Trajectories on polygonal surfaces

My research is broadly about lines on surfaces. I study billiard trajectories on polygonal billiard tables, lines on polygonal tilings, and lines on surfaces made from polygons. The simplest cases of these systems have been studied since the 1800s (see [C75, MH40]) and have very elegant dynamics, and I work on generalizing the classical results to more general systems. The simplest case is the square. Figure 1a shows a billiard path on the square billiard table, where we assume that the angle of incidence equals the angle of reflection, and the ball moves without friction.

We can unfold the table across the horizontal and vertical edges, which yields the square torus surface, on which the trajectory goes straight (Figure 1b). We record that the horizontal edges are the same (they are identified) by labeling them both 0, and the same for the vertical edges with the label 1. A linear trajectory on the square torus is essentially the same as a line on the square grid (Figure 1c), which has many copies of the square instead of just one. Thus, these three systems—billiard trajectories, lines on surfaces made from polygons, and lines on a polygonal tiling—are intimately related: by understanding one, we can understand all three.

![Figure 1: (a) a billiard path on the square table (b) the corresponding trajectory on the unfolding of the square billiard table into the square torus (c) the corresponding path on the square grid](image)

When the trajectory on the square torus or the square grid crosses an edge, we can write down the corresponding 0 or 1, which gives us a bi-infinite cutting sequence of edges crossed. For the trajectory in Figure 1 the cutting sequence contains \ldots 110110 \ldots. I have studied cutting sequences on the double pentagon [DFT11], on more general polygon surfaces [D12, D13], and on Bouw-Möller surfaces [DPU15].

I also work on geodesics on polyhedra. A straight path on the cube is essentially a straight path on the square grid, except that three squares come together at a vertex instead of four as the path wraps around the cube; I studied this in [DDTY15].
In the square grid above, the trajectory cuts through the edges of the grid and we simply record what happens. Alternatively, we can impose a refraction rule, where the trajectory bends when it crosses an edge, as light bends when it crosses an interface between air and water. It turns out that trajectories have interesting behavior when refracting through tilings under such a rule, which I have studied in [DPRS15].

Key questions about trajectories, and some answers

For linear trajectories in a given system, there are several important questions to ask. For the square torus, the answers to these questions are quite elegant. Below, I describe the answers for the square, and then describe my work that generalizes these results to other surfaces.

Which directions yield periodic trajectories? On the square billiard table, trajectories with a rational slope $p/q$ are periodic, with period $2(p + q)$. Trajectories with an irrational slope are aperiodic, and are dense in the table.

My work investigates this kind of system for other polygons, which are more complicated than the square torus but still exhibit a lot of symmetry, my favorite being the pentagon. For several years, I have been working towards answering the questions: Which directions are periodic on the regular pentagon? For a periodic direction, what is/are the corresponding period(s)?

I studied periodic trajectories in the pentagonal billiard table with Dmitry Fuchs and Sergei Tabachnikov. We described all periodic directions and gave some results on the associated periods [DFT11].

**Theorem.** Periodic directions on the double pentagon are boundary points of the tiling by ideal regular pentagons of the hyperbolic plane, which lie in $\mathbb{Q}[\phi]$.

I am now working with Samuel Lelièvre on this problem using a different approach, a recursive structure for periodic directions using a rectangular surface called the Golden L that is related to the pentagon. We put a tree structure on the periodic directions in the regular pentagon, and used this to produce many beautiful pictures of periodic paths (Figure 2).

![Periodic trajectories in the pentagon with periods 60, 150, 240 and 290](image)

Figure 2: Periodic trajectories in the pentagon with periods 60, 150, 240 and 290

Which systems have periodic trajectories? In my work as a teaching assistant in the Summer@ICERM REU with students Kelsey DiPietro, Jenny Rustad and...
Alexander St Laurent, we introduced and studied a new dynamical system called *tiling billiards*, motivated by a discovery in physics of materials with negative indices of refraction [SSS01, SPW04]. In tiling billiards, we have a linear trajectory on a polygonal tiling, with the “negative refraction rule”: when the trajectory hits an edge, it is reflected across that edge. For the square grid, the dynamics are again quite simple: trajectories either circle a vertex or zig-zag along a line (Figure 3a).

