CARTAN HOMOTOPY FORMULAS AND THE GAUSS-MANIN CONNECTION IN CYCLIC HOMOLOGY

Ezra Getzler

Department of Mathematics, MIT,
Cambridge, Mass. 02139 USA

It is well-known that the periodic cyclic homology $\mathrm{HP}_\bullet(A)$ of an algebra A is homotopy invariant (see Connes [3], Goodwillie [8] and Block [1]). Let A be an algebra over a field $\mathbf{k}$ and let A_ν be a formal deformation of A, that is, an associative product

$$m_\nu \in \mathrm{Hom}(A^{\otimes 2}, A)[\![\nu_1, \dots, \nu_n]\!]$$

such that $m|_{\nu=0}$ is the product on A. We will define a connection on the periodic cyclic bar complex of A_ν for which the differential is covariant constant, thus inducing a connection on the periodic homology $\mathrm{HP}_\bullet(A_\nu)$, thought of as a module over $\mathbf{k}[\![\nu_1, \dots, \nu_n]\!]$. This connection generalizes the classical Gauss-Manin connection, and indeed we will prove that it has curvature chain homotopic to zero.

The Gauss-Manin connection is obtained by a generalization of Rinehart's result: if D is a derivation on A, then the operator $u\mathcal{L}(D)$ on the cyclic bar complex $C(A)[\![u]\!]$ of A is chain homotopic to zero (see Rinehart [10], and also Goodwillie [8]). Inspired by the work of Nistor [9], we prove a more general result on the action of the cochains $C^\bullet(A, A)$ on the cyclic bar complex $C(A)$, where A is an A_∞-algebra. (Recall that A_∞-algebras are a generalization of differential graded algebras. It is shown in [6] that A_∞-algebras form a natural setting for the study of cyclic homology.)

The author would like to thank J. Block, V. Nistor, D. Quillen and B. Tsygan for helpful conversations on the subject of this article; in particular, the term Gauss-Manin connection is used at Quillen's suggestion. Note that some similar formulas, in the setting of Hochschild homology, have been obtained by Gelfand, Daletskiĭ and Tsygan [4].

1. Hochschild cochains and A_∞-algebras

In this paper, all vector spaces will be over a field $\mathbf{k}$. If V and W are graded vector spaces, we denote by $V \otimes W$ the graded tensor product: thus, if $A \in \mathrm{End}(V)$ and $B \in \mathrm{End}(W)$, then $(A \otimes B)(v \otimes w) = (-1)^{|B|\,|v|}(Av) \otimes (Bw)$. If V is a graded vector space, we denote by $V^{(k)}$ the tensor power $V^{\otimes k}$.

Let A be a graded vector space, and let sA be its suspension

$$(sA)_i = A_{i-1}.$$

The bar coalgebra of A is the direct sum

$$B(A) = \sum_{n=0}^{\infty} (sA)^{(n)};$$

This work was partially funded by the IHES and the NSF.

we denote the element $(sa_1)\otimes\dots\otimes(sa_n)\in B(A)$ by $[a_1|\dots|a_n]$. The coproduct is given by the formula

$$\Delta[a_1|\dots|a_n]=\sum_{i=0}^{n}[a_1|\dots|a_i]\otimes[a_{i+1}|\dots|a_n],$$

and the counit ε sends $[\,]$ to 1, and $[a_1|\dots|a_n]$ to 0 if $n\geq 1$.

Definition 1.1. *If A and B are graded vector spaces, the space of Hochschild cochains on A with values in B is*

$$C^\bullet(A,B)=\operatorname{Hom}(B(A),sB).$$

If D is a homogeneous function of A, we denote its degree of homogeneity by $d(D)$.

Given $D\in C^\bullet(B,C)$ and $D_i\in C^\bullet(A,B)$, $1\leq i\leq k$, define an element $D\{D_1,\dots,D_k\}\in C^\bullet(A,C)$ with $|D\{D_1,\dots,D_k\}|=|D|+\sum_{i=1}^k|D_i|$, given for homogeneous D_i by the formula

$$\begin{aligned}D\{D_1,\dots,D_k\}[a_1|\dots|a_n]=\sum_{(j_1,\dots,j_k)\in J}&(-1)^{\sum_{i=1}^k\omega_{j_i}|D_i|}\\ &D[a_1|\dots|a_{j_1}|D_1[a_{j_1+1}|\dots|a_{j_1+d(D_1)}]|a_{j_1+d(D_1)+1}|\dots\\ &\quad\dots|a_{j_k}|D_k[a_{j_k+1}|\dots|a_{j_k+d(D_k)}]|a_{j_k+d(D_k)+1}|\dots|a_n],\end{aligned}$$

where $\eta_i=|a_1|+\dots+|a_i|-i$ and

$$J=\{(j_1,\dots,j_k)\mid 0\leq j_1, j_i+d_i\leq j_{i+1}\text{ for }1\leq i\leq k-1, j_k\leq n-d_k\}.$$

In the case $k=1$, this operation was introduced by Gerstenhaber [5], and is denoted

$$D_0\{D_1\}=D_0\circ D_1.$$

Lemma 1.2. *If $D_0,D_1,D_2\in C^\bullet(A,A)$, then*

$$(D_0\circ D_1)\circ D_2-D_0\circ(D_1\circ D_2)=D_0\{D_1,D_2\}+(-1)^{|D_1|\,|D_2|}D_0\{D_2,D_1\}.$$

It follows from this lemma that the bracket

$$[D_0,D_1]=D_0\circ D_1-(-1)^{|D_0|\,|D_1|}D_1\circ D_0$$

gives $C^\bullet(A,A)$ the structure of a graded Lie algebra.

Recall that a coderivation on a coalgebra C is a linear map $\delta:C\to C$ such that

$$(\delta\otimes 1+1\otimes\delta)\Delta a=\Delta(\delta a)$$

for $a\in C$. The space of coderivations $\operatorname{Coder}(C)$ is a graded Lie algebra, with bracket the graded commutator.

