends from a map from $\mathbb{Z}^{E_\Gamma \cup A}$ to one from $H_1(S_\Gamma) \cong \mathbb{Z}^{E_\Gamma}/\sim$, giving a map to $H_1(L) = \mathbb{Z}^A/\sim$. $H_1(S_\Gamma)$ is rank-4 and we can take Darboux generators $T_1, T_2; B_2, B_1$ (careful about the cyclic order on the back side of the prism: $\overline{\omega}(B_2, B_1) = 1$). $H_1(L)$ is rank-2 and we can take generators γ_a, γ_b. With these generators,

$$\tau(T_1) = -\gamma_a, \qquad \tau(T_2) = \gamma_a - \gamma_b, \quad \tau(B_2) = -\gamma_a, \quad \tau(B_1) = \gamma_a + \gamma_b.$$

We can see that the kernel of τ is generated by $\mu_1 := -(T_1 + T_2 + B_1 + B_2)$ and $\mu_2 := T_1 - B_2$, and is indeed isotropic. A framing $H_1(L) \hookrightarrow H_1(S_\Gamma)$ must send γ_a to $-T_1 + \alpha\mu_1 + \beta\mu_2$ and γ_b to $B_1 + B_2 + \gamma\mu_1 + \delta\mu_2$ The image is isotropic if $\beta = \gamma$, so the different framings for this phase are parametrized by symmetric 2×2 integer matrices $\left(\begin{smallmatrix}\alpha & \beta \\ \beta & \delta\end{smallmatrix}\right)$.

6.6. **Associated Cones, Geometric Cones.** To define open Gromov-Witten conjectures, we want to express a wavefunction in a power series about a limit point of the moduli space. A phase and framing define an algebraic torus, but pinning down a limit point for the expansion requires the notion of an associated cone, which we define below after setting notation.

We write Γ for the underlying cubic graph, S_Γ for the associated Legendrian surface, $\mathbf{F}'$ for the deformed foam, and L for the corresponding Lagrangian.

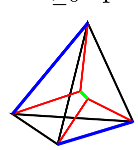
Definition 6.15. Given a splitting, i.e. a phase $K = \mathrm{Ker}(\,\tau : H_1(S_\Gamma \twoheadrightarrow H_1(L))$ and a framing $F \subset H_1(S_\Gamma)$ ω-isotropic and transverse to K (so $\tau : F \xrightarrow{\sim} H_1(L)$), an *associated cone* (or just *cone*) is an open integral convex cone $C_F \subset F \cong H_1(L)$ containing no lines.

With Remark 6.14 in mind, if Γ is simple (in particular has no bigons) and we are given a splitting, then we can specify an associated cone by choosing a spanning set of arcs.

Definition 6.16. Define a *geometric cone* of a deformed foam $\mathbf{F}'$ with $\partial\mathbf{F}' = \Gamma$ to be the $\mathbb{Z}_{\geq 0}$ span of a spanning set of arcs and edges in $\mathbb{Z}^{E_\Gamma \cup A}/\sim_{\mathbf{F}'}$. When Γ is simple, without loss of generality we take a spanning set of arcs.

Example 6.17. Let $\mathbf{F}' = \mathbf{F}$ be the foam for the necklace graph Γ^g_{neck} (see Remark 6.4), and L the corresponding Lagrangian. Label the beads 0 through g in clockwise order from some chosen starting point, and let b_i, $i = 0, ..., g$, be edges along the outer edges of the corresponding bead. Label the strands of the necklace a_i, $i = 0, ..., g$, so that the ith strand succeeds the ith bead in clockwise order and $\omega(a_i, b_i) = 1$. The b_i span the kernel of $\tau : H_1(S_{\Gamma^g_{\text{neck}}}) \to H_1(L)$, so define a phase. (The strands a_i function as arcs, albeit there are no vertices, as they connect tangle components.) The map $a_i \mapsto a_i$ defines a splitting $H_1(L) \to H_1(S_{\Gamma^g_{\text{neck}}})$. We note the following relations in $H_1(S_{\Gamma^g_{\text{neck}}})$: $2\sum a_i = 0, 2\sum b_i = 0$. So any g-element subset of $\{a_i\}$ determines a geometric cone, and by symmetry we may as well take this to be $a_1, ..., a_g$. The necklace therefore has a unique (up to symmetry) phase, framing and geometric cone.

Example 6.18. A Harvey-Lawson smoothing has a single arc and therefore a unique geometric cone. The blue edges are equivalent under the face relations and span the kernel (phase) of $\pi : H_1(S_{\Gamma_\Delta}) \to H_1(L)$. A splitting is defined by mapping the green arc to a transverse element of $H_1(\Lambda)$, and the unique associated cone is the $\mathbb{Z}_{\geq 0}$ span of this vector.



Example 6.19. Let $\mathbf{F}'$ be as in the Example of Section 6.5. Given that Γ is simple, per Remark 6.14 and the fact that the internal face relation is $a + b \sim' 0$, we conclude that there are two choices of geometric cones: $\{a, b\}$ and $\{a, c\}$.

6.7. **Mutations of Foams and Cones.** We now show that for a large class of mutations of the boundary graph, the foam filling can be mutated, along with a phase, framing and cone.

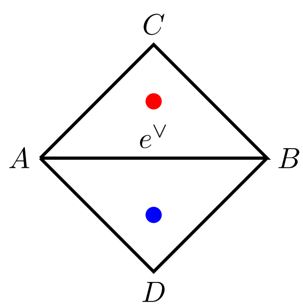
Definition 6.20. A mutation of a deformed foam at an edge $e \in \Gamma$ is *allowable* if e is not the boundary of a single tangle strand.

The reason for this definition is to exclude the case where the class $[e] \in H_1(S_\Gamma)$ is in the kernel of $\tau : H_1(S_\Gamma) \to H_1(L)$, rendering the action on the wavefunction zero. At the level of tangles, the condition ensures that the new tangle has no circle component.

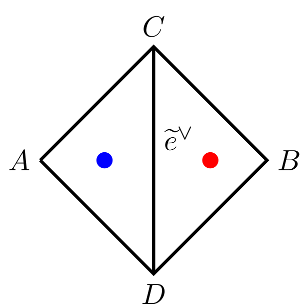
Proposition 6.21. *Let Γ be a cubic planar graph bounding a deformed foam $\mathbf{F}'$. Let Γ_e be the graph defined by performing an allowable mutation at the edge $e \in E(\Gamma)$. Then there is either one or two canonically defined deformed foams $\mathbf{F}'_{e,+}$ and $\mathbf{F}'_{e,-}$ with boundary Γ_e, corresponding to positive and negative mutations, respectively.*

Proof. The proposition follows immediately from the proof of Proposition 6.13: the allowed mutations correspond to attaching a Harvey-Lawson foam along an edge, with the allowable condition corresponding to the hypothesis that the tangle of the deformed foam have no circle components. Nevertheless, for the convenience of the reader, we provide a separate description in the language of triangulations — though they are not as general as foams (the foams of necklace-type graphs are degenerate tetrahedronizations), they are often easier to visualize.

We first mutate the ideal foam $\mathbf{F}$, then worry about its deformation $\mathbf{F}'$. On the surface $\partial B = S^2 \supset \Gamma$, the geometry near the dual edge $e^\vee$ of e is a quadrilateral as pictured here:



Define $\widetilde{\triangle}$ as follows: if the two faces above are part of a tetrahedron T, then $\widetilde{\triangle} = \triangle \setminus T$. Otherwise, let T be the tetrahedron with two faces as pictured above and the other two ACD and BCD, and set $\widetilde{\triangle} = \triangle \cup T$, In both cases, the geometry of the quadrilateral $ABCD$ at the boundary of $\widetilde{\triangle}$ is



Call the foam so constructed $\mathbf{F}_e$. It remains to describe how to deform $\mathbf{F}_e$ to $\mathbf{F}'_{e,\pm}$.

Suppose $\widetilde{\triangle}$ is formed from $\triangle$ by adding a tetrahedron as in the proof of Proposition 6.13, and so $\mathbf{F}_e$ is formed from $\mathbf{F}$ by attaching a Harvey-Lawson foam $\mathbf{F}_{\mathrm{HL}}$. We define $\mathbf{F}'_e$ by extending $\mathbf{F}'$ together with a choice of one of the three possible smoothings of $\mathbf{F}_{\mathrm{HL}}$ One of these three pairs the two tangles with endpoints at the centers of triangles ABC and ABD (pictured in red and blue) with one another, creating a new short tangle component. This is the disallowed smoothing.

The other two rotate pair these with the centers of the two new triangles ACD and BCD. The matching corresponding to the deformed foam of the positive mutation $\mathbf{F}'_{e,+}$ is shown above. $\mathbf{F}'_{e,-}$ is defined similarly.

Now suppose otherwise that $\widetilde{\triangle}$ is formed from $\triangle$ by deleting a tetrahedron T. Then $\triangle$ was the result of a mutation $\mu_{\widetilde{e},\pm}$ of $\widetilde{\triangle}$ and only the inverse mutation $\mu_{e,\mp}$ is possible. Since the case where e bounds a single tangle component is not allowed, the tangle components after deleting the tetrahedron are clear: they are truncations of the original tangle strands.

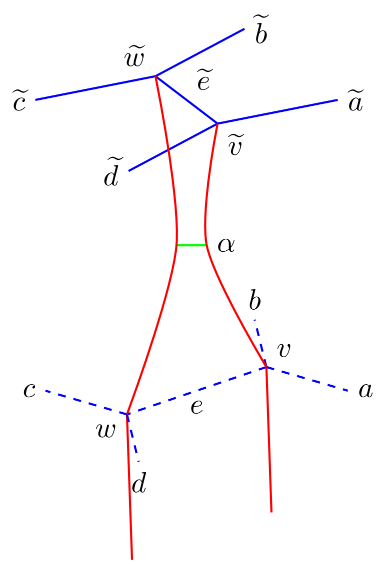
□

Proposition 6.21 will allow us to transport foams across mutations, along with phases, framings and cones. This will allow us to connect open Gromov-Witten conjectures for Lagrangian fillings related by allowed mutations which have corresponding cones, phases and framings.

Proposition 6.22. *Let $\mathbf{F}'$ and $\widetilde{\mathbf{F}}'$ be deformed foams corresponding to an allowed mutation $\Gamma \to \widetilde{\Gamma}$ of their boundaries. Then there is a canonical isomorphism*

$$\varphi : \mathbb{Z}^{E_\Gamma \cup A_{\mathbf{F}'}} / \sim_{\mathbf{F}'} \cong \mathbb{Z}^{E_{\widetilde{\Gamma}} \cup A_{\widetilde{\mathbf{F}}'}} / \sim_{\widetilde{\mathbf{F}}'}$$

Proof. We can assume that $\widetilde{\Gamma}$ is obtained by attaching a tetrahedron, as removal will give rise to the inverse isomorphism. A local study near the attachment will suffice to establish φ. We label the relevant edges and vertices as in the figure below, with Γ indicated by dashed lines.



