Honors Analysis - Homework 3

1. Suppose that $f : A \to \mathbf{R}$ is integrable. Show that the absolute value $|f| : A \to \mathbf{R}$ is also integrable.

2. Let $f : [0,1] \to \mathbf{R}$ be the function defined by

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q}, \text{ where } p, q \geqslant 0 \text{ are coprime integers} \\ 0 & \text{otherwise.} \end{cases}$$

Show that f is integrable and compute $\int_{[0,1]} f(x)\,d\mu$ with respect to Lebesgue measure.

3. Let $f_n : A \to \mathbf{R}$ be a sequence of measurable functions. Let $B \subset A$ consist of those points $x \in A$ such that the limit $\lim_{n\to\infty} f_n(x)$ exists. Show that B is measurable.

4. Give an example of an additive measure on the algebra of elementary subsets of $[0,1] \times [0,1]$, which is not σ-additive.

5. Suppose that $f : A \to \mathbf{R}$ is integrable with respect to a measure μ, and $f(x) \geqslant 0$ for all $x \in A$. For any $t \in [0, \infty)$ let

$$\Phi(t) = \mu(\{x \in A \,|\, f(x) > t\}).$$

Show that

$$\int_A f\,d\mu = \lim_{K\to\infty} \int_0^K \Phi(t)\,dt,$$

where the integral on the right can be thought of as either the Lebesgue integral, or the Riemann integral (note that Φ is monotonic, hence Riemann integrable).

6. Let μ be the Wiener measure on the space $C_0[0,1]$ of continuous functions $f : [0,1] \to \mathbf{R}$ satisfying $f(0) = 0$. For any $t \in [0,1]$ define

$$\Phi_t : C_0[0,1] \to \mathbf{R}$$
$$f \mapsto f(t)^2.$$

Compute

$$\int_{C_0[0,1]} \Phi_t\,d\mu.$$

7. Let $f : [0,1] \to [c,d]$ be a continuous function, such that for any $y \in [c,d]$ there are only finitely many x for which $f(x) = y$. Define the "solution counting" function $N : [c,d] \to \mathbf{R}$ by

$$N(y) = (\text{number of solutions of the equation } f(x) = y).$$

Show that N is a measurable function. *(Hint: try to write N as a limit of simpler functions.)*

8. Suppose that $f_n, f : A \to \mathbf{R}$ are measurable. We say that f_n converges to f in measure, if for all $\delta > 0$ we have

$$\lim_{n\to\infty} \mu\{x\,;\, |f_n(x) - f(x)| > \delta\} = 0.$$

(a) Show that if f_n converges to f almost everywhere, then f_n converges to f in measure.

(b) Give an example to show that the converse is not true.