Proposition 1.3. *There is an isomorphism of graded Lie algebras*

$$\delta:C^\bullet(A,A)\to\operatorname{Coder}(B(A)),$$

given for homogeneous D by the formula

$$\delta(D)[a_1|\dots|a_n]=\sum_{i=0}^{n-d}(-1)^{\omega_i|D|}[a_1|\dots|a_i|D[a_{i+1}|\dots|a_{i+d(D)}]|a_{i+d(D)+1}|\dots|a_n].$$

The following definition is due to Stasheff [11] (see also [6]).

Definition 1.4. *An A_∞-algebra structure on a graded vector space A is a codifferential δ of degree -1 on $B(A)$ such that $\delta[\,]=0$*

A codifferential δ on $B(A)$ corresponds to a Hochschild cochain $m \in C^\bullet(A,A)$, which may be viewed as a sequence of multilinear maps

$$m_k : A^{(k)} \to A, \quad k \geq 1,$$

of degree $k-2$. The equation $\delta^2 = 0$ translates to a sequence of identities which are summed up in the formula $m \circ m = 0$, or more explicitly, the sequence of identities for $k \geq 1$,,

$$\sum_{j=1}^{k}\sum_{i=0}^{j-1}(-1)^{\omega_i} m_j(a_1,\dots,a_i,m_{k-j+1}(a_{i+1},\dots,a_{i+k-j+1}),a_{i+k-j+2},\dots,a_k) = 0.$$

Lemma 1.5. *An A_∞-algebra such that $m_k = 0$ for $k > 2$ is the same as a differential graded algebra, with product $a_1a_2 = (-1)^{|a_1|}m_2(a_1,a_2)$ and differential m_1.*

An algebra is analogous to a connection: the cochain $m \in C^2(A,A)$ is homogeneous of degree 2, just as a connection is homogeneous of degree 1. In this language, an A_∞-structure m is the analogue of a superconnection.

The augmention A^+ of an A_∞-algebra A is the A_∞-algebra whose underlying space is $A \oplus \mathbf{k}e$, where the element e acts as an identity for A^+; that is, $m \in C^\bullet(A,A)$ is extended to A^+ by setting

$$\begin{cases} m_2(e,a) = (-1)^{|a|}m_2(a,e) = a, \\ m_2(e,e) = e, \\ m_k(\dots,e,\dots) = 0, \quad \text{for } k \neq 2. \end{cases}$$

The following lemma follows from Lemma 1.2, and is analogous to Steenrod's formula

$$a_1 \cup_0 a_2 - (-1)^{|a_1|\,|a_2|}a_2 \cup_0 a_1 = \delta(a_1 \cup_1 a_2) - (\delta a_1) \cup_1 a_2 - (-1)^{|a_1|}a_1 \cup_1 (\delta a_2).$$

In particular, it implies that the cup product is graded commutative on the Hochschild cohomology $H^\bullet(A,A)$.

Lemma 1.6.

$$(\delta D_1) \circ D_2 - \delta(D_1 \circ D_2) + (-1)^{|D_1|}D_1 \circ (\delta D_2) = m\{D_1,D_2\} + (-1)^{|D_1|\,|D_2|}m\{D_2,D_1\}$$

Let A be an A_∞-algebra, with A_∞-structure $m \in C^\bullet(A,A)$. Define a Hochschild cochain $M \in C^\bullet(C^\bullet(A,A),C^\bullet(A,A))$ by

$$M[D_1|\dots|D_k] = \begin{cases} 0, & k=0, \\ m \circ D_1 - (-1)^{|D_1|}D_1 \circ m, & k=1, \\ m\{D_1,\dots,D_k\}, & k>1. \end{cases}$$

Proposition 1.7. *The cochain M is an A_∞-structure on $C^\bullet(A,A)$.*

Proof. By the definition of $M \circ M$, we see that, if $k > 1$,

$$\begin{aligned}(M \circ M)[D_1|\dots|D_k] &= \sum_{0\leq i\leq j\leq k}(-1)^{\sum_{\ell=1}^{i}|D_\ell|}m\{D_1,\dots,D_i,m\{D_{i+1},\dots,D_j\},D_{j+1},\dots,D_k\} \\ &\quad - \sum_{i=1}^{k}(-1)^{\sum_{\ell=1}^{i}|D_\ell|}m\{D_1,\dots,D_i \circ m,D_{i+1},\dots,D_k\} \\ &\quad + (-1)^{\sum_{i=1}^{k}|D_i|}m\{D_1,\dots,D_k\} \circ m.\end{aligned}$$

The last two terms add up to

$$\sum_{i=1}^{k}(-1)^{\sum_{\ell=1}^{i}|D_\ell|}m\{D_1,\dots,D_i,m,D_{i+1},\dots,D_k\}.$$

Combining this with the first term, we obtain $(m\circ m)\{D_1,\dots,D_k\}$, which clearly vanishes. To complete the proof, we must check that $(M\circ M)[D_1]$ vanishes:

$$(M\circ M)[D_1]=[m,[m,D_1]]=[m\circ m,D_1]=0. \quad \square$$

When A is a differential graded algebra, the A_∞-algebra $C^\bullet(A,A)$ is a differential graded algebra; our construction generalizes that of Gerstenhaber [5].

The operator $D\mapsto M[D]$ is a differential on $C^\bullet(A,A)$ which is usually denoted $D\mapsto \delta D$, and its cohomology is the Hochschild cohomology $H^\bullet(A,A)$ of the A_∞-algebra A.

If A and B are A_∞-algebras, a morphism between them is a map of differential graded coalgebras $f:B(A)\to B(B)$, such that $f[\,]=[\,]$.