Consider the $\binom{4}{2} = 6$ sheets of the deformed Harvey-Lawson foam, after gluing to $\mathbf{F}'$ and deforming. They correspond to unordered pairs from among the vertices $\{v, w, \widetilde{v}, \widetilde{w}\}$. Write $f_{v,w}$ for the face determined by v and w, and likewise for the others. Let γ be the arc of the Harvey-Lawson deformed foam, and write $\alpha = \sigma(f_{v,w}, \gamma) \cdot \gamma = \pm\gamma$ for the signed contribution to the relation from $f_{v,w}$, as defined in Definition 6.12. Now suppose the face relations on $\mathbf{F}'$ relate give $e + s_e \sim 0$, $a + s_a \sim 0$, and so on. Let us list the unordered pairs along with the relations from the corresponding glued face.

$$\begin{array}{llll} f_{v,w}: & \alpha + s_e & \Rightarrow & \alpha = e \\ f_{\widetilde{v},\widetilde{w}}: & \widetilde{e} + \alpha & \Rightarrow & \widetilde{e} = -\alpha = -e \\ f_{v,\widetilde{v}}: & \widetilde{a} + s_a & \Rightarrow & \widetilde{a} = a \\ f_{w,\widetilde{w}}: & \widetilde{c} + s_c & \Rightarrow & \widetilde{c} = c \\ f_{v,\widetilde{w}}: & \widetilde{b} - \alpha + s_b & \Rightarrow & \widetilde{b} = b + e \\ f_{\widetilde{v},w}: & \widetilde{d} - \alpha + s_d & \Rightarrow & \widetilde{d} = d + e \end{array}$$

This gives the positive mutation. The other allowed matching $v \leftrightarrow \widetilde{w}$ gives the negative mutation, as follows from the interchange $\widetilde{v} \leftrightarrow \widetilde{w}$.

□

Corollary 6.23. *Suppose $\widetilde{\Gamma}$ is obtained by an allowed mutation of Γ. Let $\nu : \mathbb{Z}^{E_\Gamma} \to \mathbb{Z}^{E_{\widetilde{\Gamma}}}$ be the corresponding isomorphism of edge lattices, respecting the antisymmetric pairing. Let $\varphi : \mathbb{Z}^{E_\Gamma \cup A_{\mathbf{F}'}} / \sim_{\mathbf{F}'} \cong \mathbb{Z}^{E_{\widetilde{\Gamma}} \cup A_{\widetilde{\mathbf{F}}'}} / \sim_{\widetilde{\mathbf{F}}'}$ be the isomorphism provided by Proposition 6.22 above. Then under the isomorphisms of Equation 6.2.2 and Proposition 6.13, the maps ν and φ intertwine $\tau : H_1(S_\Gamma) \to H_1(L)$ with $\widetilde{\tau} : H_1(S_{\widetilde{\Gamma}}) \to H_1(\widetilde{L})$.*

Proof. It only remains to note that ν respects the antisymmetric pairing of edges. □

We immediately obtain the following.

Corollary 6.24. *The maps ν and φ map phases, framings and cones to phases, framings and cones.*

("Geometric"?)

7. The wavefunction

7.1. **Construction of the wavefunction.** Suppose that $\mathbf{F}'$ is a deformed ideal foam obtained from the standard necklace foam by a sequence of admissible mutations, and $\mathbf{f}$ is a framing for $\mathbf{F}'$. As explained in Section 6, the pair $(\mathbf{F}', \mathbf{f})$ gives rise to a framed seed $\underline{\mathbf{i}} = \underline{\mathbf{i}}(\mathbf{F}', \mathbf{f})$. It is convenient to visualize the framed seed as a labelling of the edges of the cubic graph Γ by elements of the Heisenberg Lie algebra spanned by $\{u_i, v_i\}_{i=1,\dots g} \cup \{c\}$.

In this section, we will show that there is a canonical wavefunction $\Psi_{\underline{\mathbf{i}}} \in \mathcal{K}$ associated to such a framed seed, thereby providing a prediction for the generating function of all-genus open Gromov-Witten invariants of the corresponding Lagrangian $L_{\mathbf{F}'} \subset \mathbb{C}^3$.

We begin with the definition of Ψ in the case of the standard necklace framed seed $\underline{\mathbf{i}}_{\text{neck}}$. The corresponding foam gives rise to an exact Lagrangian filling of the Chekanov surface, so that by Stokes' theorem all its open Gromov-Witten invariants will be zero. We therefore take the wavefunction for the standard necklace to be $\Psi_{\text{neck}} = 1$. Let us note that the necklace wavefunction depends only on the underlying deformed foam, and is completely independent of the choice of framing $\mathbf{f}$.

Now let $\mathbb{G}_{\text{ad}}$ be the sub-category of the framed seed groupoid $\mathbb{G}$ whose morphisms are given by the admissible ones, and let $\mathbb{G}_{\text{ad}}(\underline{\mathbf{i}}_{\text{neck}})$ be the connected component of $\mathbb{G}_{\text{ad}}$ containing the framed seed $\underline{\mathbf{i}}_{\text{neck}}$.

Given an object $\underline{\mathbf{i}}$ of $\mathbb{G}_{\text{ad}}(\underline{\mathbf{i}}_{\text{neck}})$, our prescription to construct $\Psi_{\underline{\mathbf{i}}}$ is as follows: choose an arbitrary path $\vec{a} : \underline{\mathbf{i}}_{\text{neck}} \to \underline{\mathbf{i}}$ in $\mathbb{G}_{\text{ad}}$. As explained in Section 3.3, the morphism $\vec{a}$ gives rise to an automorphism $\Phi_{\vec{a}}$ of $\mathcal{K}$, which we apply to Ψ_{neck} to produce a candidate for $\Psi_{\underline{\mathbf{i}}}$:

$$\Psi_{\underline{\mathbf{i}}} := \Phi_{\vec{a}} \cdot \Psi_{\text{neck}}. \tag{7.1.1}$$

What must be checked in order for this definition to make sense is that the wavefunction $\Psi_{\underline{\mathbf{i}}}$ depends only on the endpoint of the path $\vec{a}$ in the framed seeds groupoid. This path-independence is the content of the following Theorem.

Theorem 7.1. *The map*

$$\Psi \ : \ \mathrm{Ob}\left(\mathbb{G}_{\mathrm{ad}}(\underline{\mathbf{i}}_{\mathrm{neck}})\right) \longrightarrow \mathcal{O}, \qquad \underline{\mathbf{i}} \longmapsto \Psi_{\underline{\mathbf{i}}}$$

is well-defined, i.e is independent of the choice of path $\vec{a} : \underline{\mathbf{i}}_{\mathrm{neck}} \to \underline{\mathbf{i}}$ in (7.1.1).

Proof. The key observation is the following immediate consequence of Theorem 4.4: if $\Psi_{\mathbf{i}}$ satisfies the face relations in framed seed $\underline{\mathbf{i}}$ and $\underline{\mathbf{i}}' = a(\underline{\mathbf{i}})$ where a is an admissible mutation or framing shift, then $\Phi_a \cdot \Psi_{\mathbf{i}}$ satisfies the face relations for $\underline{\mathbf{i}}'$. Now suppose we have two sequences of admissible mutations and framing shifts $\vec{a}_1$ and $\vec{a}_2$ as in the statement of the theorem. Then it suffices to show that

$$\Phi_{\vec{a}_1}^{-1}\Phi_{\vec{a}_2} \cdot \Psi_{\mathrm{neck}} = \Psi_{\mathrm{neck}}. \tag{7.1.2}$$

To this end, consider the framed seed $\underline{\mathbf{i}}' = \vec{a}_1^{-1}\vec{a}_2(\underline{\mathbf{i}}_{\mathrm{neck}})$. Its underlying cubic graph Γ' is the image of the original necklace graph Γ_{neck} under an element of the mapping class group of the $(g+3)$-times punctured sphere, and moreover the labelling of the edges of Γ' by monomials in the U_i, V_i induced by its phase and framing are identical to that in the standard necklace framed seed. In particular, the face relations for $\underline{\mathbf{i}}'$ and $\underline{\mathbf{i}}_{\mathrm{neck}}$ are identical, and from the binomial face relations corresponding to the beads we deduce that

$$(1 - V_i) \cdot \left(\Phi_{\vec{a}_1}^{-1}\Phi_{\vec{a}_2} \cdot \Psi_{\mathrm{neck}}\right) = 0, \quad i = 1, \ldots, g.$$

It follows that $\Phi_{\vec{a}_1}^{-1}\Phi_{\vec{a}_2} \cdot \Psi_{\mathrm{neck}} = \Psi_{\mathrm{neck}} = 1$, which completes the proof of the Theorem. □

7.2. **Examples of wavefunctions.** We now proceed to compute the wavefunction defined in the previous section in some fundamental examples. Canoe, prism, cube with TZ's phase.

Example 7.2. The calculation in Example 4.5 shows that the wavefunction $\Psi_{\underline{\mathbf{i}}_1}$ associated to the framed seed $\underline{\mathbf{i}}_1$ for the canoe graph shown in Figure 4.2.2 is given by

$$\Psi_{\underline{\mathbf{i}}_1} = (X; q^2)_\infty.$$

It satisfies the q-difference equation

$$(1 + UV - V)\Psi_{\underline{\mathbf{i}}_1} = 0,$$

which is a scalar multiple of the face relation R' in (4.2.4). As an exercise, let us compute the effect on the wavefunction of applying the framing shift operator T_{-1}, which we recall acts on $\mathcal{A}_{2g}$ by $U \mapsto q^{-1}UV^{-1}$. The resulting framed seed is illustrated in Figure 7.2.1.

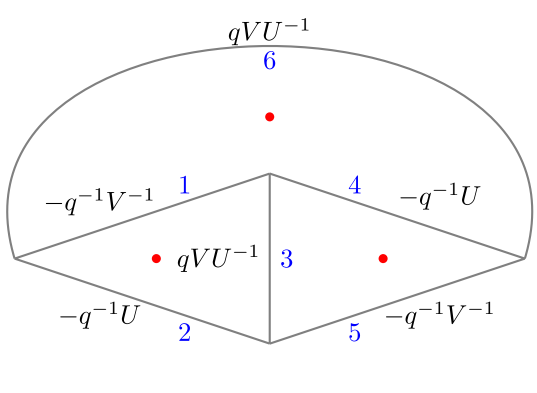


FIGURE 7.2.1. The framed seed $\underline{\mathbf{i}}_2 = (\sigma_{-1} \circ T_{-1})(\underline{\mathbf{i}}_1)$ for the canoe graph.

