Cluster structure of the quantum Coxeter–Toda system

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Motivation

I will talk about some joint work with **Alexander Shapiro** on an application of cluster algebras to quantum integrable systems.

Motivation: a conjecture from quantum higher Teichmüller theory.

Setup for higher Teichmüller theory

$$S = \text{ a marked surface}$$

$$G = PGL_n(\mathbb{C})$$

$$\mathcal{X}_{G,S} := \text{moduli of framed } G\text{–local systems on } S$$

$\mathcal{X}_{G,S}$ is a cluster Poisson variety

Fock–Goncharov: $\mathcal{X}_{G,S}$ is a **cluster Poisson variety**: it's covered up to codimension 2 by an atlas of toric charts

$$\mathcal{T}_\Sigma \colon (\mathbb{C}^*)^d \longrightarrow \mathcal{X}_{G,\widehat{S}},$$

labelled by quivers Q_Σ. The Poisson brackets are determined by the adjacency matrix ϵ_{jk} of Q_Σ:

$$\{Y_j, Y_k\} = \epsilon_{kj} Y_j Y_k.$$

Canonical quantization of cluster varieties

Let

$$q = e^{\pi i b^2}, \quad b^2 \in \mathbb{R}_{>0} \setminus \mathbb{Q}.$$

Promote each cluster chart to a quantum torus algebra

$$\mathcal{T}^q_\Sigma = \left\langle \hat{Y}_1, \ldots, \hat{Y}_d \right\rangle / \{ \hat{Y}_j \hat{Y}_k = q^{2\epsilon_{kj}} \hat{Y}_k \hat{Y}_j \}.$$

Positivity for quantum cluster varieties

Embed each quantum cluster chart $\mathcal{T}^q$ into a **Heisenberg algebra** $\mathcal{H}$ generated by $\hat{y}_1, \ldots, \hat{y}_d$ with relations

$$[\hat{y}_j, \hat{y}_k] = \frac{i}{2\pi}\epsilon_{jk},$$

by the homomorphism

$$\hat{Y}_j \mapsto e^{2\pi b \hat{y}_j}.$$

$\mathcal{H}$ has irreducible **Hilbert space representations** in which the generators $\hat{Y}_j$ act by **positive** self-adjoint operators.

Representations of $\mathcal{X}^q_{G,S}$

The quantum cluster algebra $\mathcal{X}^q_{G,S}$ has positive representations

$$V_\lambda[S]$$

labelled by all possible real eigenvalues λ of monodromies around the punctures of S.

Quantum mutation

The non–compact **quantum dilogarithm** function $\varphi(z)$ is the unique solution of the **pair** of difference equations

$$\varphi(z - ib^{\pm 1}/2) = (1 + e^{2\pi b^{\pm 1} z})\varphi(z + ib^{\pm 1}/2)$$

Quantum mutation

Since

$$z \in \mathbb{R} \implies |\varphi(z)| = 1,$$

and each $\hat{y}_k$ is self–adjoint we have **unitary** operators

$$\varphi(\hat{y}_k) \leftrightarrow \text{ quantum mutation in direction } k$$

Representation of cluster modular group

The quantum dilogarithm satisfies the **pentagon identity**:

$$[\hat{p}, \hat{x}] = \frac{1}{2\pi i}$$

$$\implies \quad \varphi(\hat{p})\varphi(\hat{x}) = \varphi(\hat{x})\varphi(\hat{p} + \hat{x})\varphi(\hat{p}).$$

So we get a unitary representation of the cluster modular group(oid).

For $\mathcal{X}_{G,S}$, this group contains the **mapping class group** of the surface (realized by flips of ideal triangulation)

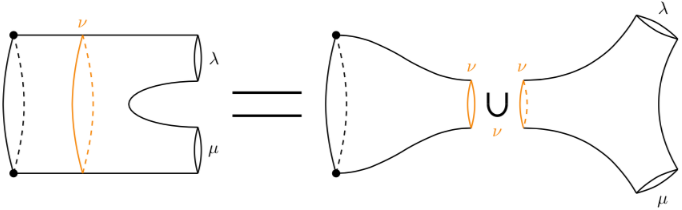
Modular functor conjecture

So we have an assignment:

$$\text{marked surface } S \longmapsto \{\text{algebra } \mathcal{X}^q_{G,S}, \text{ representation of } \mathcal{X}^q_{G,S} \text{ on } V, \\ \text{unitary action of } \mathrm{MCG}(S) \text{ on V}\}.$$

Conjecture: (Fock–Goncharov '09) This assignment is functorial under **gluing** of surfaces.