Proposition 1.8. *Let A and B be A_∞-algebras with A_∞-structures $m\in C^\bullet(A,A)$, and $n\in C^\bullet(B,B)$. A* **twisting cochain** *on A with values in B is a cochain $\rho\in C^\bullet(A,B)$ of degree zero such that $\rho[\,]=0$ and*

$$n_1(\rho)+n_2(\rho,\rho)+n_3(\rho,\rho,\rho)+\dots=\rho\circ m.$$

There is a correspondence between A_∞-morphisms and twisting cochains, given by the formula

$$f(\rho)[a_1|\dots|a_k]=\sum_{\ell=1}^{k}\sum_{0\le j_1\le\dots\le j_{\ell-1}\le k}[\rho[a_1|\dots|a_{j_1}]|\dots|\rho[a_{j_{i-1}+1}|\dots|a_{j_i}]|\dots|\rho[a_{j_{\ell-1}+1}|\dots|a_k]].$$

If B is a differential graded algebra, the formula for a twisting cochain becomes $\delta\rho+\rho\cup\rho=0$, where

$$(f_1\cup f_2)(a_1,\dots,a_k)=\sum_{i=1}^{k}(-1)^{\eta_i|f_2|}f_1(a_1,\dots,a_i)f_2(a_{i+1},\dots,a_k),$$
$$(\delta f)(a_1,\dots,a_k)=df(a_1,\dots,a_k)-(-1)^{|f|}(f\circ m)(a_1,\dots,a_k).$$

The Hochschild chain complex $C(A)=\sum_{n=0}^{\infty}C_n(A)$ of a graded vector space A is the graded vector space such that

$$C(A)=\begin{cases}A, & n=0,\\ A^+\otimes(sA)^{(n)}, & n>0.\end{cases}$$

The element $a_0\otimes\dots\otimes a_n$ of $C_n(A)$ will be denoted $(a_0,\dots,a_n)$; it has degree $|a_0|+\dots+|a_n|+n$. For such an element, let $\eta_j=|a_0|+\dots+|a_j|-j$. In the rest of this section, we will construct a twisting cochain of the A_∞-algebra $C^\bullet(A,A)$ with values in $\mathrm{End}(C(A))$.

Given $D_1,\dots,D_k\in C^\bullet(A,A)$, define an operator $\mathbf{b}\{D_1,\dots,D_k\}$ on $C(A)$ by the formula

$$\begin{aligned}\mathbf{b}\{D_1,\dots,D_k\}&(a_0,\dots,a_n)\\ &=\sum_{\ell=k+1}^{\infty}\sum_{(j_0,\dots,j_k)\in J(\ell)}\varepsilon(j_0,\dots,j_k)\\ &\quad(m_\ell(a_{j_0+1},\dots,D_1[\dots],\dots,D_k[\dots],\dots,a_n,a_0,\dots),\dots,a_{j_0})\\ &\quad+\begin{cases}\displaystyle\sum_{\ell=0}^{\infty}\sum_{j=0}^{n-\ell}(-1)^{\eta_j-1}(a_0,\dots,m_\ell(a_{j+1},\dots,a_{j+\ell}),\dots,a_n), & k=0,\\ 0, & k>0,\end{cases}\end{aligned}$$

where $\varepsilon(j_0,\dots,j_k) = (-1)^{\eta_n(\eta_n-\eta_{j_0})+\sum_{i=1}^k |D_i|(\eta_{j_i}-\eta_{j_0})}$, we write $D_i[\dots]$ as an abbreviation for $D_i[a_{j_i+1}|\dots|a_{j_i+d(D_i)}]$, and

$$J(\ell) = \Big\{(j_0,\dots,j_k) \mid n-(\ell-1)-\sum_{i=1}^k |D_i| \le j_0 \le j_1, \\ j_i + d(D_i) \le j_{i+1} \text{ for } 1 \le i \le k-1, j_k \le n-d(D_k)\Big\}.$$

For $k=0$, $\mathbf{b}\{\}$ is the Hochschild boundary b on $C(A)$. For $k=1$, we obtain an operator $\mathbf{b}\{D\}$, which in the special case where A is a differential graded algebra may be written

$$\mathbf{b}\{D\}(a_0,\dots,a_n) \\ = (-1)^{(\eta_n-1)(\eta_n-\eta_{n-d(D)})+|D|+1}(D[a_{n-d(D)+1}|\dots|a_n]a_0,\dots,a_{n-d(D)})$$

This operator, with $D \in C^1(A,A)$, is considered by Rinehart, where it is denoted by $e(D)$.

Theorem 1.9. *$\mathbf{b}$ is a twisting cochain of $C^\bullet(A,A)$ with values in* $\mathrm{End}(C(A))$*.*

Proof. Apply the cochains $\delta\mathbf{b}$ and $\mathbf{b}\cup\mathbf{b}$ to $\{D_1,\dots,D_k\}$. If $k=0$, the term $\delta\mathbf{b}\{\}$ vanishes, and we must show that $\mathbf{b}\{\}\mathbf{b}\{\}=0$. This is the standard fact that the Hochschild boundary $b=\mathbf{b}\{\}$ is a differential on $C(A)$, and follows from the formula $m\circ m=0$. Thus, take $k\ge 1$. We use a rather abbreviated notation in the proof, but the reader will have little difficulty in reconstituting the full formulas, if so desired.

Observe that $(\delta\mathbf{b})\{D_1,\dots,D_k\} = P+Q$, where

$$P = \sum_{0\le i<j\le k} (-1)^{\sum_{\ell=1}^i |D_\ell|}\mathbf{b}\{D_1,\dots,m\{D_{i+1},\dots,D_j\},\dots,D_k\},$$
$$Q = -\sum_{1\le i\le k} (-1)^{\sum_{\ell=1}^{i-1} |D_\ell|}\mathbf{b}\{D_1,\dots,D_i\circ m,\dots,D_k\}.$$

Using the formula $m\circ m=0$, we see that $\mathbf{b}\{\}\mathbf{b}\{D_1,\dots,D_k\}=R$, where

$$R = \sum_{(j_0,\dots,j_k)}\sum_j \varepsilon(j_0,\dots,j_k)\,(-1)^{\sum_{i=1}^k |D_i|+(\eta_n-\eta_{j_0})+\eta_j} \\ (m(\dots,D_1[\dots],\dots,D_k[\dots],\dots,a_0,\dots),\dots,m(a_{j+1},\dots),\dots).$$

Similarly, we check that $(-1)^{\sum_{\ell=1}^i |D_\ell|}\mathbf{b}\{D_1,\dots,D_i\}\mathbf{b}\{D_{i+1},\dots,D_k\}=S$, where

$$S = \sum_{(j_0,\dots,j_k)}\sum_j \varepsilon(j_0,\dots,j_k)\,(-1)^{\sum_{i=1}^k |D_i|+(\eta_j-\eta_{j_0})} \\ (m(\dots,D_1[\dots],\dots,m(a_{j+1},\dots,D_{i+1}[\dots],\dots,D_k[\dots],\dots,a_0,\dots),\dots),\dots).$$

