Putnam Questions, Week 5

1. $[\sqrt{44}] = 6$ and $[\sqrt{4444}] = 66$. Generalize.

2. Prove that $(2+\sqrt{5})^{1/3} + (2-\sqrt{5})^{1/3}$ is rational.

3. Let $\alpha = \frac{3+\sqrt{13}}{2}$. What is the last digit of $[\alpha^{2009}]$?

4. Suppose that α, β, and γ are real numbers such that

$$\alpha + \beta + \gamma = 2,$$

$$\alpha^2 + \beta^2 + \gamma^2 = 14,$$

$$\alpha^3 + \beta^3 + \gamma^3 = 17.$$

   Find $\alpha\beta\gamma$.

5. How many ways are their to make 1 with quarters, dimes, and nickels? How about if one also allows 1 cent coins?

6. Find a formula for a_n, where $a_0 = a_1 = 2$, and $a_{n+1} = 4a_n - 4a_{n-1}$.

7. Find $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n$.

8. Suppose that $x_0 = 18$, $x_{n+1} = \dfrac{10x_n}{3} - x_{n-1}$, and that the sequence $\{x_n\}$ converges to some real number. Find x_1.

9. Let $\{x_n\}_{n\geq 0}$ be a sequence of nonzero real numbers such that $x_n^2 - x_{n-1}x_{n+1} = 1$ for all $n \geq 1$. Prove there exists a real number a such that $x_{n+1} = ax_n - x_{n-1}$ for all $n \geq 1$.

10. Let 1, 2, 3, ... 2006, 2007, 2009, 2012, 2016, ... be a sequence defined by $x_k = k$ for $k = 1$ to 2006 and $x_{k+1} = x_k + x_{k-2005}$ for $k \geq 2006$. Show that the sequence has 2005 consecutive terms each divisible by 2006