Modular functor conjecture



i.e. if $S = S_1 \cup_\gamma S_2$, there should be a mapping–class–group equivariant isomorphism

$$V[S] \simeq \int_\nu^\oplus V_\nu[S_1] \otimes V_\nu[S_2]$$

$\nu =$ eigenvalues of monodromy around loop γ

Relation with quantum Toda system

So we need to understand the **spectral theory** of the operators $\hat{H}_1, \ldots, \hat{H}_n$ quantizing the functions on $\mathcal{X}_{G,S}$ sending a local system to its eigenvalues around γ.

If we can find a **complete set of eigenfunctions** for $\hat{H}_1, \ldots, \hat{H}_n$ that are simultaneous eigenfunctions for the **Dehn twist** around γ, we can construct the isomorphism and prove the conjecture.

Relation with quantum Toda system

Theorem (S.–Shapiro '17)

There exists a cluster for $\mathcal{X}_{G,S}$ in which the operators $\hat{H}_1, \ldots, \hat{H}_n$ are identified with the Hamiltonians of the **quantum Coxeter–Toda** *integrable system.*

Classical Coxeter–Toda system

Fix $G = PGL_n(\mathbb{C})$, equipped with a pair of opposite Borel subgroups $B_\pm$, torus $H = B_+ \cap B_-$, and Weyl group $W \simeq S_n$.

We have **Bruhat cell decompositions**

$$G = \bigsqcup_{u \in W} B_+ u B_+ \\ = \bigsqcup_{v \in W} B_- v B_- .$$

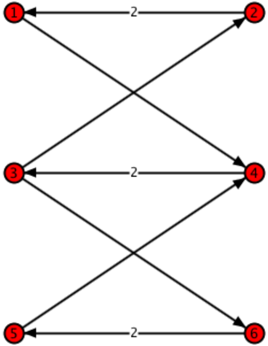
The **double Bruhat cell** corresponding to a pair $(u, v) \in W \times W$ is

$$G^{u,v} := (B_+ u B_+) \cap (B_- v B_-) .$$

Double Bruhat cells

Berenstein–Fomin–Zelevinsky: the double Bruhat cells $G^{u,v}$, and their quotients $G^{u,v}/Ad_H$ are a cluster varieties.

e.g. $G = PGL_4$, $u = s_1 s_2 s_3$, $v = s_1 s_2 s_3$. The quiver for $G^{u,v}/Ad_H$ is



Poisson structure on G

There is a natural Poisson structure on G with the following key properties:

- $G^{u,v} \subset G$ are all **Poisson subvarieties**, whose Poisson structure descends to quotient $G^{u,v}/Ad_H$.
- The algebra of conjugation invariant functions $\mathbb{C}[G]^{Ad_G}$ (generated by traces of finite dimensional representations of G) is **Poisson commutative**:

$$f_1, f_2 \in \mathbb{C}[G]^{Ad_G} \implies \{f_1, f_2\} = 0$$

Cluster Poisson structure on G

- **Gekhtman–Shapiro–Vainshtein:** the Poisson bracket on $\mathbb{C}[G^{u,v}]$ is **compatible** with the cluster structure: if $Y_1, \ldots, Y_d$ are cluster $\mathcal{X}$–coordinates,

$$\{Y_j, Y_k\} = \epsilon_{jk} Y_j Y_k,$$

Classical Coxter–Toda system

Now suppose u, v are both **Coxeter** elements in W (each simple reflection appears exactly once in reduced decomposition)
e.g. $G = PGL_4, u = s_1 s_2 s_3, v = s_1 s_2 s_3.$

Then

$$\begin{aligned} \dim(G^{u,v}/Ad_H) &= l(u) + l(v) - \dim(H) \\ &= 2\dim(H), \end{aligned}$$

and we have $\dim(H)$–many independent generators of our Poisson commutative subalgebra $\mathbb{C}[G]^G$

$\Longrightarrow$ we get an **integrable system** on $G^{u,v}/Ad_H$, called the **Coxter–Toda** system.