Finally, we check that $(-1)^{\sum_{i=1}^k |D_i|}\mathbf{b}\{D_1,\dots,D_k\}\mathbf{b}\{\}+Q+R=T+U+V$, where

$$T = \sum_{i=0}^k (-1)^{\sum_{\ell=1}^i |D_\ell|}\mathbf{b}\{D_1,\dots,D_i,m,D_{i+1},\dots,D_k\},$$
$$U = \sum \varepsilon(j_0,\dots,j_k)\,(-1)^{\sum_{i=1}^k |D_i|+\eta_j-\eta_{j_0}} \\ (m(\dots,D_1[\dots],\dots,D_k[\dots],\dots,m(a_{j+1},\dots,a_0,\dots),\dots),\dots),$$
$$V = \sum \varepsilon(j_0,\dots,j_k)\,(-1)^{\sum_{i=1}^k |D_i|+(\eta_n-\eta_{j_0})+(\eta_j-1)} \\ (m(\dots,D_1[\dots],\dots,D_k[\dots],\dots,a_0,\dots,m(a_{j+1},\dots),\dots),\dots).$$

It only remains to observe that

$$P+S+T+U+V=\sum_{\ell=k+1}^{\infty}\sum_{(j_0,\dots,j_k)\in J(\ell)}\varepsilon(j_0,\dots,j_k)$$
$$((m\circ m)_\ell(a_{j_0+1},\dots,D_1[\dots],\dots,D_k[\dots],\dots,a_0,\dots),\dots,a_{j_0}),$$

which vanishes, because $m\circ m=0$. □

2. The Cartan homotopy formula

Let t be the operator on $C(A)$ defined by the formulas

$$t(a_0,\dots,a_n)=(-1)^{\eta_n(|a_n|-1)}(a_n,a_0,\dots,a_{n-1}),$$
$$t(e,a_1,\dots,a_n)=0.$$

If D is a Hochschild k-cochain on A, define the operator $D:C_n(A)\longrightarrow C_{n-k+1}(A)$ by

$$D(a_0,\dots,a_n)=(D[a_0|\dots|a_{k-1}],a_k,\dots,a_n).$$

Given $D_1,\dots,D_k\in C^\bullet(A,A)$, define an operator $\mathbf{B}\{D_1,\dots,D_k\}$ on $C(A)$ by the formula

$$\mathbf{B}\{D_1,\dots,D_k\}=\sum_{(j_0,j_1,\dots,j_k)\in J}\sigma\cdot t^{j_1-j_0}\cdot D_1\cdot t^{j_2-j_1}\cdot D_2\cdot\dots\cdot t^{j_k-j_{k-1}}\cdot D_k\cdot t^{-j_k-1},$$

where $\sigma(a_0,\dots,a_n)=(e,a_0,\dots,a_n)$, and

$$J=\{(j_0,\dots,j_k)\mid 0\le j_0,j_i+d(D_i)\le j_{i+1}\text{ for }1\le i\le k,j_k\le n-d(D_k)\}.$$

More explicitly, we may write

$$\mathbf{B}\{D_1,\dots,D_k\}(a_0,\dots,a_n)=\sum_{(j_0,\dots,j_k)\in J}(-1)^{\eta_n(\eta_{j_0}-1)+\sum_{i=1}^k|D_i|(\eta_{j_i}-\eta_{j_0})}$$
$$(e,a_{j_0+1},\dots,D_1[\dots],\dots,D_k[\dots],\dots,a_0,\dots,a_{j_0}),$$

where we write $D_i[\dots]$ as an abbreviation for $D_i[a_{j_i+1}|\dots|a_{j_i+d(D_i)}]$. For $k=0$, the operator $\mathbf{B}\{\}$ is Connes's differential B. For $k=1$, we obtain an operator $\mathbf{B}\{D\}$. This operator, with $D\in C^1(A,A)$, is considered by Rinehart, where it is denoted by $E(D)$.

Let $C(A)[\![u]\!]$ be the space of power series in a variable u of degree -2. Consider $\mathbf{b}$ and $\mathbf{B}$ to be cochains on $C^\bullet(A,A)$ with values in the algebra $\operatorname{End}(C(A))[\![u]\!]$, extending it linearly over $k[\![u]\!]$.

Definition 2.1.
(1) *Let* $\iota\in C^-(C^\bullet(A,A),\operatorname{End}(C(A))[\![u]\!])$ *equal* $\mathbf{b}-u\mathbf{B}$.
(2) *Let* $\mathcal{L}\in C^+(C^\bullet(A,A),\operatorname{End}(C(A))[\![u]\!])$ *be the* **curvature** *of* ι*, defined by the formula*

$$\delta\iota+\iota\cup\iota=u\mathcal{L}.$$

Since $\mathcal{L}$ is the curvature of ι, it satisfies the **Bianchi identity**

$$\delta\mathcal{L}+\iota\cup\mathcal{L}-\mathcal{L}\cup\iota=0.$$

For example, this shows that $[b-uB,\mathcal{L}\{D\}]=\mathcal{L}\{\delta D\}$.

From the definition of the curvature $\mathcal{L}\{D\}$, we see that

$$[b-uB,\iota\{D\}]=u\mathcal{L}\{D\}-\iota\{\delta D\}. \tag{2.1}$$

This formula is the non-commutative analogue of the Cartan homotopy formula in differential geometry: if X is a vector field on a smooth manifold, $[d,\iota(X)]=\mathcal{L}(X)$.

The main results of this paper is the calculation of $\mathcal{L}$.