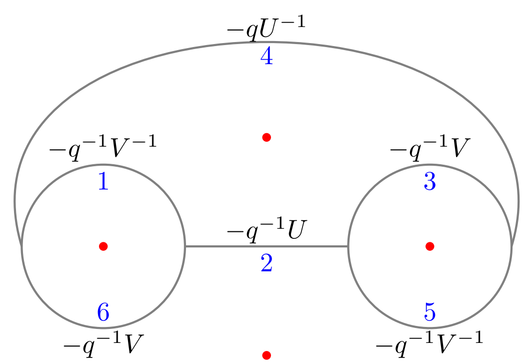


FIGURE 7.2.2. The framed seed $\mu_4^+(\underline{\mathbf{i}}_2) \simeq \underline{\mathbf{i}}_0$.

Lemma 7.3. *We have*

$$(\sigma_{-1} \circ T_{-1}) \cdot (U; q^2)_\infty = (U; q^2)_\infty^{-1}. \tag{7.2.1}$$

Proof. Since

$$(\sigma_{-1} \circ T_{-1})(1 + UV - V) = (1 - U - V)(\sigma_{-1} \circ T_{-1}),$$

the Lemma follows by observing that both sides satisfy the q-difference equation

$$(1 - U - V)\Psi = 0,$$

which is easily seen to have a unique formal power series solution of the form $\Psi \in 1 + \mathfrak{m} \in \mathcal{O}$. $\square$

Now observe that applying to the framed seed $\underline{\mathbf{i}}_2$ the positive mutation at edge 4 returns us to the framed seed shown in Figure 7.2.2, which coincides with the standard necklace $\underline{\mathbf{i}}_0$ up to a permutation of the numbering of its edges. Hence we have a loop in the framed seed groupoid

$$\begin{tikzcd} & \underline{\mathbf{i}}_0 \arrow[dl, "\mu_3^+"'] & \\ \underline{\mathbf{i}}_1 \arrow[rr, "\sigma_{-1}\circ T_{-1}"] & & \underline{\mathbf{i}}_2 \arrow[ul, "\mu_4^+"'] \end{tikzcd} \tag{7.2.2}$$

and we indeed see that

$$\begin{aligned}\Psi_{\mu_4^+(\underline{\mathbf{i}}_2)} &= \Phi(-q^{-1}U) \cdot (U; q^2)_\infty^{-1} \\ &= (U; q^2)_\infty \cdot (U; q^2)_\infty^{-1} \\ &= 1,\end{aligned}$$

in accordance with Theorem 7.1.

More generally, we can consider the framed seed $\underline{\mathbf{i}}_{\text{canoe}}$ obtained from the standard genus g necklace framed seed $\underline{\mathbf{i}}_{\text{neck}}$ by performing positive mutations at all g beads labelled $-q^{-1}U_j$, $j = 1, \ldots g$ under the framing isomorphism. The corresponding wavefunction is then

$$\Psi_{\mathbf{i}_{\text{canoe}}} = \prod_{i=1}^{g} (X_i; q^2)_\infty^{-1}, \tag{7.2.3}$$

which is annihilated by the left ideal in $\mathcal{D}_{2g}$ generated by

$$R_i = 1 + U_i V_i - V_i, \qquad\qquad i = 1, \ldots, g.$$

Let us write $\Psi_{\mathbf{i}^{(1)}_{\text{canoe}}}$ for the wavefunction obtained by applying the operator $\sigma_{(-1,\ldots,-1)} \circ T_{-I_g}$ to $\Psi_{\mathbf{i}_{\text{canoe}}}$, where I_g is the $g \times g$ identity matrix. Then we again have

$$\Psi_{\mathbf{i}^{(1)}_{\text{canoe}}} = \prod_{i=1}^{g} (X_i; q^2)_\infty$$

Lemma 7.4. *The explicit power series of the superpotential* (7.2) *is*

$$\Psi_{\mathbf{i}^{(1)}_{\text{canoe}}} = \sum_{\mathbf{v} \in \mathbb{Z}^g_{\geq 0}} \frac{1}{(q^2)_{\mathbf{v}}} X^{\mathbf{v}}, \qquad \textit{where } (q^2)_{\mathbf{v}} = \prod_{i=1}^{g} \prod_{k=1}^{v_i} (1 - q^{2k}).$$

Proof. Set

$$\Psi := \sum_{\mathbf{v} \in \mathbb{Z}^g_{\geq 0}} \frac{C_{\mathbf{v}}(q)}{(q^2)_{\mathbf{v}}} X^{\mathbf{v}}, \qquad \text{where } C_0(q) = 1$$

Let $e_i \in \mathbb{Z}^g$ be the i^{th} unit vector. At the level of the coefficients of $X^{\mathbf{v}}$, the equation $(1-U_i-V_i)\Psi = 0$ is equivalent to the recurrence

$$\frac{C_{\mathbf{v}}(q)}{(q^2)_{\mathbf{v}}} - \frac{C_{\mathbf{v}-e_i}(q)}{(q^2)_{\mathbf{v}-e_i}} - \frac{C_{\mathbf{v}}(q)}{(q^2)_{\mathbf{v}}} q^{2v_i} = 0$$

Note that $(q^2)_{\mathbf{v}} = (q^2)_{\mathbf{v}-e_i}(1-q^{2v_i})$. Therefore we have

$$C_{\mathbf{v}}(q) = C_{\mathbf{v}-e_i}(q) = \ldots = C_0 = 1.$$

□

Example 7.5 (Partition function for unknot of AENV?)**.** Set

$$\Psi(X) = (X;q)_\infty^{-1}(QX;q)_\infty,$$

where Q is a formal variable commuting with all the other variables. The $\Psi(X)$ is annihilated by

$$\mathcal{L} = (1-U) - (1-QU)V = 1 - U - V + QUV$$

Lemma 7.6. *We have*

$$\Psi(X) := \sum_{k\geq 0} \frac{(Q;q)_k}{(q;q)_k} X^k.$$

Proof. Set

$$\Psi(X) := \sum_{k\geq 0} C_k(Q,q)X^k, \qquad\qquad \text{where } C_0 = 1$$

By computing the coefficients of $\mathcal{L}\cdot\Psi$, we get

$$C_k - C_{k-1} - q^k C_k + Qq^{k-1}C_{k-1} = (1-q^k)C_k - (1-Qq^{k-1})C_{k-1} = 0$$

Therefore

$$C_k = \frac{\prod_{i=1}^k (1-Qq^{i-1})}{\prod_{i=1}^k (1-q^i)} = \frac{(Q;q)_k}{(q;q)_k}.$$

□

7.3. **Open Gromov-Witten Conjectures.** We can now propose an interpretation of the wave-function of a geometric seed: it is the generating function of open Gromov-Witten invariants of the Lagrangian filling defined by the deformed foam. To be more precise, we recall the geometric framework.

Let $\underline{\mathbf{i}} \in \mathbb{G}_{\mathrm{ad}}$, meaning there is a path $\vec{a} : \underline{\mathbf{i}}_{neck} \to \underline{\mathbf{i}}$ in the admissible framed seed groupoid. By Theorem 7.1, there is a well-defined wavefunction $\Psi_{\underline{\mathbf{i}}} = \Phi_{\vec{a}} \cdot \Psi_{\underline{\mathbf{i}}_{\mathrm{neck}}} = \Phi_{\vec{a}} \cdot 1$.

The framing of $\underline{\mathbf{i}}$ has geometric content. Recall from Example 6.17 that $\underline{\mathbf{i}}_{\mathrm{neck}}$ is canonical. By Proposition 6.22 and especially Corollary 6.24, we learn $\underline{\mathbf{i}} = \vec{a}\cdot\underline{\mathbf{i}}_{\mathrm{neck}}$ is a geometric seed, i.e. has a geometric phase and framing, and cone. That is, there is a corresponding cubic graph Γ, deformed foam filling $\mathbf{F}'$, and Lagrangian L, along with phase $K = \mathrm{K}er(\tau : H_1(S_\Gamma) \to H_1(L))$ and transverse isotropic framing $F \subset H_1(S_\Gamma)$, as well as a cone $C \subset F$. We choose a basis e_i, $i = 1, ..., g$ for C. The sequence $0 \to K \to H_1(S_\Gamma) \to \pi(F) \to 0$ and basis e_i then defines a framing for $\mathbf{i}$ in the sense of Section 3.1. (The central character is....)

The geometric seed identifies the quantum torus $\mathcal{T}^q_{\mathbf{i}}$ with the quantization of the symplectic lattice $H^1(S_\Gamma)$ endowed with its intersection form. In particular, a monomial $x^d = \prod_{i=1}^g x_i^{d_i}$ in the ring of power series $\mathbb{C}[[\{x_i\}]]$ has exponent d lying in $\mathbb{Z}^g_{\geq 0} \cong C \subset H_1(L)$. Each such d determines an open Gromov-Witten problem of counting holomorphic maps from Riemann surfaces with one boundary component mapping to the pair $(\mathbb{C}^3, L)$, such that the image of the boundary lies in homology class d. Such open Gromov-Witten problems depend on additional data known as a framing, and while there is not yet a rigorous definition of these open Gromov-Witten invariants

Let e_i, $i = 1, ..., g$ be a basis for C. Then x_i are the corresponding coordinates for $H_1(L, \mathbb{C}^*)$, and the dual basis e_j^* defines coordinates y_j such that x_i, y_j provide the identification of the framed seed charts...

A rigorous definition of open Gromov-Witten invariants still awaits definition, but it is anticipated that it will involve framings as constructed here, generalizing the well-studied cases of Aganagic-Vafa branes [AKV, KL, FL].[6] We then conjecture that the wavefunction $\Psi_{\underline{\mathbf{i}}} \in \mathbb{C}[q, q^{-1}][[\{x_i\}]]$ is the all-genus generating function of open Gromov-Witten invariants and obeys Ooguri-Vafa integrality, which expresses the invariants in terms of the quantum dilogarithm $\Phi(z) = \prod_{n \geq 0}(1 - q^n z)^{-1}$.

Conjecture 7.7. *Let $\underline{\mathbf{i}} \in \mathbb{G}_{\mathrm{ad}}$ be a framed seed with wavefunction $\Psi_{\underline{\mathbf{i}}}$. Write A for the framing and L for the Lagrangian of the deformed foam. Then*

$$\Psi_{\underline{\mathbf{i}}} = \prod_{d \in \mathbb{Z}_{\geq 0} \setminus \{0\}} \prod_{s \geq 0} \Phi(x^d q^s)^{n_{d,s}^{(A)}},$$

with $n_{d,s}^{(A)} \in \mathbb{Z}$ the Ooguri-Vafa invariants.

Remark 7.8. The Ooguri-Vafa invariants are related to open Gromov-Witten invariants as follows. Write $q = e^\lambda$ and expand $\Psi_{\underline{\mathbf{i}}}$ as a power series in λ (and the x_i). Then the coefficient of $x^d \lambda^h$ is the genus-h open Gromov-Witten invariant of L in framing A, in class $d \in H_1(L) \cong H_2(\mathbb{C}^3, L)$. See [Za, Sections 2 and 4] for further discussion of these variables.

