Putnam Problems: Integrals, Wednesday February 14th

Compute the following integrals.

1. $\displaystyle\int_0^\infty \frac{x\,dx}{e^x - 1}$
2. $\displaystyle\int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx$
3. $\displaystyle\int_0^1 \frac{\ln(1+x)}{x}\,dx$

Prove the following:

1. $\displaystyle\int_0^1 \frac{dx}{x^x} = 1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \frac{1}{5^5} + \dots$
2. $\displaystyle\int_0^1 \left(\frac{1}{1-x} + \frac{1}{\ln(x)}\right) dx = \gamma$, where

$$\gamma = \lim_{k\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k} - \ln(k)\right).$$

Hint: what is $\displaystyle\int_0^1 \frac{1-x^k}{1-x}\,dx$?