

## Putnam Questions, Week 2

1. Prove that the number of subsets of  $\{1, 2, \dots, n\}$  with odd cardinality is equal to the number of subsets of even cardinality.
2. In how many ways can two squares be selected from an 8-by-8 chessboard so that they are not in the same row or the same column?
3. In how many ways can four squares, not all in the same row or column, be selected from an 8-by-8 chessboard to form a rectangle?
4. Find the number of subsets of  $\{1, 2, \dots, n\}$  that contain no two consecutive elements of  $\{1, 2, \dots, n\}$ .
5. Define a **selfish** set to be a set which has its own cardinality (number of elements) as an element. Find, with proof, the number of subsets  $\{1, 2, \dots, n\}$  which are *minimal* selfish sets, that is, selfish sets none of whose proper subsets is selfish.
6. For a partition  $\pi$  of  $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ , let  $\pi(x)$  be the number of elements in the part containing  $x$ . Prove that for any two partitions  $\pi$  and  $\pi'$ , there are two distinct numbers  $x$  and  $y$  in  $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$  such that  $\pi(x) = \pi(y)$  and  $\pi'(x) = \pi'(y)$ . [A *partition* of a set  $S$  is a collection of disjoint subsets (parts) whose union is  $S$ .]
7. Let  $S$  be a set of  $n$  distinct real numbers. Let  $A_S$  be the set of numbers that occurs as averages of two distinct elements of  $S$ . For a given  $n \geq 2$ , what is the smallest possible number of distinct elements in  $A_S$ .