# BASIC IDEAS OF ABSTRACT MATHEMATICS

## Propositions

A *proposition* is a statement that is either true or false. In our course, we will usually call a mathematical proposition a *theorem*. A theorem is a main result. A proposition that is mainly of interest to prove a larger theorem is called a *lemma*. Some intermediate results are called *propositions*. Finally, a *corollary* is a statement that follows easily from a theorem.

The following sentences are examples of propositions. Two plus two is four. Two plus two is zero. Bicycles have three wheels. The first of these propositions is true and the last two are false.

Some sentences are not propositions: "How are you doing?" "Picasso's painting Guernica is obedient." "Picasso's painting Guernica is beautiful." Sometimes a sentence does not provide enough information to determine whether it is true or false, so it is not a proposition. An example is the following: "He is an Olympic swimmer." The sentence does not tell us who we are talking about. If we identify the person, then the sentence becomes a proposition: "Michael Phelps is an Olympic swimmer."

Assume P and Q are each propositions. The proposition "P and Q" is true exactly when both P and Q are true; e.g., "the numbers 5 and 7 are odd integers". The proposition "P or Q" is true exactly when at least one of P or Q is true, i.e., either one or both are true; e.g., "either 5 or 7 is an odd integer" or "either 5 or 6 is an odd integer". Also, "for any integer $n$, either $n$ or $n+1$ is an odd integer."

## Implications

Mathematical propositions are often of the form "If P, then Q". Such a statement is called an *implication*. "P" is called the *hypothesis* and "Q" is called the *conclusion*. For example, "If $x$ and $y$ are odd integers, then $xy$ is an odd integer." If the hypothesis is false, then the implication is called true no matter whether the conclusion is true or false. The following is a true proposition: "If cows could fly then the moon is made of blue cheese." The implication "if P then Q" is sometime written as or "Q whenever P". The statement "If a test was given then all the students came to class" is equivalent to the following statements: "All the students came to class if a test was given," or "All the students came to class whenever a test was given." A more mathematical example of equivalent statements are the following: "If $0 < x < 2$, then $x^2 < 4$", "$x^2 < 4$ whenever $0 < x < 2$," and "$x^2 < 4$ for all $x$ such that $0 < x < 2$." (See quantifiers below.)

The *converse* of a proposition "if P then Q" is the statement "if Q then P". The converse of a statement is not equivalent to a statement. For example the proposition "if $x$ is positive then $x^2$ is positive" is true, while "if $x^2$ is positive then $x$ is positive" is false.

The implication "P if and only if Q" is equivalent to the two implications "if P then Q" and "if Q then P". The implication "P if and only if Q" is often written as "P iff Q".

The use of "if" in definitions in our book and most other mathematics books really means "iff". For this reason, in definitions, we will use "provided that" instead of "if". For example, "A system of linear equations is said to be *consistent* provided that it has at least one solution" means that "if there is at least one solution then the system of linear equations is consistent" and "if the system of linear equations is consistent then there is at least one solution".

## Quantifiers

There is a difference between the following assertions: (i) For any positive number $x$, $x^2 > x$. (ii) There exists a positive number $x$ such that $x^2 > x$. The first statement is false and the second is true. The first sentence contains what is called the universal quantifier "for any." The second sentence contains what is called the existential quantifier " there exists."

Quantifiers can come up in situations that are non-mathematical. Notice that the following pairs of sentences do not mean the same thing. "All the children are above average" and "some of the children are above average". "All children are human" and "some of the children are human". "All classes at Northwestern are hard" and "some classes at Northwestern are hard". It is important to pay attention to the quantifiers in a theorem.

The *universal quantifier* can be expressed as "for any $x$", "for each $x$", "for every $x$", or "for all $x$". These three phrases mean the same thing, but the first three are singular and have the connotation of taking one $x$ at a time. The last phrase is plural and has the connotation of taking all the possible $x$ at once.

The *existential quantifier* can be expressed as 'there exists", "there is", or "for some": "There exists a positive number $x$ such that $x^2 > x$", "There is a positive number $x$ such that $x^2 > x$", and "For some $x$, $x^2 > x$."

## Negation

The *negation* "not P" is true exactly when P is false and is written as "~P".

It is important how negation works with quantifiers, with "and", and with "or." The negation of "all classes at Northwestern are hard" is "there is a class at Northwestern that is not hard". The negation of "Bob is tall and handsome" is "'Bob is either not tall or not handsome". The negation "~(P and Q)" is "~P or ~Q". For "Mary is either smart or observant" to be true, Mary needs only have one of the characterizations true. Therefore, for the negation, both must be false: "Mary is neither smart nor observant." The negation "~(P or Q)" is "~P and ~Q". The negation of "For all $x$, if P($x$) then Q($x$)" is "there exists $x$ such that P($x$) is true and Q($x$) is false". For example, the negation of "For all real $x$ such that $1 < x < 2$, $1 < x^2 < 3$" is "there exists a real $x$ with $1 < x < 2$ such that $x^2 < 1$ or $x^2 > 3$".

The *contrapositive* of the statement "if P then Q" is the statement "if not Q then not P". The contrapositive of a statement is equivalent to a statement. In particular, the statement "if P($x$) is true for a real value $x$, then Q($x$) is true for that value of $x$" is the same as "if Q($x$) is false for some real value $x$, then P($x$) is false for that value of $x$".

## Examples and Counterexamples

A *counterexample* is an example that shows a proposition is false. We only need one counterexample to show that the proposition is false. A counterexample does not "prove the rule". If there is one person taller than 7 feet, this shows that the statement "all people are shorter than 7 feet" is false. A counterexample can help you understand what assumptions you really need to argue that the conclusion is true. This type of reasoning can be helpful in economics or business.

By contrast, one example shows why a proposition is "reasonable" or what it means but itself does not prove the proposition.

## Proofs

Proofs can be *constructive* or *non-constructive*. Thus in linear algebra, we will learn an algorithm to find the solution of a system of equations. This not only says that a solution exists, but shows how to find or construct a solution. In other situations, it can be proved that ordinary differential equations have solutions without necessarily showing how to find an expression for the solution.

In general, it is difficult to find a proof. Mathematician think for months, years, or sometimes centuries to discover why a theorem is true or false. In our course, we will mainly read proofs and try to understand why they show that the theorem is true. The true-false questions will also force you to read statements carefully and decide whether they are true or false based on results we have obtained in the course.

There are different approaches to proofs.

- A *direct* proof of "if P then Q" starts with the assumption P and argues directly that Q is true.
- A proof by using the *contrapositive* assumes Q is false and argues directly that P is false.
- Closely related to the contrapositive is proof by *contradiction*. To prove "if P then Q", we assume P and "~Q" and show that this contradicts some basic fact of mathematics. The difference we can use the statement P in the proof and show that it leads to some false conclusion like $1 = 0$.
- There are other methods or complexities of proofs that will arise as we look at proofs in the course.