Homework 4, due 2/27
Only your **four** best solutions will count towards your grade.

1. Define the form
$$\omega = \frac{\sqrt{-1}}{2}\sum_{i=1}^{n} dz_i \wedge d\bar{z}_i$$
on $\mathbf{C}^n$, i.e. the associated (1,1)-form of the Euclidean metric. Find a function $\phi : \mathbf{C}^n \to \mathbf{R}$ satisfying $\omega = \sqrt{-1}\partial\overline{\partial}\phi$.

2. Show that if $u : \mathbf{C}^n \to \mathbf{R}$ satisfies $\partial\bar{\partial}u = 0$, then u is the real part of a holomorphic function.

3. Define the form
$$\omega = \sqrt{-1}\partial\overline{\partial}\log(1 + \|z\|^2)$$
on $\mathbf{C}^n$. Show that ω is the associated (1,1)-form of a metric g, i.e. that the corresponding symmetric tensor g defines a positive definite inner product. *Hint: by symmetry it is enough to check this along a coordinate axis.*

4. Show that $\mathbf{C}^n$ does not have any compact complex submanifolds of positive dimension.

5. Let $f : U \to \mathbf{C}$ be a smooth function, where $U \subset \mathbf{C}^n$ is an open set. Let $\Gamma_f \subset \mathbf{C}^{n+1}$ denote the graph of f. Show that f is holomorphic if and only if at each point $p \in \Gamma_f$ the tangent space $T_p\Gamma_f$ is a complex subspace of $\mathbf{C}^{n+1}$.

6. Let X be a complex manifold, and $Y \subset X$ a smooth (real) submanifold. Show that X is a complex submanifold if and only if each tangent space $T_pY \subset T_pX$ for $p \in Y$ is invariant under the complex structure map $I : T_pX \to T_pX$.