Theorem 2.2. *For $k=0$, $\mathcal{L}\{\}=0$ (that is, $(b-uB)^2=0$). For $k\geq 1$, $\mathcal{L}\{D_1,\dots,D_k\}$ is given by the formula*

$$\begin{aligned}\mathcal{L}\{D_1,\dots,D_k\}&(a_0,\dots,a_n)\\ &=\sum_{(j_1,\dots,j_k)\in J}(-1)^{\sum_{i=1}^k|D_i|(\eta_{j_i}-1)}(a_0,\dots,D_1[\dots],\dots,D_k[\dots],\dots,a_n)\\ &+\sum_{i=1}^k\sum_{(j_1,\dots,j_k)\in J(i)}(-1)^{\eta_n(\eta_n-\eta_{j_1}+\sum_{\ell=2}^i|D_\ell|)+\sum_{\ell=2}^k|D_\ell|(\eta_{j_\ell}-\eta_{j_1}+\sum_{\ell=2}^i|D_\ell|)}\\ &\qquad(D_1[a_{j_1+1}|\dots|D_{i+1}[\dots]|\dots|D_k[\dots]|\dots|a_0|\dots],\dots\\ &\qquad\qquad\dots,D_2[\dots],\dots,D_i[\dots],\dots,a_{j_1}),\end{aligned}$$

where

$$\begin{aligned}J&=\{0\leq j_1\ ,\ j_\ell+d(D_\ell)\leq j_{\ell+1}\ ,\ j_k+d(D_k)\leq n\},\\ J(i)&=\{0\leq j_2\ ,\ j_\ell+d(D_\ell)\leq j_{\ell+1}\ ,\ j_k+d(D_k)\leq n,\ j_i+d(D_i)\leq j_1\leq j_{i+1}\}.\end{aligned}$$

Proof. We first calculate that

$$\begin{aligned}\mathbf{b}\{\}\mathbf{B}\{D_1,\dots,D_k\}+\sum_{i=1}^k(-1)^{\sum_{\ell=1}^i|D_\ell|}\mathbf{B}\{D_1,\dots,D_i\}\mathbf{b}\{D_{i+1},\dots,D_k\}\\ =P+Q+R+S,\end{aligned}$$

where

$$\begin{aligned}P&=-\sum_{0\leq i\leq j\leq k}(-1)^{\sum_{\ell=1}^i|D_\ell|}\mathbf{B}\{D_1,\dots,m\{D_{i+1},\dots,D_j\},\dots,D_k\},\\ Q&=\sum_{i=1}^k(-1)^{\sum_{\ell=1}^i|D_\ell|}\mathbf{B}\{D_1,\dots,D_i\circ m,\dots,D_k\},\\ R&=-\sum_{(j_1,\dots,j_k)\in J}(-1)^{\sum_{i=1}^k|D_i|(\eta_{j_i}-1)}(a_0,\dots,D_1[\dots],\dots,D_k[\dots],a_n),\\ S&=\sum_{\{(j_0,\dots,j_k)\in J|j_0=j_1\}}(-1)^{\eta_n(\eta_n-\eta_{j_0})+\sum_{\ell=2}^k|D_\ell|(\eta_{j_\ell}-\eta_{j_0})}\\ &\qquad\qquad(D_1[\dots],\dots,D_k[\dots],\dots,a_0,\dots).\end{aligned}$$

Next, we check that $(\delta\mathbf{B})\{D_1,\dots,D_k\}+P+Q=0$, and that

$$(-1)^{|D_1|}\mathbf{b}\{D_1\}\mathbf{B}\{D_2,\dots,D_k\}+R+S=-\mathcal{L}\{D_1,\dots,D_k\}.$$

The proof is completed by observing that $\mathbf{b}\{D_1,\dots,D_i\}\mathbf{B}\{D_{i+1},\dots,D_k\}=0$ for $i>1$, and $\mathbf{B}\{D_1,\dots,D_i\}\mathbf{B}\{D_{i+1},\dots,D_k\}=0$. □

We will need an explicit formula for $\mathcal{L}\{D\}$:

$$\begin{aligned}\mathcal{L}\{D\}(a_0,\dots,a_n)=&\sum_{j=0}^{n-d(D)}(-1)^{|D|(\eta_j-1)}(a_0,\dots,D[a_{j+1}|\dots|a_{j+d(D)}],\dots,a_n)\\ &+\sum_{j=n-d(D)}^{n}(-1)^{\eta_n(\eta_n-\eta_j)}(D[a_{j+1}|\dots|a_0|\dots|a_{j+d(D)-n-1}],\dots,a_j).\end{aligned}$$

The Hochschild boundary $b = \mathbf{b}\{\}$ on $C(A)$ is equal to $\mathcal{L}(m)$, where $m \in C^\bullet(A,A)$ defines the A_∞-structure on A.

Given cochains D_1 and D_2 on A, define $\rho\{D_1, D_2\}$ by the formula

$$\rho\{D_1,D_2\}(a_0,\dots,a_n) = \sum_{j_1\le j_2} (-1)^{\eta_n(\eta_n-\eta_{j_1})+|D_2|(\eta_{j_2}-\eta_{j_1})} (D_1[a_{j_1+1}|\dots|D_2[a_{j_2+1}|\dots]|\dots|a_0|\dots],\dots,a_{j_1}).$$

Note that $\rho\{m, D\} = \mathbf{b}\{D\}$.

Lemma 2.3.

$$\mathcal{L}\{D_1,D_2\} + (-1)^{|D_1|\,|D_2|}\mathcal{L}\{D_2,D_1\} + \mathcal{L}\{D_1\circ D_2\} = \mathcal{L}\{D_1\}\mathcal{L}\{D_2\} + \rho\{D_1,D_2\} + (-1)^{|D_1|\,|D_2|}\rho\{D_2,D_1\}$$

Proof. It is easily checked that

$$\mathcal{L}\{D_1\}\mathcal{L}\{D_2\} = \mathcal{L}\{D_1\circ D_2\} + P_1 + (-1)^{|D_1|\,|D_2|}P_2 + Q_1 + (-1)^{|D_1|\,|D_2|}Q_2,$$

where

$$\begin{aligned}
P_1 &= \sum (-1)^{|D_1|(\eta_n-\eta_{j_1})+|D_2|(\eta_n-\eta_{j_2})}(a_0,\dots,D_1[\dots],\dots,D_2[\dots],\dots,a_n),\\
P_2 &= \sum (-1)^{|D_1|(\eta_n-\eta_{j_1})+|D_2|(\eta_n-\eta_{j_2})}(a_0,\dots,D_2[\dots],\dots,D_1[\dots],\dots,a_n),\\
Q_1 &= \sum (-1)^{\eta_n(\eta_n-\eta_{j_1})+|D_2|(\eta_{j_2}-\eta_{j_1})}(D_1[\dots|a_0|\dots],\dots,D_2[\dots],\dots),\\
Q_2 &= \sum (-1)^{\eta_n(\eta_n-\eta_{j_2})+|D_2|(\eta_{j_1}-\eta_{j_2})}(D_2[\dots|a_0|\dots],\dots,D_1[\dots],\dots).
\end{aligned}$$

