

Putnam Problems: Functional Equations, Wednesday February 21th

1. Find all functions f defined for real numbers $x \geq 0$ such that $f(x^2 + y^2) = f(x)^2 + f(y)^2$.
2. Find all functions $f : \mathbf{N} \rightarrow \mathbf{N}$ such that $f(nm) = f(n)f(m)$, and such that

$$\lim_{n \rightarrow \infty} \frac{\log(f(n))}{\log(n)} = 1.$$

3. The set of all positive integers is the union of disjoint subsets $\{f(1), f(2), \dots\}$ and $\{g(1), g(2), \dots\}$ where

$$\begin{aligned} f(1) &< f(2) < \dots, \\ g(1) &< g(2) < \dots, \\ g(n) &= f(f(n)) + 1 \text{ for all } n \geq 1. \end{aligned}$$

Determine $f(2007)$.