Remark 7.9. Recall $\Phi(z) \sim e^{\mathrm{Li}_2(z)/\lambda}$ as $\lambda \to 0$. Conjecture 7.7 therefore reduces to the conjecture of [TZ] for disk invariants, described in the Introduction in Section 1.3. More specifically, writing $\Psi_{\underline{\mathbf{i}}} \sim e^{W_{\underline{\mathbf{i}}}}$, in the semiclassical limit $v_i = \partial_{u_i} W$, meaning W is a local prepotential for the Lagrangian subspace $\mathscr{M}_\Gamma \subset \mathscr{P}_\Gamma$.

Remark 7.10. In the next section, we provide evidence for the conjecture by arguing that the wavefunctions $\Psi_{\underline{\mathbf{i}}}$ obey integrality. The Harvey-Lawson brane in $\mathbb{C}^3$ with its various framings, as studied in [AKV, Section 6.1], gives further evidence. This example enjoys a $U(1)$ symmetry, permitting localization techniques for open Gromov-Witten calculations [KL], while the Lagrangians for for cubic graphs Γ generally do not. Further tests of the conjecture must therefore await rigorous defitions of open Gromov-Witten invariants and the development of new techniques.

7.4. **Integrality of the wavefunction.** The infinite *q-Pochhammer* symbol is a formal power series

$$(x; q)_\infty := \prod_{n=0}^{\infty}(1 - q^n x) \tag{7.4.1}$$
$$= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k q^{k(k-1)/2}}{\prod_{i=1}^k (1 - q^i)} x^k$$
$$= \exp\Big(\sum_{k=1}^{\infty} \frac{x^k}{k(q^k - 1)}\Big)$$
$$= \Phi(-q^{-1}x)^{-1} \qquad \in \mathbb{Q}(q)[[x]].$$

It is the unique formal power series starting from 1 and satisfying the difference relation

$$(x; q)_\infty = (1 - x) \cdot (qx; q)_\infty. \tag{7.4.2}$$

[6] We thank Jake Solomon and Sara Tukachinski for explaining this perspective. See also [I] for a more general definition of framings.

For $m \in \mathbb{Z}$, we define the finite *q-Pochhammer symbol* by

$$(x;q)_m := \frac{(x;q)_\infty}{(q^m x;q)_\infty}$$

The wavefunction Ψ constructed in the previous section is an element of the commutative local ring $\mathcal{O}_{\mathbb{Q}} := \mathcal{Q} \otimes_{\mathbb{Z}} \mathcal{O}$ of formal power series in $X_1, \ldots, X_g$ with coefficients in the field $\mathbb{Q}((q))$. Let $\mathfrak{m}$ be the unique maximal ideal in the ring $\mathcal{O}_{\mathbb{Q}}$. By considering the quotients $\mathcal{O}_{\mathbb{Q}}/\mathfrak{m}^k$, it is easy to show that every $F \in 1 + \mathfrak{m}$ admits a unique factorization

$$F = \prod_{\mathbf{v} \in \mathbb{Z}^g_{\geq 0} - \{0\}} \prod_{k \in \mathbb{Z}} \left((-q)^k X^{\mathbf{v}}; q^2 \right)_\infty^{c_{\mathbf{v},k}}, \qquad c_{\mathbf{v},k} \in \mathbb{Q}. \tag{7.4.3}$$

The coefficients $c_{\mathbf{v},k}$ for each $\mathbf{v} \in \mathbb{Z}^g_{\geq 0} \setminus \{0\}$ can be packaged in a Laurent series

$$P_{F,\mathbf{v}}(t) := \sum_{k \in \mathbb{Z}} c_{\mathbf{v},k} t^k \in \mathbb{Q}((t)).$$

Following [KS, §6.1], a series $F \in 1 + \mathfrak{m}$ is called *admissible* if the $P_{F,\mathbf{v}}(t)$ are Laurent polynomials with integral coefficients for all $\mathbf{v}$.

Recall the logarithm

$$\log : \ 1 + \mathfrak{m} \longrightarrow \mathfrak{m}, \qquad \log(1+f) := \sum_{k \geq 1} \frac{(-1)^k f}{k}.$$

Lemma 7.11. *For each admissible series F, we have*

$$\lim_{q^{\frac{1}{2}} \to 1} (q-1) \log F = \sum_{\mathbf{v} \in \mathbb{Z}^n_{\geq 0} - \{0\}} P_{F,\mathbf{v}}(1) \mathrm{Li}_2(X^{\mathbf{v}}).$$

Proof. By the third formula of (7.4.1), we have

$$\lim_{q^{\frac{1}{2}} \to 1} (q-1) \log (x,q)_\infty = \lim_{q^{\frac{1}{2}} \to 1} \sum_{k=1}^{\infty} \frac{(q-1)x^k}{k(q^k-1)} = \sum_{k=1}^{\infty} \frac{x^k}{k^2} = \mathrm{Li}_2(x). \tag{7.4.4}$$

The rest is clear. $\square$

The property of admissibility is clearly preserved under the action of any automorphism $\sigma_{\mathbf{d}}$ of $\mathcal{O}$ of the form

$$\sigma_{\mathbf{d}}(F)(U_1, \ldots, U_g) = F\left((-q)^{d_1} U_1, \ldots, (-q)^{d_g} U_g \right), \qquad d_1, \ldots, d_g \in \mathbb{Z}.$$

In [KS], Kontsevich and Soibelman proved that admissibility is also preserved under another, much less trivial family of automorphisms: the changes of framing.

Theorem 7.12 ([KS, Th.6.1])**.** *A power series $F \in \mathcal{O}$ is admissible if and only if $T_\Omega \cdot F$ is admissible for all integral symmetric matrices Ω.*

Recall the notion of an admissible mutation introduced in Section 3.3. As explained there, each such mutation a gives rise to an element $\Phi_a \in \mathcal{A}_{2g}$ which acts on the power series ring $\mathcal{O}$. Let us say that an admissible mutation in direction e_k is *primitive* if in the monomial

$$M_{\mathbf{f}}(e_k) = (-q)^r \ \mathbf{:} \prod_{j=1}^{g} U_j^{m_j} V_j^{n_j} \mathbf{:}, \qquad m_j, n_j, r \in \mathbb{Z}$$

the vector

$$\mathbf{m} = (m_1, \ldots, m_g) \tag{7.4.5}$$

is a primitive vector in $\mathbb{Z}^g$.

Lemma 7.13. *Suppose that the mutation a is both admissible and primitive, and that $F \in \mathcal{O}$ is an admissible formal power series. Then the power series $\Phi_a \cdot F$ is also admissible.*

Proof. Since the exponent vector $\mathbf{m}$ in (7.4.5) is primitive we can choose a basis for $\mathbb{Z}^g$ containing $\mathbf{m}$ as one of its elements. Hence (*cf.* Remark 3.1) we may reduce to proving the Lemma in the case that $M_{\mathbf{f}}(e_k) = (-q)^r \prod_{j=1}^{g} U_1 V_j^{n_j}$. In this case, let Ω be any symmetric matrix whose first column is $-\mathbf{n} = (-n_1, \ldots, -n_g)$. By Theorem 7.12, it suffices to show $T_\Omega \cdot \Phi_a \cdot F$ is admissible, so we compute

$$\begin{aligned} T_\Omega \cdot \Phi_a \cdot F &= T_\Omega \cdot \Phi\left(M_{\mathbf{f}}(e_k)\right)^{\epsilon_k} \cdot F \\ &= \Phi\left((-q)^r U_1\right)^{\epsilon_k} \cdot T_\Omega \cdot F \end{aligned}$$

But $T_\Omega \cdot F$ is admissible by Theorem 7.12, and so is $\Phi\left((-q)^r U_1\right)^{\epsilon_k}$ by the last formula in (7.4.1). Since the product of two admissible series is clearly admissible, this implies that $\Phi_a \cdot F$ is also admissible, thereby proving the Lemma. □

8. Towards an analytic wavefunction

In this section we will discuss the problem of promoting the algebraic construction of the wavefunction from Section 7 to an analytic one. Doing this in general would necessitate extending the theory of representations of quantum cluster varieties beyond the "principal series", a task we do not take on in the present work. Nonetheless, we will present several examples which we believe provide nontrivial evidence for the existence of a well-defined analytic wavefunction associated to a smoothed ideal foam.

Let us first recall some of the elements of the theory of unitary representations of quantum cluster varieties as developed in [FG2]. A representation of a quantum cluster variety is, by definition, a functor $\mathcal{G}_{\mathscr{X}} \to \mathrm{Hilb}$ from the cluster modular groupoid to the category of Hilbert spaces with morphisms given by unitary isomorphisms. The representations constructed by Fock and Goncharov depend on a quantization parameter $\hbar \in \mathbb{R}$. To each object $\mathbf{i}$ of $\mathcal{G}_{\mathscr{X}}$ is associated a pair of quantum tori $\mathcal{T}^q$ and $\mathcal{T}^{q^\vee}$, generated respectively by $\{e^{2\pi\hbar x_k}\}$ and $\{e^{2\pi\hbar^{-1}x_k}\}$, where $\{x_k\}$ are the logarithmic cluster variables associated to the seed $\mathbf{i}$. For each $\mathbf{i}$, the generators of these quantum tori act by unbounded, self-adjoint operators in the Hilbert space $\mathcal{H}_{\mathbf{i}}$. The latter space comes equipped with the additional data of a dense subspace $\mathcal{S}_{\mathbf{i}}$, the Fock-Goncharov Schwartz space, defined to be the maximal joint domain of the algebras $\mathbb{L}^q_{\mathscr{X}}, \mathbb{L}^{q^\vee}_{\mathscr{X}}$. The unitary isomorphism $\mathcal{K}_{\mathbf{i}\to\mathbf{i}'}$ corresponding to an arrow $a : \mathbf{i} \to \mathbf{i}'$ in the cluster modular groupoid preserves the corresponding Schwartz spaces, where it intertwines the action of $\mathcal{G}_{\mathscr{X}}$ on $\mathbb{L}^q_{\mathscr{X}}$ by cluster transformations.

When the skew form on Λ has a nontrivial kernel, the Fock-Goncharov unitary representations of the quantum cluster variety are labelled by *central characters* $\lambda \in \mathrm{Hom}(\ker(\epsilon), \mathbb{R})$, and thus can be thought of as a kind of principal series. The reality condition is required to ensure that all elements of the underlying Heisenberg algebra act by self-adjoint operators. This self-adjointness is crucial for the entire construction: indeed, it guarantees that for each logarithmic cluster variable x_k its noncompact quantum dilogarithm $\varphi(x_k)$ defines a unitary automorphism of $\mathcal{H}_{\mathbf{i}}$, which forms the key ingredient in defining the intertwiner $\mathcal{K}_{\mathbf{i}\to\mathbf{i}'}$.

