

Putnam Questions, Week 3

1. If $2n + 1$ and $3n + 1$ are both perfect squares, prove that n is divisible by 40.
2. How many zeros does $1000!$ end with?
3. For how many k is the binomial coefficient $\binom{100}{k}$ odd?
4. Let n be a positive integer. Suppose that 2^n and 5^n *begin* with the same digit. What is the digit?
5. Prove that there are no four consecutive non-zero binomial coefficients $\binom{n}{r}$, $\binom{n}{r+1}$, $\binom{n}{r+2}$, $\binom{n}{r+3}$ in arithmetic progression.
6. Show that every positive integer is a sum of one or more numbers of the form $2^r 3^s$, where r and s are nonnegative integers and no summand divides another.
7. Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers, $n = a_1 + a_2 + \dots + a_k$, with k an arbitrary positive integer, and $a_1 \leq a_2 \leq \dots \leq a_k \leq a_1 + 1$? For example, with $n = 4$, there are four ways: 4 , $2 + 2$, $1 + 1 + 2$, $1 + 1 + 1 + 1$.
8. Let n be an even positive integer. Write the numbers $1, 2, \dots, n^2$ in the squares of an $n \times n$ grid so that the k th row, from left to right is
$$(k - 1)n + 1, (k - 1)n + 2, \dots, (k - 1)n + n.$$
Color the square of the grid so that half the squares in each row and in each column are red and the other half are black (a checkerboard coloring is one possibility). Prove that for each such coloring, the sum of the numbers on the red squares is equal to the sum of the numbers in the black squares.
9. How many primes among the positive integers, written as usual in base 10, are such that their digits are alternating 1s and 0s, beginning and ending with 1?
10. Show that if n is an integer greater than 1, then n does not divide $2^n - 1$.