Here, we abbreviate $D_i[a_{j_i+1}|\dots|a_{j_i+d(D_i)}]$ to $D_i[\dots]$, and in the definitions of Q_i, the sum is taken over $j_i > n - d(D_i)$. Since $\mathcal{L}\{D_1,D_2\} = P_1 + Q_1 + \rho\{D_1,D_2\}$ and $(-1)^{|D_1|\,|D_2|}\mathcal{L}\{D_2,D_1\} = P_2 + Q_2 + \rho\{D_2,D_1\}$, the proof of the lemma is completed. □

It follows from this lemma that

$$[\mathcal{L}(D_1),\mathcal{L}(D_2)] = \mathcal{L}([D_1,D_2])$$

which gives another proof that $b^2 = 0$.

If W is a graded module over the ring $\mathbf{k}[u]$, the cyclic homology $\mathrm{HC}_\bullet(A;W)$ with coefficients in W is the homology of the complex

$$(C(A)\otimes W, b - uB)\,.$$

Let us list some examples with different coefficients W.

(1) $W = \mathbf{k}[u]$ gives the negative cyclic homology $\mathrm{HC}^-_\bullet(A)$;
(2) $W = \mathbf{k}[u,u^{-1}]$ gives the periodic cyclic homology $\mathrm{HP}_\bullet(A)$;
(3) $W = \mathbf{k}[u,u^{-1}]/u\mathbf{k}[u]$ gives the positive cyclic homology $\mathrm{HC}_\bullet(A)$;
(4) $W = \mathbf{k}[u]/u\mathbf{k}[u]$ gives the Hochschild homology $\mathrm{HH}_\bullet(A)$.

The following theorem is a consequence of the results of Sections 1 and 2.

Theorem 2.4.

(1) *The graded Lie algebra $H^\bullet(A,A)$ acts on $\mathrm{HC}_\bullet(A;W)$ by the Lie derivative $D\mapsto\mathcal{L}(D)$, for any coefficients W.*
(2) *If $D \in Z^\bullet(A,A)$ is a cocycle, then $u\mathcal{L}(D)$ is chain homotopic to zero on the cyclic bar complex $C(A)[\![u]\!]$. In particular, $\mathcal{L}(D)$ acts as zero on the periodic cyclic homology $\mathrm{HP}_\bullet(A)$.*

3. The Gauss-Manin connection

Let A_ν be an n-parameter formal deformation of the A_∞-algebra A; in other words, the A_∞-structure on A_ν is defined by a cochain $m_\nu \in C^\bullet(A, A^+)[\![\nu_1, \dots, \nu_n]\!]$ such that $m|_{\nu=0}$ is the product on A and $m_\nu \circ m_\nu = 0$. The cohomology of the periodic cyclic bar complex $C(A)[\![\nu_1, \dots, \nu_n]\!]((u))$ with differential $b_\nu - uB$ is the periodic cyclic homology $\mathrm{HP}_\bullet(A)$, which is a module over $\mathbf{k}[\![\nu_1, \dots, \nu_n]\!]((u))$. Let

$$\mathcal{A}_i = \frac{\partial m_\nu}{\partial \nu_i} \in C^\bullet(A, A)[\![\nu_1, \dots, \nu_n]\!].$$

Proposition 3.1. *The* **Gauss-Manin connection**

$$\nabla = d + u^{-1} \sum_{i=1}^{n} \iota_\nu\{\mathcal{A}_i\}\, d\nu_i$$

commutes with $b_\nu - uB$, and thus induces a connection on the module $\mathrm{HP}_\bullet(A_\nu)$.

Proof. Taking a partial derivative of the formula $m_\nu \circ m_\nu = 0$ with respect to ν_i, we see that $[m_\nu, \mathcal{A}_i] = 0$. Observe that

$$\frac{\partial(b_\nu - uB)}{\partial \nu_i} = \mathcal{L}\{\mathcal{A}_i\},$$

since $b_\nu = \mathcal{L}\{m_\nu\}$. Thus, it follows from (2.1) that

$$\begin{aligned}[\nabla, b_\nu - uB] &= \sum_{i=1}^{n} \left(\frac{\partial(b_\nu - uB)}{\partial \nu_i} - [\iota_\nu\{\mathcal{A}_i\}, b_\nu - uB] \right) d\nu_i \\ &= \sum_{i=1}^{n} (\mathcal{L}\{\mathcal{A}_i\} - \mathcal{L}\{\mathcal{A}_i\})\, d\nu_i = 0. \quad \square\end{aligned}$$

As an example, suppose we have a one-parameter family of associative products $a_1 *_\nu a_2$ on the ungraded vector space A. Then $\iota_\nu\{\mathcal{A}_\nu\}$ is given by the formula

$$\begin{aligned}\iota_\nu\{\mathcal{A}_\nu\}(a_0, \dots, a_n) &= (\mathcal{A}_\nu(a_{n-1}, a_n) *_\nu a_0, a_1, \dots, a_{n-2}) \\ &\quad - \sum_{1 \le i \le j \le n-1} (-1)^{ni+(j-i)} (e, a_i, \dots, \mathcal{A}_\nu(a_j, a_{j+1}), \dots, a_0, \dots, a_{i-1}).\end{aligned}$$

In the remainder of this section, we give an expression for the curvature of the Gauss-Manin connection ∇. We show that it has the form $[b_\nu - uB, P]$ for a certain operator P, and hence that it induces a flat connection on the periodic cyclic homology.