Quantization of Coxeter–Toda system

Consider the Heisenberg algebra $\mathcal{H}_n$ generated by

$$x_1, \ldots, x_n; \quad p_1, \ldots, p_n,$$

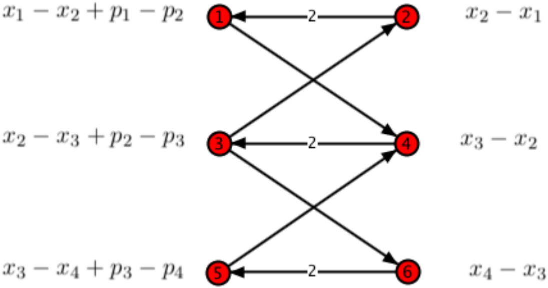
$$[p_j, x_k] = \frac{\delta_{jk}}{2\pi i}$$

acting on $L^2(\mathbb{R}^n)$, via

$$p_j \mapsto \frac{1}{2\pi i} \frac{\partial}{\partial x_j}$$

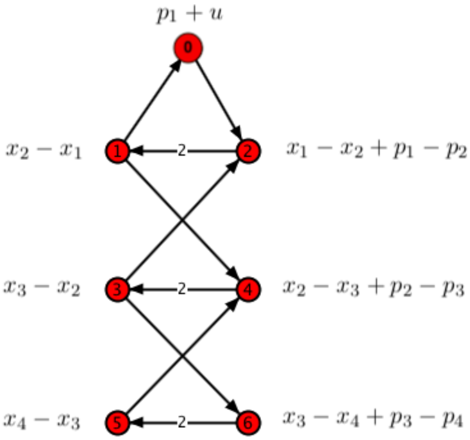
Quantization of Coxeter–Toda system

A representation of the quantum torus algebra for the Coxeter–Toda quiver:



e.g. $\hat{Y}_2$ acts by multiplication by $e^{2\pi b(x_2 - x_1)}$.

Let's add an extra node to our quiver, along with a "spectral parameter" u:



Construction of quantum Hamiltonians

Theorem (S.–Shapiro)

Consider the operator $Q_n(u)$ obtained by mutating consecutively at $0, 1, 2, \dots, 2n$. Then

1. *The quiver obtained after these mutations is isomorphic to the original;*
2. *The unitary operators $Q_n(u)$ satisfy*

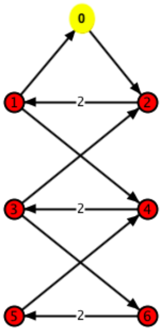
$$[Q_n(u), Q_n(v)] = 0,$$

3. *If $A(u) = Q_n(u - ib/2)Q_n(u + ib/2)^{-1}$, then one can expand*

$$A(u) = \sum_{k=0}^{n} H_k U^k, \quad U := e^{2\pi bu}$$

and the commuting operators $H_1, \dots, H_n$ quantize the Coxeter–Toda Hamiltonians.

Example: $G = PGL_4$



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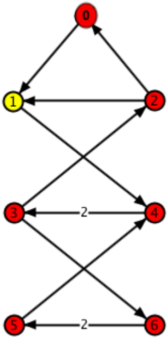

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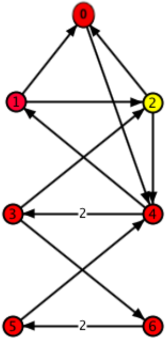

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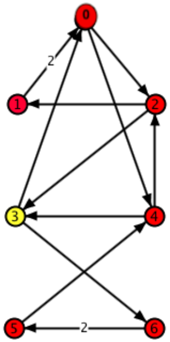

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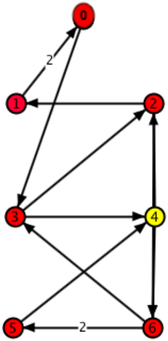

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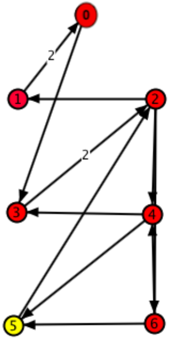

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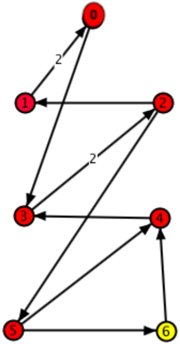