In the context of moduli spaces of framed local systems on surfaces with punctures, recall that the central characters parametrize the eigenvalues of the local system's monodromy around the punctures. As we have seen in Section 4.1, however, the quantization (4.2.1) of the defining constraints for $\mathcal{P}_g$, which impose that the monodromy around each puncture be unipotent, forces a sum of logarithmic cluster variables to act by a pure imaginary scalar, a constraint which cannot be satisfied if each such variable acts by a self-adjoint operator.

Thus we cannot appeal to the standard theory of principal series in order quantize the chromatic Lagrangian – a new kind of representation of the quantum cluster variety is required. Although we do not currently know how such representations should be defined, let us sketch out some features we would desire of them in order to define an analytic wavefunction.

To a framed seed $\underline{\mathbf{i}}(\mathbf{F}')$ with underlying deformed foam $\mathbf{F}'$, we would like to associate a space of meromorphic functions $\mathcal{V}_{\underline{\mathbf{i}}}$ in g variables $z_1, \ldots, z_g$, defined by appropriate conditions on their asymptotic behavior along with the possible locations of their poles. The framed seed determines a natural action of $\mathbb{L}^q_{\mathscr{X}}$ by q-difference operators on the space of all meromorphic functions on $\mathbb{C}^g$, and the subspace $\mathcal{V}_{\underline{\mathbf{i}}}$ should be preserved under this action.

To each admissible mutation or framing shift $a : \underline{\mathbf{i}}_1 \to \underline{\mathbf{i}}_2$, there should correspond an isomorphism between the spaces $\mathcal{V}_{\underline{\mathbf{i}}_1}, \mathcal{V}_{\underline{\mathbf{i}}_2}$. These isomorphisms should again intertwine the action of $\mathbb{L}^q_{\mathscr{X}}$, and their composites corresponding to trivial cluster transformations should act by scalar multiples of the identity.

Given a representation of the quantum cluster variety $\mathcal{P}_g$ in this sense, one could then attempt to define the wavefunction associated to a framed seed obtained from the standard necklace by a sequence of admissible mutations as in (7.1.1). To verify that this prescription is indeed well-defined would again amount to showing that all half Dehn twists fix the necklace wavefunction $\Psi_{\text{neck}} = 1$.

Let us note that there is another regime for $\hbar$ which is nicely compatible with the analytic properties of the noncompact quantum dilogarithm – namely, when $|\hbar| = 1$. In this regime, Faddeev [F] has constructed discrete series-type representations of the modular double of $U_q(\mathfrak{sl}_2)$ whose central characters also correspond to a sum of logarithmic cluster variables acting by a pure imaginary scalar. Thus the regime $|\hbar| = 1$ may be most suitable one in which to try to carry out the construction of such representations of quantum cluster varieties associated to punctured surfaces.

In the following subsections, we present some explicit calculations in $g = 1, 2$ which indicate how one might try to define the action of admissible mutations and framing shifts in the non-unitary case, and provide examples of candidate analytic wavefunctions.

8.1. **Analytic wavefunctions for $g = 1$.**

We begin with the standard necklace framed seed $\underline{\mathbf{i}}_0$ for $g = 1$ as shown in Figure 8.1.1.

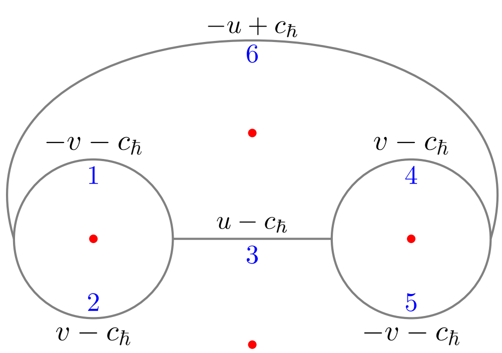


FIGURE 8.1.1. The standard necklace framed seed $\underline{\mathbf{i}}_0$ for $g = 1$

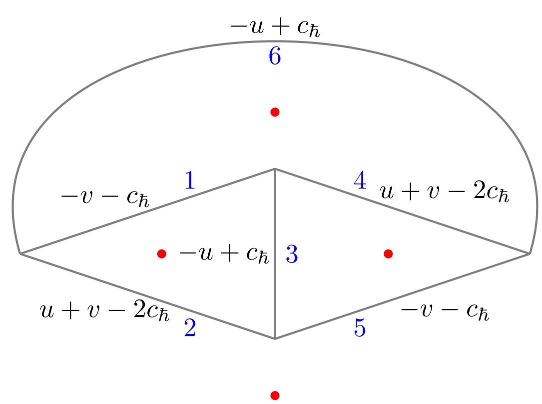


FIGURE 8.1.2. The framed seed $\underline{\mathbf{i}}_1 = \mu_3^+(\underline{\mathbf{i}}_0)$ for the canoe graph.

In this figure, we have decorated the edges of the cubic graph by the Heisenberg algebra elements corresponding to the logarithmic cluster variables, and we have set (see Appendix A)

$$c_\hbar := \frac{i}{2}\left(\hbar + \hbar^{-1}\right) \in i\mathbb{R}.$$

Now consider the following loop in the framed seeds groupoid. First, observe that the positive mutation μ_3^+ at edge 3 yields the canoe framed seed $\underline{\mathbf{i}}_1$ shown in Figure 8.1.2. Performing the change of framing σ conjugating all Heisenberg algebra elements by $e^{\pi i(v-c_\hbar)^2}$, thereby effecting the shift $\sigma : u \mapsto u - v + c_\hbar$, we arrive at the framed seed $\underline{\mathbf{i}}_2 = \sigma(\underline{\mathbf{i}}_1)$ shown in Figure 8.1.3.

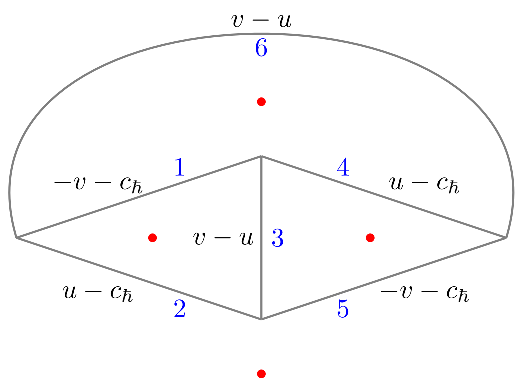


FIGURE 8.1.3. The framed seed $\underline{\mathbf{i}}_2 = \sigma(\underline{\mathbf{i}}_1)$ for the canoe graph.

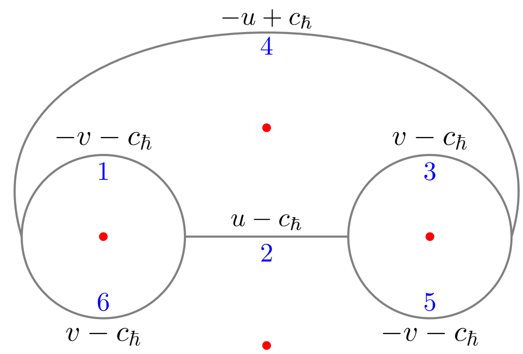


FIGURE 8.1.4. The framed seed $\mu_4^+(\underline{\mathbf{i}}_2) \simeq \underline{\mathbf{i}}_0$.

Finally, performing a positive mutation μ_4^+ at edge 4 in $\underline{\mathbf{i}}_2$ results in the framed seed shown in Figure 8.1.4, which represents the same framed seed as the initial one $\underline{\mathbf{i}}_0$. We therefore have a loop in the framed seeds groupoid

$$\begin{tikzcd} & \underline{\mathbf{i}}_0 \arrow[dl, "\mu_3^+"'] & \\ \underline{\mathbf{i}}_1 \arrow[rr, "\sigma"] & & \underline{\mathbf{i}}_2 \arrow[ul, "\mu_4^+"'] \end{tikzcd} \tag{8.1.1}$$

We once again take the wavefunction for the standard necklace framed seed $\underline{\mathbf{i}}_0$ to be $\psi_{\underline{\mathbf{i}}_0} = 1$, but now regarded as an entire function on $\mathbb{C}$ rather than as a formal power series. We now explain how the mutations and framing shifts in (8.1.1) give rise to operators on spaces of meromorphic functions with appropriate analytic properties, and verify that the composite of these operators indeed preserves $\psi_{\underline{\mathbf{i}}_0}$ up to a phase.

By analogy with the Fock-Goncharov construction in the unitary case, we take the positive mutation μ_3^+ at edge 3 carrying Heisenberg element by $u - c_\hbar$ to correspond to the operator of multiplication by the meromorphic function $\varphi(z - c_\hbar)$, which has simple poles at $\{in\hbar + im\hbar^{-1}\}_{n,m\in\mathbb{Z}_{\geq 1}}$. We thus obtain

$$\begin{aligned} \psi_{\underline{\mathbf{i}}_1} &= \varphi(z - c_\hbar) \cdot \psi_{\underline{\mathbf{i}}_0} \\ &= \varphi(z - c_\hbar) \end{aligned}$$

which now satisfies the dual pair of face relations

$$\begin{aligned} e^{-2\pi\hbar z}\psi_{\underline{\mathbf{i}}_1}(z) + (1 - e^{-2\pi\hbar z})\psi_{\underline{\mathbf{i}}_1}(z + i\hbar) = 0 \\ e^{-2\pi\hbar^{-1} z}\psi_{\underline{\mathbf{i}}_1}(z) + (1 - e^{-2\pi\hbar^{-1} z})\psi_{\underline{\mathbf{i}}_1}(z + i\hbar^{-1}) = 0. \end{aligned}$$

Let us regard the function $\psi_{\underline{\mathbf{i}}_1}$ as an element of the space $\mathcal{V}_{\underline{\mathbf{i}}_1}$ consisting of functions $f(z)$ analytic outside of the cone $\{in\hbar + im\hbar^{-1}\}_{n,m\in\mathbb{R}_{\geq 1}}$, and having prescribed asymptotic behavior

$$f(z)\big|_{z\to\infty} \sim \begin{cases} A_- & |\arg(z)| > \frac{\pi}{2} + \arg(\hbar) \\ A_+ e^{\pi i(z-c_\hbar)^2} & |\arg(z)| < \frac{\pi}{2} - \arg(\hbar) \end{cases} \tag{8.1.2}$$

for some $A_\pm \in \mathbb{C}$.

