Putnam Problems: Polynomials, Wednesday February 7th

1. Let $f(x)$ be a polynomial, and suppose that $f(x) + f'(x) > 0$ for all x. Prove that $f(x) > 0$ for all x.

2. Let $f(x)$ be a polynomial. Suppose for every rational number p/q, the value of $f(p/q)$ is rational. Prove that all the coefficients of $f(x)$ are rational.

3. Prove that two of the roots of the polynomial $x^4 + 12x - 5$ add up to two.

4. Let $f(x)$ be a polynomial with complex coefficients such that all its roots lie in the upper half plane. Prove that all the roots of $f'(x)$ also lie in the upper half plane.

5. Let $f(x)$ be a polynomial that takes on integer values for all $x \in \mathbf{Z}$. Can it be the case that $f(n)$ is prime for all sufficiently large n?

6. Let $n > 2$, and let $f(x)$ be the polynomial $x^{2006} + x + n$. Prove that all the roots α of $f(x)$ have absolute value $|\alpha| > 1$. Prove that if $n = p$ is prime then $f(x)$ is irreducible.