Let $\sigma\{D_1, D_2\}$ be the operator on $C(A)$ be defined by the formula

$$\sigma\{D_1, D_2\} = \iota\{D_1, D_2\} + (-1)^{|D_1|\,|D_2|} \iota\{D_2, D_1\} - \iota\{D_1 \circ D_2\}.$$

The following lemma will enable us to calculate the curvature of ∇.

Lemma 3.2.

$$\begin{aligned}[b - uB, \sigma\{D_1, D_2\}] + \sigma\{\delta D_1, D_2\} + (-1)^{|D_1|} \sigma\{D_1, \delta D_2\} + (-1)^{|D_1|} [\iota\{D_1\}, \iota\{D_2\}] \\ = u(\mathcal{L}\{D_1\}\mathcal{L}\{D_2\} + \rho\{D_1, D_2\} + (-1)^{|D_1|\,|D_2|} \rho\{D_2, D_1\})\end{aligned}$$

Proof. The definition of $\mathcal{L}$ in terms of ι shows that

$$[b-uB,\iota\{D_1,D_2\}]+\iota\{\delta D_1,D_2\}+(-1)^{|D_1|}\iota\{D_1,\delta D_2\} \\ +(-1)^{|D_1|}\iota\{D_1\}\,\iota\{D_2\}+\iota\{m\{D_1,D_2\}\}=u\mathcal{L}\{D_1,D_2\}.$$

If we let $\sigma_0\{D_1,D_2\}=\iota\{D_1,D_2\}+(-1)^{|D_1|\,|D_2|}\iota\{D_2,D_1\}$, we see that

$$[b-uB,\sigma_0\{D_1,D_2\}]+\sigma_0\{\delta D_1,D_2\}+(-1)^{|D_1|}\sigma_0\{D_1,\delta D_2\} \\ +(-1)^{|D_1|}[\iota\{D_1\},\iota\{D_2\}]+\iota\big\{m\{D_1,D_2\}+(-1)^{|D_1|\,|D_2|}m\{D_2,D_1\}\big\} \\ =u\big(\mathcal{L}\{D_1,D_2\}+(-1)^{|D_1|\,|D_2|}\mathcal{L}\{D_2,D_1\}\big).$$

On the other hand, by Lemma 1.6, we have

$$[b-uB,\iota\{D_1\circ D_2\}]+\iota\{\delta D_1\circ D_2\}+(-1)^{|D_1|}\iota\{D_1\circ\delta D_2) \\ =\iota\big\{m\{D_1,D_2\}+(-1)^{|D_1|\,|D_2|}m\{D_2,D_1\}\big\}+u\mathcal{L}(D_1\circ D_2).$$

Combining these two equations with Lemma 2.4 proves the lemma. □

It is now easy to prove the following theorem.

Theorem 3.3. *The curvature ∇^2 of the Gauss-Manin connection is given by the formula*

$$\nabla^2=u^{-2}\sum_{1\le i\le j\le n}\big([b_\nu-uB,\sigma_\nu\{\mathcal{A}_i,\mathcal{A}_j\}]-u\mathcal{L}\{\mathcal{A}_i\}\mathcal{L}\{\mathcal{A}_j\}\big)\,d\nu_i\wedge d\nu_j \\ =u^{-2}\sum_{1\le i\le j\le n}[b_\nu-uB,\sigma_\nu\{\mathcal{A}_i,\mathcal{A}_j\}+u\iota\{\mathcal{A}_i\}\mathcal{L}\{\mathcal{A}_j\}]\,d\nu_i\wedge d\nu_j,$$

and hence is chain homotopic to zero.

Proof. Observe that

$$\frac{\partial\iota_\nu\{\mathcal{A}_j\}}{\partial\nu_i}=\iota\left\{\frac{\partial^2m_\nu}{\partial\nu_i\partial\nu_j}\right\}+\rho\{\mathcal{A}_i,\mathcal{A}_j\}.$$

The formula for the curvature of ∇ is seen as follows:

$$\Big[\frac{\partial}{\partial\nu_i}+\iota_\nu\{\mathcal{A}_i\},\frac{\partial}{\partial\nu_j}+\iota_\nu\{\mathcal{A}_j\}\Big] \\ =\iota\Big(\frac{\partial^2m_\nu}{\partial\nu_j\partial\nu_i}\Big)-\iota\Big(\frac{\partial^2m_\nu}{\partial\nu_i\partial\nu_j}\Big)+\rho\{\mathcal{A}_i,\mathcal{A}_j\}-\rho\{\mathcal{A}_j,\mathcal{A}_i\}+[\iota_\nu(\mathcal{A}_i),\iota_\nu(\mathcal{A}_j)].$$

The first two terms cancel, and the remaining terms are shown by Lemma 3.2 to equal

$$[b_\nu-uB,\sigma_\nu\{\mathcal{A}_i,\mathcal{A}_j\}]-u\mathcal{L}\{\mathcal{A}_i\}\mathcal{L}\{\mathcal{A}_j\}.\quad\square$$

4. The Gauss-Manin connection and iterated integrals

In this section, we will illustrate the Gauss-Manin connection of the last section in a simple example. Let A be a differential graded algebra. There are three commuting differentials on the cyclic bar complex $C(A)$, which we denote

$$\begin{aligned} d(a_0,\dots,a_k) &= \sum_{i=0}^{k}(-1)^{\eta_{i-1}+1}(a_0,\dots,da_i,\dots,a_k), \\ b(a_0,\dots,a_k) &= \sum_{i=0}^{k-1}(-1)^{\eta_i}(a_0,\dots,a_ia_{i+1},\dots,a_k) \\ &\quad + (-1)^{\eta_k(|a_k|+1)}(a_ka_0,a_1,\dots,a_{k-1}), \\ B(a_0,\dots,a_k) &= \sum_{i=0}^{k}(-1)^{\eta_k(\eta_{i-1}+1)}(e,a_i,\dots,a_k,a_0,\dots,a_{i-1}); \end{aligned}$$

as usual, $\eta_i = |a_0| + \cdots + |a_i| - i$. The total differential on $C(A)$ is $d + b - B$.

