Homework 4, due 10/1

Only your **four** best solutions will count towards your grade.

1. Let $\Omega \subset \mathbf{C}$ be connected, and $f_k, f : \Omega \to \mathbf{C}$ holomorphic functions, such that $f_k \to f$ locally uniformly on Ω. Show that if each f_k is injective, then f is either injective, or constant.

2. Let $\Omega \subset \mathbf{C}$ be connected, and $f : \Omega \to \mathbf{C}$ a non-constant holomorphic function. Let $U \Subset \Omega$ have compact closure in Ω, such that $|f|$ is constant on ∂U. Show that f must have a zero in U.

3. Prove that if $\Omega \subset \mathbf{C}$ is open and $f : \Omega \to \mathbf{C}$ is holomorphic, then $f^{-1}(\mathbf{R})$ cannot be a non-empty compact subset of Ω.

4. Let $\omega_1, \omega_2 \in \mathbf{C}$ be linearly independent over $\mathbf{R}$, and let

$$L = \{n_1\omega_1 + n_2\omega_2 \,:\, n_1, n_2 \in \mathbf{Z}\}.$$

   Let f be a meromorphic function on $\mathbf{C}$ that is doubly periodic. I.e. $f(z + l) = f(z)$ for all $l \in L$ and $z \in \mathbf{C}$ where f is holomorphic.

   For $z_0 \in \mathbf{C}$ denote by $P(z_0)$ the parallelogram

$$P(z_0) = \{z_0 + t_1\omega_1 + t_2\omega_2 \,:\, t_1, t_2 \in [0, 1]\}.$$

   If $\partial P(z_0)$ contains no zeros or poles of f, prove that there are the same number of zeros as there are poles in $P(z_0)$, counted with multiplicity.

5. In the same setting as the previous problem, suppose that $\partial P(z_0)$ contains no zeros or poles of f. Let $z_1, \ldots, z_n$ be the zeros in $P(z_0)$, and $w_1, \ldots, w_n$ be the poles in $P(z_0)$, repeated according to multiplicities. Considering the integral of $zf'(z)/f(z)$ prove that

$$\sum_{k=1}^{n} (z_k - w_k) \in L.$$

6. Suppose that f is holomorphic on an open set containing the closed disk $\overline{D} = \overline{D(0, 1)}$. In this problem we give a proof, using the maximum principle, of Cauchy's inequality for f' in the form that there exists a constant $C > 0$, independent of f, such that

$$|f'(0)| \le C \sup\{|f(z)| \,:\, |z| = 1\}. \tag{1}$$

   (a) Show that given a cutoff function $\eta : \overline{D} \to \mathbf{R}$, vanishing on the boundary of the disk (for instance we can take $\eta(z) = 1 - |z|^2$), we have

$$\Delta(\eta^2|f'|^2 + D|f|^2) \ge 0, \tag{2}$$

   where $\Delta u = \partial^2 u/\partial x^2 + \partial^2 u/\partial y^2$ is the Laplacian. Here D is a constant depending on η.

   (b) Using the inequality (1) above, show that the function $\eta^2|f'|^2 + D|f|^2$ achieves its maximum at the boundary of D, and hence deduce the estimate (2).