However, other tilings have quite rich dynamics; for example, triangle tilings (Figure 3b) have many periodic trajectories with very stable behavior, and the trihexagonal tiling (Figure 3c) has wildly unstable trajectories.

![Figure 3: Tiling billiards trajectories in (a) the square grid, (b) a triangle tiling, and (c) the trihexagonal tiling](image)

Our work [DPRS15] was a preliminary exploration of this system. We showed that periodic trajectories exist for large classes of tilings, and constructed many such trajectories. We define a *triangle tiling* as a tiling by congruent triangles where every vertex has valence 6 and half-turn symmetry, and in a right triangle tiling, we assume that the diagonals are parallel.

**Theorem.**
- Every triangle tiling has a trajectory of period 6 around each vertex.
- Every triangle tiling, except those of isosceles triangles with a vertex angle greater than or equal to $\pi/3$, has a trajectory of period 10 around two vertices.
- Every right triangle tiling has an escaping trajectory.

We also showed that dense trajectories exist in tiling billiards, essentially by modifying trajectories similar to the example in Figure 3c:

**Theorem.** The trihexagonal tiling exhibits trajectories that are dense in an infinite region of the plane.

Pat Hooper and I have ideas about how to prove some of the things that we discovered computationally, such as the diffusion rate of aperiodic trajectories in the trihexagonal tiling. We plan to work on these later this fall.

This system also has many special cases that would be good for student research:
- In the space of triangle tilings, or of parallelogram tilings, etc., which regions have which classes of periodic trajectories?
- What are the dynamics of aperiodic tilings, such as the Penrose tiling?
Given a cutting sequence, can we recover the direction of the trajectory? For the square torus, we repeatedly apply a particular derivation rule to the cutting sequence, and the information we get at each step of this process gives us the coefficients of the continued fraction expansion of the slope of the trajectory. Series developed this structure in [S85a, S85b, S91], and it is explored in depth in my expository book [D15a].

Here again, I study surfaces that are more complicated than the square but still have certain “lattice” symmetries: other regular polygon surfaces, and Bouw-Möller surfaces [BM06].

In my thesis work, I studied the dynamics of trajectories on the double pentagon surface, which is two copies of a regular pentagon with opposite parallel edges identified. Using the methods that John Smillie and Corinna Ulcigrai developed for the regular octagon surface [SU11, SU10], I used the “lattice” symmetries of the regular polygons to derive an analogous derivation rule for all double odd polygons [D12]:

**Theorem.** For the double regular \(n\)-gon, the result of applying the shear \[
\begin{bmatrix}
-1 & 2 \cos \frac{\pi}{n} \\
0 & 1
\end{bmatrix},
\] as induced on the cutting sequence corresponding to a linear trajectory, is to keep only the sandwiched symbols (those with the same symbol on each side).

In [D13], I determined the derivation rule for certain trajectories on the larger family of Bouw-Möller surfaces. The rule reduces to the “sandwiching” rule for regular polygons, but because the edge labels have less symmetry, the statement is not as elegant:

**Theorem.** For the \((m,n)\) Bouw-Möller surface, let \(M(m,n) = 2 \cot \frac{\pi}{n} + 2 \frac{\cos \frac{\pi}{m}}{\sin \frac{\pi}{n}}\). The result of applying the shear \[
\begin{bmatrix}
-1 & M(m,n) \\
0 & 1
\end{bmatrix}
\] to the surface, as induced on the cutting sequence corresponding to a linear trajectory whose angle is less than that of any positive diagonal of the surface, is to keep only the symbols that are the middle letter of a three-letter sequence where the corresponding edge transitions are either both horizontal or both in direction \(\pi/n\).