If $A_{-i} = \Omega^i(M)$ is the differential graded algebra of differential forms on a smooth manifold M, there is a map of complexes

$$\begin{array}{ccc} C(\Omega(M)) & \xrightarrow{\sigma} & \Omega(LM) \\ {\scriptstyle d+b-B}\downarrow & & \downarrow{\scriptstyle d-\iota(T)} \\ C(\Omega(M)) & \xrightarrow{\sigma} & \Omega(LM) \end{array}$$

called the **iterated integral** (see Chen [2] and Getzler-Jones-Petrack [7]). If Δ^k is the k-simplex $0 \le t_0 \le \cdots \le t_k \le 1$, and $a(t)$ is the pull-back of the differential form $a \in \Omega(M)$ by the evaluation map $\gamma \mapsto \gamma(t)$, then σ is defined on $C(\Omega(M))$ by the formula

$$\sigma(a_0,\dots,a_k) = (-1)^k \int_{\Delta^k} a_0(0)\,\iota(T)a_1(t_1)\dots\iota(T)a_k(t_k)\,dt.$$

The cyclic bar complex algebra $(C(C^\infty(M)), b - B)$ of the algebra of smooth functions $C^\infty(M)$ on M maps to the complex of differential forms $(\Omega(M), d)$ by the map

$$(f_0,\dots,f_k) \overset{\alpha}{\longmapsto} \frac{(-1)^k}{k!} f_0\,df_1\dots df_k.$$

Now consider the operator $\iota\{d\} = \mathbf{b}\{d\} - \mathbf{B}\{d\} : C(\Omega(M)) \to C(\Omega(M))$ of Section 2; the operators $\mathbf{b}\{d\}$ and $\mathbf{B}\{d\}$ are given by the formulas

$$\begin{aligned} \mathbf{b}\{d\}(a_0,\dots,a_k) &= (-1)^{(\eta_k-1)(|a_k|+1)}(da_k\,a_0,a_1,\dots,a_{k-1}) \\ \mathbf{B}\{d\}(a_0,\dots,a_k) &= \sum_{1\le i\le j\le k}(-1)^{\eta_k(\eta_{i-1}-1)+(\eta_{j-1}-\eta_{i-1})} \\ &\qquad (e,a_i,\dots,da_j,\dots,a_0,\dots,a_{i-1}). \end{aligned}$$

We will be interested in the operator $e^{-\iota\{d\}}$, which may be rewritten as a Volterra series

$$e^{-\iota\{d\}} = \sum_{k=0}^{\infty} \int_{\Delta^k} e^{-t_1\mathbf{b}\{d\}}\mathbf{B}\{d\}e^{-(t_2-t_1)\mathbf{b}\{d\}}\dots e^{-(t_k-t_{k-1})\mathbf{b}\{d\}}\mathbf{B}\{d\}e^{-(1-t_k)\mathbf{b}\{d\}}\,dt,$$

using the formula $\mathbf{B}\{d\}^2 = 0$.

Proposition 4.1. *Let $i^* : \Omega(LM) \to \Omega(M)$ denote the restriction of differential forms under the inclusion $M \subset LM$. We have a commuting diagram of complexes*

$$\begin{array}{ccc} \big(C(C^\infty(M)), b - B\big) & \xrightarrow{\alpha} & \big(\Omega(M), d\big) \\ {\scriptstyle e^{-\iota\{d\}}}\big\downarrow & & \big\uparrow{\scriptstyle i^*} \\ \big(C(\Omega(M)), d + b - B\big) & \xrightarrow{\sigma} & \big(\Omega(LM), d - \iota(T)\big) \end{array}$$

Proof. In Section 3, we proved that

$$(d + b - B) \cdot e^{-\iota\{d\}} = e^{-\iota\{d\}} \cdot (b - B).$$

Thus, it only remains to show that if $f_i \in C^\infty(M)$, then

$$i^*\sigma \cdot e^{-\iota\{d\}}(f_0, \dots, f_k) = \frac{(-1)^k}{k!} f_0\, df_1 \dots df_k.$$

The key observation is that $i^*\sigma(a_0, \dots, a_k) = 0$ if $k > 0$. Thus, only the term proportional to $\mathbf{b}\{d\}^k$ contributes, and we see that

$$\begin{aligned} i^*\sigma \cdot e^{-\iota\{d\}}(f_0, \dots, f_k) &= \frac{1}{k!} i^*\sigma \cdot \mathbf{b}\{d\}^k (f_0, \dots, f_k) \\ &= \frac{1}{k!} i^*\sigma(df_1 \dots df_k\, f_0) \\ &= \frac{(-1)^k}{k!} f_0\, df_1 \dots df_k \quad \square \end{aligned}$$

References

1. J.L. Block, *Cyclic homology of filtered algebras*, K-Theory **1** (1987), 515–518.
2. K.T. Chen, *Iterated integrals of differential forms and loop space homology*, Ann. Math. **97** (1973), 217–246.
3. A. Connes, *Non-commutative differential geometry*, Publ. Math. IHES **62** (1985), 41–144.
4. I.M. Gel'fand, Yu.L. Daletskiĭ, and B.L. Tsygan, *On a variant of noncommutative geometry*, Soviet Math. Dokl. **40** (1990), 422-426.
5. M. Gerstenhaber, *The cohomology structure of an associative ring*, Ann. Math. **78** (1963), 59–103.
6. E. Getzler and J.D.S. Jones, *A_∞-algebras and the cyclic bar complex*, Ill. Jour. Math. **34** (1989), 256–283.
7. E. Getzler, J.D.S. Jones and S. Petrack, *Differential forms on loop spaces and the cyclic bar complex*, Topology **30** (1991), 339–371.
8. T. Goodwillie, *Cyclic homology, derivations and the free loopspace*, Topology **24** (1985), 187–215.
9. V. Nistor, *An bivariant Chern character for p-summable quasihomomorphisms*, K-theory **5** (1991), 193–211.
10. G. Rinehart, *Differential forms on general commutative algebras*, Trans. Amer. Math. Soc. **108** (1963), 195–222.
11. J.D. Stasheff, *Homotopy associativity of H-spaces, II*, Trans. Amer. Math. Soc. **108** (1963), 293–312.