Let us now consider the effect of performing the change of framing σ. As in the case of mutation, we again define its action on our wavefunction by analytic continuation of the integral transform representing the action of $e^{\pi i v^2}$ in the unitary case. Indeed, consider the integral

$$f \longmapsto \int e^{-\pi i (z-t-c_\hbar)^2} f(t+2c_\hbar)dt, \tag{8.1.3}$$

where the contour of integration stays within the domain of analyticity of f and escapes to infinity in the sectors $|\arg(t)| > \frac{\pi}{2} + \arg(\hbar)$ and $|\arg(t)| < \frac{\pi}{2} - \arg(\hbar)$. It follows from the asymptotics (8.1.2) that the integral converges absolutely for $|\arg(z) - \frac{\pi}{2}| < \pi - \arg(\hbar)$, so that $\hat{f}(z)$ defines an analytic function on the complement of the cone $\{-in\hbar - im\hbar^{-1}\}_{n,m\in\mathbb{R}_{\geq 0}}$.

Applying (8.1.3) to ψ_{canoe} and using the inversion and Fourier transformation properties (A.1.1) and (A.2.1) of the noncompact quantum dilogarithm, we obtain

$$\begin{aligned}
\psi_{\underline{\mathbf{i}}_2} &= \int e^{-\pi i (z-t-c_\hbar)^2} \psi_{\underline{\mathbf{i}}_1}(t+2c_\hbar)dt \\
&= e^{-\pi i z^2 + 2\pi i c_\hbar z} \int e^{2\pi i z t} e^{-\pi i (t+c_\hbar)^2} \varphi(t+c_\hbar)dt \\
&= \zeta_{inv} e^{-\pi i z^2 + 2\pi i c_\hbar z} \int \frac{e^{2\pi i z t}}{\varphi(-t-c_\hbar)} dt \\
&= \zeta_{inv}\zeta e^{-\pi i z^2 + 2\pi i c_\hbar z} \varphi(-z+c_\hbar) \\
&= e^{\pi i c_\hbar^2} \zeta_{inv}^2 \zeta \cdot \varphi(z-c_\hbar)^{-1}.
\end{aligned}$$

Finally, the positive mutation μ_4^+ at edge 4 of $\underline{\mathbf{i}}_2$ which carries Heisenberg element $u - c_\hbar$ acts by the operator of multiplication by the meromorphic function $\varphi(z - c_\hbar)$, and hence under our proposed action for framing shifts and admissible mutations the loop (8.1.1) does indeed act trivially on the analytic wavefunction $\psi_{\underline{\mathbf{i}}_0}$, up to a constant phase.

8.2. **Analytic wavefunctions for $g = 2$.** In the genus 2 case, the combinatorics of ideal foams and framed seeds becomes richer. To illustrate this, we will describe a loop in the framed seeds groupoid that reflects a 3-2 Pachner move for deformed foams, and verify that this loop acts trivially on our proposed analytic wavefunction.

Again we begin with the standard necklace framed seed $\underline{\mathbf{i}}_{\mathrm{neck}}$ for $g = 2$, for which $\psi_{\mathrm{neck}} = 1$. Performing positive mutations at the (commuting) edges labelled $u_1 - c_\hbar$, $u_2 - c_\hbar$, we obtain the framed seed for the canoe graph shown $\underline{\mathbf{i}}_2$ in Figure 8.2.1, whose underlying deformed foam consists of two tetrahedra. The corresponding wavefunction is

$$\psi_{\underline{\mathbf{i}}_2} = \varphi(z_1 - c_\hbar)\varphi(z_2 - c_\hbar).$$

On the other hand, consider the framed seed $\underline{\mathbf{i}}_3$ obtained from $\underline{\mathbf{i}}_{\mathrm{neck}}$ by instead performing *negative* mutations at the edges labelled $u_1 - c_\hbar, u_2 - c_\hbar$, followed by a *positive* mutation at the edge labelled $-u_1 - u_2 + 3c_\hbar$. This framed seed is illustrated in Figure 8.2.2. The corresponding deformed foam now consists of three deformed Harvey-Lawson tetrahedra, and the wavefunction is

$$\psi_{\underline{\mathbf{i}}_3} = \frac{\varphi(-z_1 - z_2 + 3c_\hbar)}{\varphi(-z_1 + c_\hbar)\varphi(-z_2 + c_\hbar)}.$$

Introducing the framing shift

$$\begin{aligned}
\sigma \ : \ u_1 &\longmapsto u_1 + v_2 - v_1 + 3c_\hbar \\
u_2 &\longmapsto u_2 - v_2 + v_1 + 3c_\hbar
\end{aligned}$$

and the change of coordinates

$$\tau \ : \ u_j \mapsto -u_j, \quad v_j \mapsto -v_j,$$

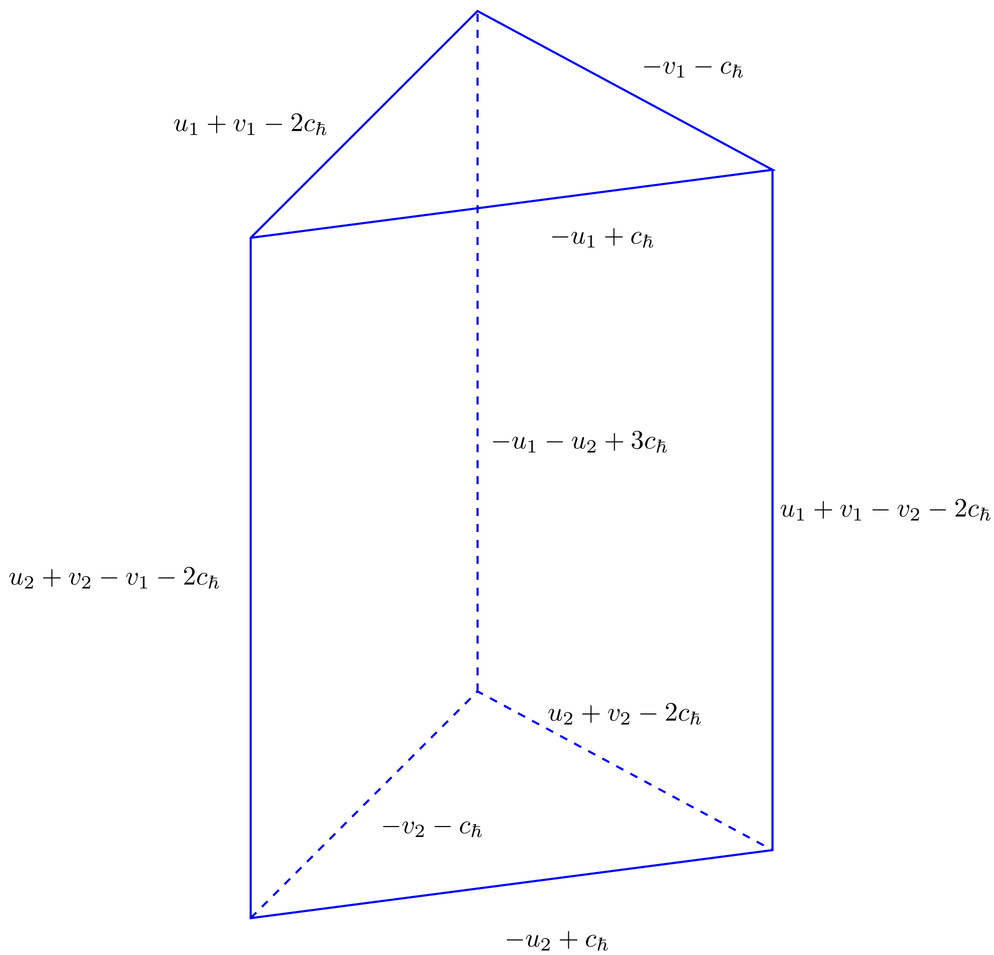


FIGURE 8.2.1. The framed seed $\underline{\mathbf{i}}_2$

we observe that the framed seed $(\tau \circ \sigma) \cdot \underline{\mathbf{i}}_2$ coincides with $\underline{\mathbf{i}}_3$ up to a re-labelling of edges of the cubic graph. We will now confirm that the corresponding wavefunctions are indeed projectively equal.

To understand the action of the framing shift operator σ on $\psi_{\underline{\mathbf{i}}_2}$, note that since σ acts trivially on the necklace wavefunction $\psi_{\mathrm{neck}} = 1$, it will map $\psi_{\underline{\mathbf{i}}_2} = \varphi(u_1 - c_\hbar)\varphi(u_2 - c_\hbar) \cdot \psi_{\mathrm{neck}}$ to

$$\begin{aligned} \sigma \cdot \psi_{\underline{\mathbf{i}}_2} &= \varphi(u_1 + v_2 - v_1 + 2c_\hbar)\varphi(u_2 - v_2 + v_1 + 2c_\hbar) \cdot \sigma \cdot \psi_{\mathrm{neck}} \\ &= \varphi(u_1 + v_2 - v_1 + 2c_\hbar)\varphi(u_2 - v_2 + v_1 + 2c_\hbar) \cdot \psi_{\mathrm{neck}}. \end{aligned}$$

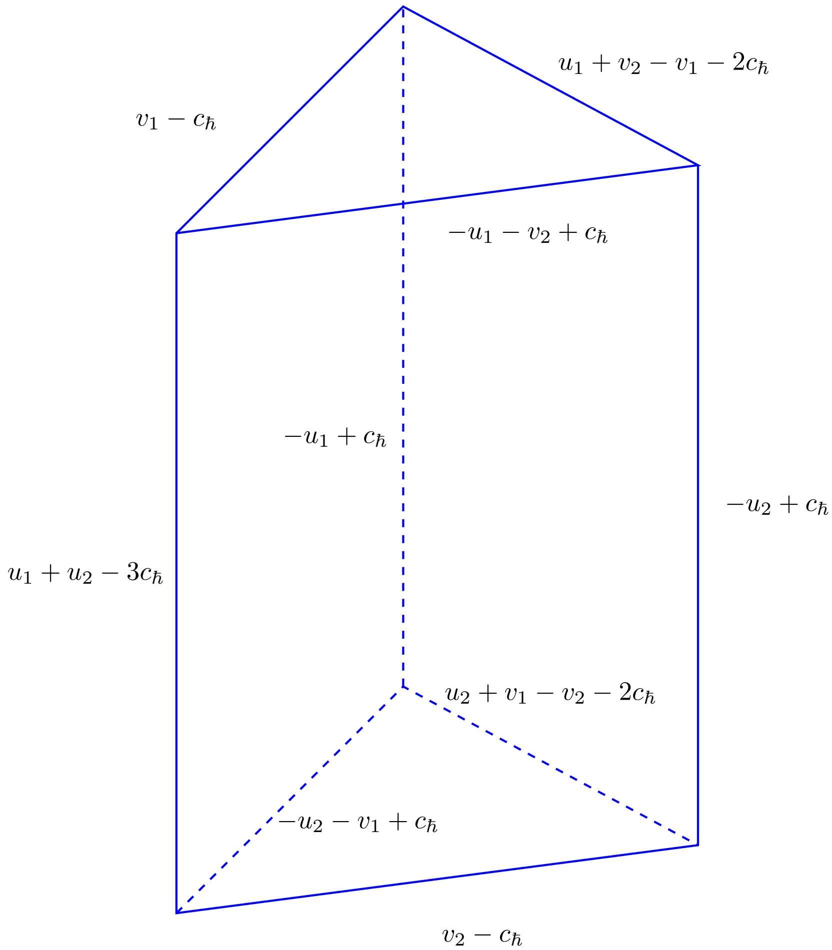


FIGURE 8.2.2. The framed seed $\underline{\mathbf{i}}_3$

The latter operators are once again understood by means of the Fourier self-duality (A.2.1), so that we have, e.g.

