Putnam Questions, Week 4

1. Evaluate the following determinant: $\begin{vmatrix} 1 & 1 & 1 & 1 \\ w & x & y & z \\ w^2 & x^2 & y^2 & z^2 \\ w^3 & x^3 & y^3 & z^3 \end{vmatrix}$.

2. Evaluate the following determinant: $\begin{vmatrix} 1 & 1 & 1 & 1 \\ w & x & y & z \\ w^2 & x^2 & y^2 & z^2 \\ w^4 & x^4 & y^4 & z^4 \end{vmatrix}$.

3. Do there exist polynomials a, b, c, d such that $1 + xy + x^2y^2 = a(x)b(y) + c(x)d(y)$?

4. Determine all polynomials such that $P(0) = 0$ and $P(x^2 + 1) = P(x)^2 + 1$.

5. Consider all lines that meet the graph
$$y = 2x^4 + 7x^3 + 3x - 5$$
in four distinct points $P_i = [x_i, y_i]$, $i = 1, 2, 3, 4$. Prove that
$$\frac{x_1 + x_2 + x_3 + x_4}{4}$$
is independent of the line, and compute its value.

6. Let k be the smallest positive integer for which there exist distinct integers a, b, c, d, e such that
$$(x - a)(x - b)(x - c)(x - d)(x - e)$$
has exactly k nonzero coefficients. Find, with proof, a set of integers for which this minimum k is achieved.

7. Fint the maximum value of $f(x) = x^3 - 3x$ on the set of all real numbers x satisfying $x^4 + 36 \leq 13x^2$.

8. Find polynomials $f(x)$, $g(x)$, and $h(x)$ such that
$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1, & \text{if } x < -1, \\ 3x + 2, & \text{if } -1 \leq x \leq 0 \\ -2x + 2, & \text{if } x > 0 \end{cases}$$