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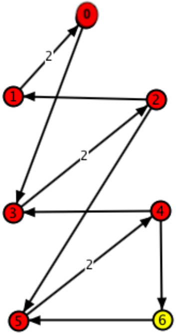
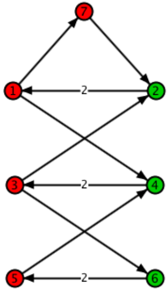


The Dehn twist operator D_n from quantum Teichmüller theory corresponds to mutation at all even vertices $2, 4, \ldots, 2n$:



We have

$$[D_n, Q_n(u)] = 0$$

Problem: Construct complete set of joint eigenfunctions (*i.e.* b–Whittaker functions) for operators $Q_n(u), D_n$.

e.g. $n = 1$.

$$Q_1(u) = \varphi(p_1 + u)$$

If $\lambda \in \mathbb{R}$,

$$\Psi_\lambda(x_1) = e^{2\pi i \lambda x_1}$$

satisfies

$$Q_1(u)\Psi_\lambda(x_1) = \varphi(\lambda + u)\Psi_\lambda(x_1).$$

(equivalent to formula for Fourier transform of $\varphi(z)$)

Recursive construction of the eigenfunctions

Recursive construction: let

$$\mathcal{R}_n^{n+1}(\lambda) = Q_n\left(\lambda^*\right) \frac{e^{2\pi i \lambda x_{n+1}}}{\varphi(x_{n+1} - x_n)},$$

where $\lambda^* = \frac{i(b+b^{-1})}{2} - \lambda$.

Pengaton identity $\Longrightarrow$

$$Q_{n+1}(u)\ \mathcal{R}_n^{n+1}(\lambda) = \varphi(u+\lambda)\ \mathcal{R}_n^{n+1}(\lambda)\ Q_n(u).$$

Recursive construction of the eigenfunctions

So given a $\mathfrak{gl}_n$ eigenvector $\Psi_{\lambda_1,\ldots,\lambda_n}(x_1,\ldots,x_n)$ satisfying

$$Q_n(u)\Psi_{\lambda_1,\ldots,\lambda_n} = \prod_{k=1}^{n} \varphi(u+\lambda_k)\ \Psi_{\lambda_1,\ldots,\lambda_n},$$

we can build a $\mathfrak{gl}_{n+1}$ eigenvector

$$\Psi_{\lambda_1,\ldots,\lambda_{n+1}}(x_1,\ldots,x_{n+1}) := \mathcal{R}_n^{n+1}(\lambda_{n+1}) \cdot \Psi_{\lambda_1,\ldots,\lambda_n}$$

satisfying

$$Q_{n+1}(u)\Psi_{\lambda_1,\ldots,\lambda_{n+1}} = \prod_{k=1}^{n+1} \varphi(u+\lambda_k)\ \Psi_{\lambda_1,\ldots,\lambda_{n+1}}.$$

A modular b–analog of Givental's integral formula

Writing all the $R_n^{n+1}(\lambda)$ as integral operators, we get an explicit Givental–type integral formula for the eigenfunctions:

$$\Psi_\lambda^{(n)}(x) = e^{2\pi i \lambda_n \underline{x}} \int \prod_{j=1}^{n-1} \left(e^{2\pi i \underline{t}_j(\lambda_j - \lambda_{j+1})} \prod_{k=2}^{j} \varphi(t_{j,k} - t_{j,k-1}) \right.$$
$$\left. \prod_{k=1}^{j} \frac{dt_{j,k}}{\varphi(t_{j,k} - t_{j+1,k} - c_b)\varphi(t_{j+1,k+1} - t_{j,k})} \right),$$

where $t_{n,1} = x_1, \ldots, t_{n,n} = x_n$.

e.g. $n = 4$ we integrate over all but the last row of the array

$$\begin{array}{llll} t_{11} & & & \\ t_{21} & t_{22} & & \\ t_{31} & t_{32} & t_{33} & \\ x_1 & x_2 & x_3 & x_4 \end{array}$$

Unitarity of the b–Whittaker transform

Using the cluster construction of the b–Whittaker functions, we can prove

Theorem (S.–Shapiro)

The b–Whittaker transform

$$(\mathcal{W}[f])(\lambda) = \int_{\mathbb{R}^n} \Psi_{\lambda}^{(n)}(x) f(x) dx$$

is a unitary equivalence.

This completes the proof of the Fock–Goncharov conjecture for $G = PGL_n$.