$$\begin{aligned}
\varphi(u_1 + v_2 - v_1 + 2c_\hbar) \cdot f(z_1, z_2) &= \zeta^{-1} \int \frac{e^{2\pi i t(u_1+v_2-v_1+c_\hbar)}}{\varphi(t - c_\hbar)} f(z_1, z_2) dt \\
&= \zeta^{-1} \int \frac{e^{\pi i t^2} e^{2\pi i t(u_1+c_\hbar)}}{\varphi(t - c_\hbar)} e^{2\pi i t(v_2 - v_1)} \cdot f(z_1, z_2) dt \\
&= \zeta^{-1} \int \frac{e^{\pi i t^2} e^{2\pi i t(u_1+c_\hbar)}}{\varphi(t - c_\hbar)} \cdot f(z_1 + t, z_2 - t) dt \\
&= e^{-\pi i c_\hbar^2} (\zeta_{inv} \zeta)^{-1} \int e^{-2\pi i t(u_1+2c_\hbar)} \varphi(t + c_\hbar) \cdot f(z_1 - t, z_2 + t) dt
\end{aligned}$$

Hence we see that up to multiplicative phase constants,

$$\begin{aligned}
\sigma \cdot \psi_{\underline{\mathbf{i}}_2} &\equiv \int e^{-2\pi i t(z_1+2c_\hbar)} e^{-2\pi i s(z_2+t+2c_\hbar)} \varphi(t + c_\hbar) \varphi(s + c_\hbar) ds dt \\
&\equiv \int \frac{e^{-2\pi i t(z_1+2c_\hbar)} \varphi(t + c_\hbar)}{\varphi(t + z_2 + c_\hbar)} dt,
\end{aligned}$$

where the Fourier integral over s is again performed using (A.2.1), and the symbol $\equiv$ denotes projective equality modulo phase constants. On the other hand, the resulting integral over t may be computed by means of the 'pentagon' integral evaluation (A.2.4), with the result

$$\sigma \cdot \psi_{\underline{\mathbf{i}}_2} \equiv \frac{\varphi(z_1 + z_2 + 3c_\hbar)}{\varphi(z_2 + c_\hbar)\varphi(z_1 + c_\hbar)}.$$

Hence we conclude that

$$(\tau \circ \sigma) \cdot \psi_{\underline{\mathbf{i}}_2} \equiv \psi_{\underline{\mathbf{i}}_3}.$$

9. Framing Duality

9.1. **Classical limit.** If we mutate the necklace graph Γ_g^{neck} at g of the $g+1$ edges between beads we get the "canoe"-shaped graph $\Gamma_g^{\mathrm{canoe}}$

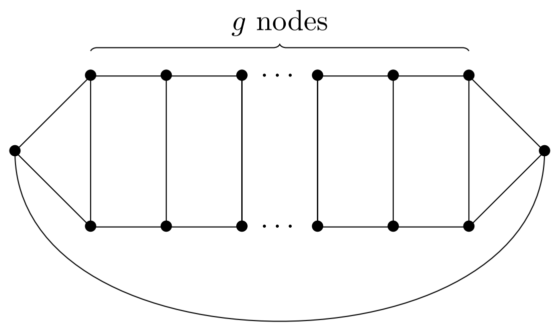


Figure 9.1.1. The "canoe" graph $\Gamma_g^{\mathrm{canoe}}$.

In fact, $\Gamma_1^{\mathrm{canoe}}$ is the tetrahedron, a blow-up of the Θ graph with two-nodes and three edges, and more generally $\Gamma_g^{\mathrm{canoe}}$ is an iterated g-fold blow-up: $\Gamma_{g+1}^{\mathrm{canoe}}$ is obtained from blowing up $\Gamma_g^{\mathrm{canoe}}$ at either of the two vertices at the ends of the canoe. Recall from [TZ, Section 5.2] that if $\hat{\Gamma}$ is the blow-up of Γ at a vertex, then $\mathscr{M}_{\hat{\Gamma}} = H \times \mathscr{M}_\Gamma$, where H is the pair of pants $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. So $\mathscr{M}_{\Gamma_g^{\mathrm{canoe}}} \cong H^g$, i.e. g copies of the tetrahedron moduli space — and more specifically the superpotential is determined in the induced phase with canonical "zero" framing from the mutations of $\Gamma_{g+1}^{\mathrm{neck}}$ as the sum of g copies of the tetrahedron superpotential:

$$W^{(0)}_{\Gamma_g^{\mathrm{canoe}}} = \sum_{i=1}^{g} \mathrm{Li}_2(U_i), \tag{9.1.1}$$

where $U_i = e^{u_i}$. The lift of $\mathscr{M}_{\Gamma_g^{\mathrm{canoe}}}$ is cut out from $\mathscr{P}$ by the graph of $W^{(0)}_{\Gamma_g^{\mathrm{canoe}}}$. We can also consider the framing defined by a $g \times g$ symmetric integral matrix A. This defines new coordinates $v_i' = v_i, u_i' = u_i + A_{i,j}v_j$. Then the lift of $\mathscr{M}_{\Gamma_g^{\mathrm{canoe}}}$ is cut out from $\mathscr{P}$ in these coordinates as the graph of the associated superpotential $W^{(A)}_{\Gamma_g^{\mathrm{canoe}}}$. We write

$$W^{(A)}_{\Gamma_g^{\mathrm{canoe}}} = \sum_{d \neq 0 \in (\mathbb{Z}_{\geq 0})^g} a(d, A) \mathrm{Li}_2(U'^d), \tag{9.1.2}$$

where $U'^d = \prod_{i=1}^g U_i'^{d_i}$ and $U_i' = e^{u_i'}$. For later reference, we define $V_i = V_i' = e^{v_i}$. Note that the *single* geometric object $\mathscr{M}_{\Gamma_g^{\mathrm{canoe}}} \hookrightarrow \mathscr{P}$ defines *all* superpotentials $W^{(A)}_{\Gamma_g^{\mathrm{canoe}}}$.

9.2. **A-polynomial of a quiver.** Before stating the framing duality conjecture, we will need to recall the A-polynomial and DT series of a quiver. Let B be an $n \times n$ matrix and let Q_B be the quiver with n nodes labeled $1, ..., n$, and $B_{i,j}$ arrows between node i and node j. The A-polynomial of Q_B is defined as follows. Let $d \neq 0 \in (\mathbb{Z}_{\geq 0})^n$ be a dimension vector. Then

$$A_d(q) = \#\{\text{absolutely irreducible representations of } Q_B \text{ over } \mathbf{F}_q \text{ modulo isomorphism}\}.$$

Remark 9.1. The A-polynomials are DT-invariants for quivers with potential (as Eric has pointed out in email!). Let Q be a quiver with arrows a_j. Let $\hat{Q}$ be the double quiver by adding arrows a_j^* of opposite direction and a new loop c_i for each vertex. Take the potential

$$W = \sum_i c_i \sum_j [a_j, a_j^*].$$

The A-polynomials for Q are the DT-invariants for $(\hat{Q}, W)$.

Conjecture 9.2. *Let $h \geq 0$ and let $A = (h)$ be the one-by-one matrix with single entry h. Let Q_A be the quiver with one node and h arrows. Write $W^{(A)}_{\Gamma_h^{\text{canoe}}} = \sum_d a(d,h)\mathrm{Li}_2(U'^d)$. Then*

$$A_d(1) = a(d,h).$$

[According to [HLRV], Kac polynomials and DT-invariants of quivers are in general different. In particular, see Cor 1.5. of [HLRV]. We should have $\mathrm{DT}_d(1) = \pm a_d$ instead of $A_d(1)$.]

Proof. We justify the conjecture here. First, it holds when $A = 0$, for then the graph has no arrows and there is a unique irreducible representation for each of the g basis dimension vector se $d_j^{(i)} = \delta_{i,j}$, with no others: Setting $a(d,0) = A_d(1)$ reproduces $W^{(0)}$ of Equation (9.1.1). More generally, we refer to Equation (4.3.1) and Proposition 4.2.1 of [RV], where the notations V, x, and N are here W, U and n. In our notation, Equation (4.3.1) says $dW^{(A)} = \sum_i v_i du_i$. Writing $W^{(M)}_{\Gamma_g^{\text{canoe}}}$ as in Equation (9.1.2), this says $V_i = \prod_d (1 - U^d)^{d_i a(d,A)}$. Comparison with Equation (4.3.1) of [RV] says $a(d,A) = A_d(1)$. We note that in [RV] the function $W(U)$ is called Schläffi's differential by analogy with the volume of hyperbolic polyhedra, which is part of a dual superpotential computation in [DGGo]. ☐

Remark 9.3. It is a conjecture of Hausel and Rodriguez Villegas [HRV] that for the one-node quiver with h arrows — i.e., Q_A, $A = (h)$ — we have that $A_d(1)$ is the dimension of the middle cohomology of the twisted GL_d-character variety $\mathscr{M}_h$ of a genus-h.

9.3. **DT invariants for symmetric quivers.** Let $A = (a_{ij})$ be an $n \times n$ symmetric matrix with nonnegative integral entries and let Q_A be its corresponding symmetric quiver. Set

$$\chi_A(\mathbf{v}, \mathbf{w}) := \mathbf{v}^t (I - A)\mathbf{w}.$$

By [KS, §5.6], the generating function for the COHA $\mathcal{H}$ of Q_A (also called DT series) is

$$H_A(q^{1/2}, X) = \sum_{\mathbf{v} \in \mathbb{Z}^n_{\geq 0}, k \in \mathbb{Z}} (-1)^k \dim(\mathcal{H}_{\mathbf{v},k}) q^{k/2} X^{\mathbf{v}} \tag{9.3.1}$$

$$= \sum_{\mathbf{v} \in \mathbb{Z}^n_{\geq 0}} \frac{(-q^{\frac{1}{2}})^{\chi_A(\mathbf{v},\mathbf{v})}}{(q)_\mathbf{v}} X^{\mathbf{v}} \qquad \in \ 1 + \mathfrak{m}.$$

where

$$\mathcal{H}_{\mathbf{v},k} = H^{k - \chi_A(\mathbf{v},\mathbf{v})}(BG_\mathbf{v}), \qquad G_\mathbf{v} = \prod_i GL_{v_i}(\mathbb{C}),$$

and
$$(q)_{\mathbf{v}} := \prod_{i=1}^{n} (1-q)(1-q^2)...(1-q^{v_i}).$$
Set
$$H_A(q^{1/2}, X) := \prod_{\mathbf{v}\in\mathbb{Z}^n_{\geq 0}} \prod_{k\in\mathbb{Z}} (q^{k/2}X^{\mathbf{v}}; q)_{\infty}^{\delta_{\mathbf{v},k}}$$
Equivalently, we get
$$(1-q^{-1})\mathrm{Log}\, H_A(q^{1/2}, X) = \sum_{\mathbf{v}\in\mathbb{Z}^n_{\geq 0}, k\in\mathbb{Z}} \delta_{\mathbf{v},k} \mathrm{Li}_2(q^{\frac{k}{2}} X^{\mathbf{v}})$$
By [E, Corollary 4.1], the $(-1)^{k-1}\delta_{\mathbf{v},k}$ are non-negative integers and are nonzero only for finitely many $k \in \mathbb{Z}$.

