Homework 2, due 9/9

1. (a) Verify that the series defining the complex exponential function

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

   has radius of convergence ∞.

   (b) Show that $\exp'(z) = \exp(z)$.

2. Suppose that $\gamma : [0,1] \to \mathbf{C}$ is a differentiable curve parametrizing the boundary $\partial\Omega$ of an open set $\Omega \subset \mathbf{C}$ counterclockwise. Show that the area $A(\Omega)$ is given by

$$A(\Omega) = \frac{1}{2i} \int_\gamma \bar{z}\, dz.$$

3. Compute the degree $\mathrm{Deg}(f,0)$ of the function $f(z) = z^n$ with respect to the origin, where the degree is defined by

$$\mathrm{Deg}(f,0) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz,$$

   with γ denoting the unit circle counterclockwise.

4. Let $f : \mathbf{C}\setminus\{0\} \to \mathbf{C}$ be defined by $f(z) = z^{-n}$ for an integer n. Show that there exists a holomorphic F on $\mathbf{C}\setminus\{0\}$ with $F' = f$ if and only if $n \neq 1$.

5. Show that there is a holomorphic function F on the disk $D(1,1)$ such that $F'(z) = 1/z$.

6. Let $\gamma : [0,1] \to \mathbf{C}$ be a piecewise differentiable path such that $\gamma(0) = 2$, and $\gamma^{-1}([0,\infty)) = \{0\}$ (i.e. γ only meets the non-negative real axis once, at $t = 0$). Prove that

$$0 < \left| \mathrm{Im} \left[\int_0^1 \frac{\gamma'(t)}{\gamma(t)}\, dt \right] \right| < 2\pi.$$

7. Let $\Omega \subset \mathbf{C}$ be connected, and $f : \Omega \to \mathbf{C}$ a non-constant holomorphic function. Let $U \Subset \Omega$ be such that $|f|$ is constant on ∂U. Show that f must have a zero in U.

8. Prove that if $\Omega \subset \mathbf{C}$ and $f : \Omega \to \mathbf{C}$ is holomorphic, then $f^{-1}(\mathbf{R})$ cannot be a non-empty compact subset of Ω.