Recently, in joint work with Corinna Ulcigrai and Irene Pasquinelli, we were able to treat all cutting sequences on Bouw-Möller surfaces, and determine a derivation rule for them [DPU15]. The key was the structure of transition diagrams:

The transition diagram for the \((m,n)\) Bouw-Möller surface is a directed graph whose nodes are edge labels of the surface, with an arrow between two edge labels when a straight trajectory can cut through the first edge label and then the second edge label. If an edge of the \((n,m)\) dual surface is crossed in between, the arrow is labeled with its label.

It turns out that the diagrams have an organized, intertwined structure, which comes from the affine equivalence of the \((m,n)\) and \((n,m)\) Bouw-Möller surfaces:

**Theorem.** The transition diagram is a \(n \times (m-1)\) rectangle, with \(n\) labels on each row snaking left and right. The arrow labels follow the same pattern but snaking down

\[1\] I made a video that explains the double pentagon surface, and this result, using colors and dance: [http://vimeo.com/47049144](http://vimeo.com/47049144)
and up in an overlapping grid. Two examples is shown in Figure 4; the diagrams have the same structure for larger values of \( m \) and \( n \), extended down and to the right.

Figure 4: The transition diagrams for the \((4, 3)\) and \((3, 4)\) Bouw-Möller surfaces

Understanding this structure allowed us to identify the direction of a trajectory corresponding to a given cutting sequence, analogous to the continued fractions on the square torus.

**Theorem.** To a cutting sequence corresponding to a trajectory on a Bouw-Möller surface, we alternately apply a derivation rule (which changes the direction of the trajectory), and then a normalization rule (which rotates it into a standard sector). The infinite sequence of the list of normalization rules we apply (based on the direction of the derived trajectory) at each step defines the direction of the original trajectory.

The transition diagrams also allow us to give the derivation rule for all trajectories:

**Theorem.** For the \((m, n)\) Bouw-Möller surface, let \( M(m, n) \) be as above. The result of applying the shear \([-1 \ M(m,n) \ 0 \ 1]\) to the surface, as induced on a cutting sequence corresponding to a linear trajectory, is as follows: find the cutting sequence as a path on the transition diagram for the surface, and collect the labels of the arrows along the path; the sequence of these arrow labels is the derived sequence.

**Can we describe all possible cutting sequences?** For the square torus, cutting sequences are those sequences on which we can apply the derivation rule infinitely many times.\(^2\)

In my work with Pasquinelli and Ulcigrai, we give a similar characterization for cutting sequences on Bouw-Möller surfaces. For the square torus, we apply just one derivation rule to cutting sequences, but on Bouw-Möller surfaces we apply two different derivation rules: we use the \((m, n)\) transition diagram and the \((n, m)\) transition diagram alternately.

**Theorem.** The closure of the set of cutting sequences on Bouw-Möller surfaces is the set of sequences on which we can apply the alternating derivation rules infinitely many times.

\(^2\)Actually, this characterizes the closure of the set of cutting sequences, but the boundary cases are easily described.
How many ways are there to get from one point to another? Given two points $A$ and $B$ on the square torus, we want to know which directions we can go from $A$ to get to $B$ (possibly wrapping around many times). This is equivalent to marking a point $A'$ on a square grid, and drawing trajectories to every lattice point (the unfoldings of $B$); these directions are easy to describe.

I have studied this question on the cube, which has the simple structure of the square grid, but more complicated adjacencies. In joint work with Victor Dods, Cindy Traub and Jed Yang, we studied vertex-to-vertex paths on the cube [DDTY15].

**Theorem.** Given a vertex on the cube, there are other vertices that are a distance $1$, $\sqrt{2}$ and $\sqrt{3}$ away. The (infinite) number of paths to vertices of each type are in a $4:3:6$ ratio.

To do this, we used the structure of the Stern-Brocot tree on the rational numbers, and computer programming to perform large calculations.

Dmitry Fuchs and Ekaterina Fuchs have related results for closed trajectories on regular polyhedra, using different methods [FF10], and D. Fuchs continues to work in this area. In the future, I would like to explore this question for general rectangular boxes, and then for general polyhedra.

References


