Honors Analysis - Homework 1

1. Prove the following inclusions of sets.

(a) $A \triangle B \subset (A \triangle C) \cup (C \triangle B)$

(b) $(A_1 \setminus A_2) \triangle (B_1 \setminus B_2) \subset (A_1 \triangle B_1) \cup (A_2 \triangle B_2)$

(c) $(A_1 \cap A_2) \triangle (B_1 \cap B_2) \subset (A_1 \triangle B_1) \cup (A_2 \triangle B_2)$

2. Let $\mathcal{E}$ be the set of elementary subsets of the unit square $[0,1] \times [0,1]$. For $A, B \in \mathcal{E}$, define

$$d(A, B) = \tilde{m}(A \triangle B).$$

Define an equivalence relation on $\mathcal{E}$ by letting $A \sim B$ if $d(A, B) = 0$. Denote by $\mathcal{E}/\sim$ the set of equivalence classes.

(a) Show that d defines a metric on $\mathcal{E}/\sim$.

(b) Is $\mathcal{E}/\sim$ with this metric a complete metric space?

3. Let $C \subset [0,1]$ be the Cantor set (defined last semester). Let $A \subset [0,1] \times [0,1]$ be the set defined by

$$A = \{(x, y) \mid x \in C, y \in [0,1]\}.$$

Show that A is measurable, and $\mu(A) = 0$.

4.

(a) For any elementary set $A \subset [0,1] \times [0,1]$ show that

$$\mu(A) = \lim_{n \to \infty} \frac{1}{n^2} \# \left(A \cap \frac{1}{n} \mathbf{Z}^2 \right).$$

Here $\#(E)$ denotes the number of elements of a set E, and $\frac{1}{n}\mathbf{Z}^2$ is the set of elements $\left(\frac{a}{n}, \frac{b}{n}\right)$ where $a, b \in \mathbf{Z}$.

(b) Show that the above equation is not true for more general measurable sets A.