Remark 9.4. The paper [HLRV] gives a cohomological interpretation of DT-invariants of quivers. Let Γ be a quiver with r vertices and let $\mathbf{v} = (v_1, ..., v_r) \in Z^r_{\geq 0}$ be a dimension vector. Associate to $(\Gamma, \mathbf{v})$ a new quiver $\tilde{\Gamma}$ by attaching a leg of length $v_i - 1$ at the vertex i. We extend the dimension vector $\mathbf{v}$ to $\tilde{\mathbf{v}}$ by placing decreasing dimensions $v_i - 1, v_i - 2, ..., 1$ at the extra leg. Let $W_{\mathbf{v}}$ be the Weyl group of type $A_{v_1-1} \times \ldots \times A_{v_r-1}$ that is generated by the reflections at the extra vertices. Let $\mathcal{Q}_{\tilde{\mathbf{v}}}$ be the smooth generic complex quiver variety associated to $(\tilde{\Gamma}, \tilde{\mathbf{v}})$. The Weyl group $W_{\mathbf{v}}$ acts on $H^*_c(\mathcal{Q}_{\tilde{\mathbf{v}}}, \mathbb{C})$ and hence gives a natural decomposition of the latter into isotypical components. According to [HLRV, Cor 1.5], after a slight renormalization, we have
$$\mathrm{DT}_{\Gamma,\mathbf{v}}(t) = \sum_i \dim \left(H^{2i}_c(\mathcal{Q}_{\tilde{\mathbf{v}}}, \mathbb{C})^{W_{\mathbf{v}}}\right) t^{i-d_{\tilde{\mathbf{v}}}}.$$

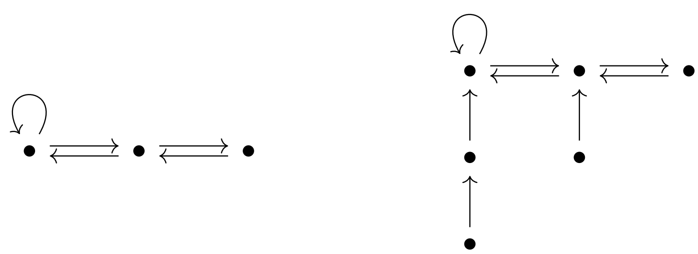


FIGURE 9.3.1. A new quiver associated to $\mathbf{v} = (3, 2, 1)$

Comparing (**??**) with (7.4.4), we have
$$V_A(X) := \lim_{q^{\frac{1}{2}}\to 1} (1-q^{-1}) \cdot \log\mathrm{DT}_A = \sum_{\mathbf{v}\in\mathbb{Z}^n_{\geq 0}\backslash\{0\}} \mathrm{DT}_{\mathbf{v}}(1)\mathrm{Li}_2(X^{\mathbf{v}})$$
Then
$$dV_A(X) = \sum_{i=1}^{n} \log Z_i \, d\log X_i$$
where
$$Z_i := \lim_{q^{1/2}\to 1} \frac{H_A(q^{1/2}; X_1, ..., qX_i, ..., X_n)}{H_A(q^{1/2}; X_1, ..., X_i, ..., X_n)}$$
By [KS, Theorem 5.3], $Z_1, ..., Z_n$ are solutions to the system of equations
$$X_i(-Z_i)^{1-a_{ii}}\Big(\prod_{j\neq i} Z_j^{-a_{ij}}\Big) + Z_i = 1, \qquad i = 1, ..., n.$$

There is a basic identity

$$(X, q^{-1})_\infty = (qX, q)_\infty^{-1}.$$

Let us assume that the DT series is defined by (9.3.1) for arbitrary symmetric matrix A. Let us set

$$\sigma(\mathbf{v}) = \sum_{i=1}^{n} v_i$$

Lemma 9.5. *We have*

$$H_A(q^{-1/2}, -q^{-\frac{1}{2}}X) = H_{I-A}(q^{1/2}, X)$$

Further

$$\mathrm{DT}_{\mathbf{v}}^{I-A}(t) = -t^{-\sigma(\mathbf{v})-2}\mathrm{DT}_{\mathbf{v}}^{A}(t^{-1}).$$

Proof. Note that

$$(q^{-1})_{\mathbf{v}} = (q)_{\mathbf{v}} q^{-\frac{1}{2}\mathbf{v}^t\mathbf{v}}(-q^{\frac{1}{2}})^{-\delta(\mathbf{v})}$$

Therefore

$$\frac{(-q^{\frac{1}{2}})^{-\mathbf{v}^t(I-A)\mathbf{v}}}{(q^{-1})_{\mathbf{v}}}(-q^{-\frac{1}{2}}X)^{\mathbf{v}} = \frac{q^{\frac{1}{2}\mathbf{v}^tA\mathbf{v}}}{(q)_{\mathbf{v}}}X^{\mathbf{v}}$$

The Lemma follows. □

Appendix A. Non-compact quantum dilogarithms

In this appendix, we recall some important properties of the non-compact quantum dilogarithm that we use in the paper. For further background and details regarding this function, we refer the reader to [FKV, Ka, V]. We assume that $\hbar \in \mathbb{C}$ is such that $\hbar + \hbar^{-1} \in \mathbb{R}$, and lies in the first quadrant $\Re(\hbar) > 0, \Im(\hbar) \geq 0$. Let us also write

$$c_\hbar = \frac{i(\hbar + \hbar^{-1})}{2}$$

as well as

$$\zeta = e^{\pi i(1-4c_\hbar^2)/12} \qquad \text{and} \qquad \zeta_{\mathrm{inv}} = \zeta^{-2}e^{-\pi i c_\hbar^2}.$$

A.1. The non-compact quantum dilogarithm.

Definition A.1. Let C be the contour going along the real line from $-\infty$ to $+\infty$, surpassing the origin in a small semi-circle from above. The *non-compact quantum dilogarithm function* $\varphi_\hbar(z)$ is defined in the strip $|\Im(z)| < c_\hbar$ by the following formula [Ka]:

$$\varphi_\hbar(z) = \exp\left(\frac{1}{4}\int_C \frac{e^{-2izt}}{\sinh(t\hbar)\sinh(t\hbar^{-1})}\frac{dt}{t}\right).$$

The non-compact quantum dilogarithm can be analytically continued to the entire complex plane as a meromorphic function with an essential singularity at infinity. The resulting function $\varphi_\hbar(z)$ enjoys the following properties [Ka]:

Poles and zeros:

$$\varphi_\hbar(z)^{\pm 1} = 0 \quad \Leftrightarrow \quad z = \mp\left(c_\hbar + im\hbar + in\hbar^{-1}\right) \quad \text{for} \quad m, n \in \mathbb{Z}_{\geq 0};$$

Behavior around poles and zeros:

$$\varphi_\hbar(z \pm c_\hbar) \sim \pm\zeta^{-1}(2\pi i z)^{\mp 1} \qquad \text{as} \qquad z \to 0;$$

Asymptotic behavior:

$$\varphi_\hbar(z)\big|_{z\to\infty} \sim \begin{cases} \zeta_{\mathrm{inv}}e^{\pi i z^2}, & |\arg(z)| < \frac{\pi}{2} - \arg(\hbar), \\ 1, & |\arg(z)| > \frac{\pi}{2} + \arg(\hbar); \end{cases}$$

Symmetry:

$$\varphi_\hbar(z) = \varphi_{-\hbar}(z) = \varphi_{\hbar^{-1}}(z);$$

Inversion formula:

$$\varphi_\hbar(z)\varphi_\hbar(-z) = \zeta_{\mathrm{inv}} e^{\pi i z^2}; \tag{A.1.1}$$

Functional equations:

$$\varphi_b\left(z - i\hbar^{\pm 1}/2\right) = \left(1 + e^{2\pi\hbar^{\pm 1} z}\right)\varphi_b\left(z + i\hbar^{\pm 1}/2\right); \tag{A.1.2}$$

Unitarity:

$$\overline{\varphi_\hbar(z)}\varphi_\hbar(\overline{z}) = 1;$$

In what follows we will drop the subscript $\hbar$ from the notation for the quantum dilogarithm, and simply write $\varphi(z)$.

A.2. **Integral identites for $\varphi(z)$.** The quantum dilogarithm function $\varphi(z)$ satisfies many important integral identities. Before describing some of them, let us fix a useful convention.

Remark A.2. Throughout the paper, we will often consider contour integrals of the form

$$\int_C \prod_{j,k} \frac{\varphi(t-a_j)}{\varphi(t-b_k)} f(t)dt,$$

where $f(t)$ is some entire function. Unless otherwise specified, the contour C in such an integral is always chosen to be passing below the poles of $\varphi(t-a_j)$ for all j, above the poles of $\varphi(t-b_k)^{-1}$ for all k, and escaping to infinity in such a way that the integrand is rapidly decaying.

The Fourier transform of the quantum dilogarithm can be calculated explicitly by the following integrals:

$$\zeta\varphi(w) = \int \frac{e^{2\pi i x(w-c_b)}}{\varphi(x-c_b)}dx, \tag{A.2.1}$$

$$\frac{1}{\zeta\varphi(w)} = \int \frac{\varphi(x+c_b)}{e^{2\pi i x(w+c_b)}}dx. \tag{A.2.2}$$

It was shown in [FKV] that φ satisfies the following integral analogs of Ramanujan's ${}_1\psi_1$ summation formula:

$$\frac{\varphi(a)\varphi(w)}{\varphi(a+w-c_b)} = \zeta^{-1}\int \frac{\varphi(x+a)}{\varphi(x-c_b)} e^{2\pi i x(w-c_b)}dx, \tag{A.2.3}$$

$$\frac{\varphi(a+w+c_b)}{\varphi(a)\varphi(w)} = \zeta\int \frac{\varphi(x+c_b)}{\varphi(x+a)} e^{-2\pi i x(w+c_b)}dx. \tag{A.2.4}$$

Each of these integral evaluations is equivalent to the non-commutative pentagon identity for φ – for further details, see [FKV].

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