Homework 9, due 12/5
Only your **four** best solutions will count towards your grade.

1. Suppose that α is a (1,0)-form on a compact Riemann surface X.
   (a) If in a local holomorphic chart $\alpha = \alpha_z dz$, define $\overline{\alpha} = \overline{\alpha_z} d\bar{z}$. Show that $\overline{\alpha}$ defines a (0,1)-form on X, i.e. check that the coordinate representations of $\overline{\alpha}$ satisfy the right compatibility condition.
   (b) Show that
   $$\int_X \frac{i}{2}\alpha \wedge \overline{\alpha} \geq 0,$$
   with equality only if $\alpha = 0$.
   (c) Suppose that $f : X \to \mathbf{C}$ satisfies $\partial\bar{\partial} f = 0$ (and X is compact). Show that f is constant, by considering the integral of $\partial f \wedge \overline{\partial f}$ and using Stokes' Theorem.

2. Let X be a compact Riemann surface, and for any (1,0)-form $\theta \in \Omega^{1,0}_X$, define the norm $\|\theta\|$ by
   $$\|\theta\|^2 = i \int_X \theta \wedge \overline{\theta}.$$
   From the previous question we know that this is a non-negative real number, which vanishes only if $\theta = 0$. Denote by $[\theta]$ the equivalence class of θ in $\Omega^{1,0}/(\mathrm{im}\,\partial)$.

   Show that if $\alpha \in [\theta]$ has minimal norm among the elements in the class $[\theta]$, then $\overline{\partial}\alpha = 0$, i.e. α is a holomorphic one-form. (Note that this gives another approach to proving the isomorphism $H^{0,1} = \overline{H^{1,0}}$ from class.)

3. Let α be a 2-form supported in a chart U on a Riemann surface. Suppose that z, w are two local coordinates on U, and $\alpha = f(z) dz \wedge d\bar{z}$ and $\alpha = g(w) dw \wedge d\bar{w}$ are the expressions of α in these coordinates. Show that the integral $\int_U \alpha$ defined in class is independent of the coordinate representation chosen for α.

4. (a) Let α be any meromorphic one-form on $\mathbf{P}^1$. Show that
   $$\sum_{p \in \mathbf{P}^1} \mathrm{ord}_p \alpha = -2.$$
   Hint: show that $\alpha = f dz$ for a meromorphic function f.
   (b) Let $p_1, \ldots, p_k \in \mathbf{P}^1$, and $a_1, \ldots, a_k \in \mathbf{Z}$ satisfy $\sum_i a_i = -2$. Can you find a meromorphic one-form α on $\mathbf{P}^1$ such that $\mathrm{ord}_{p_i}\alpha = a_i$ for each i, and $\mathrm{ord}_p \alpha = 0$ for all other p?

5. Consider the one-form $\alpha = \bar{z} dz$ on $\mathbf{C}$.
   (a) Does there exist a function $f : \mathbf{C} \to \mathbf{C}$ such that $\alpha = df$?

(b) Does there exist $f : \mathbf{C} \to \mathbf{C}$ such that $\alpha = \partial f$?

6. In class we showed that $\dim H_X^{1,0} \leq g$, where g is the genus of the compact Riemann surface X. Let

$$H^1(X, \mathbf{R}) = \frac{\ker(d : \Omega^1(X) \to \Omega^2(X))}{d\Omega^0(X)}$$

denote the De Rham cohomology of X. Show that $\dim_{\mathbf{R}} H_X^{1,0} = \dim_{\mathbf{R}} H^1(X, \mathbf{R})$ by showing that the map $H_X^{1,0} \to H^1(X, \mathbf{R})$ given by $\alpha \mapsto \alpha + \bar{\alpha}$ is a (real linear) isomorphism. This can be used to show that $\dim H_X^{1,0} = g$.