

**PUTNAM TRAINING
EASY PUTNAM PROBLEMS**

(Last updated: December 7, 2023)

REMARK. This is a list of exercises on Easy Putnam Problems —Miguel A. Lerma

EXERCISES

1. **2023-A1.** For a positive integer n , let $f_n(x) = \cos(x)\cos(2x)\cos(3x)\cdots\cos(nx)$. Find the smallest n such that $|f_n''(0)| > 2023$.
2. **2022-B1.** Suppose that $P(x) = a_1x + a_2x^2 + \cdots + a_nx^n$ is a polynomial with integer coefficients, with a_1 odd. Suppose that $e^{P(x)} = b_0 + b_1x + b_2x^2 + \cdots$ for all x . Prove that b_k is nonzero for all $k \geq 0$.
3. **2021-A1.** A grasshopper starts at the origin in the coordinate plane and makes a sequence of hops. Each hop has length 5, and after each hop the grasshopper is at a point whose coordinates are both integers; thus, there are 12 possible locations for the grasshopper after the first hop. What is the smallest number of hops needed for the grasshopper to reach the point $(2021, 2021)$?
4. **2020-A1.** How many positive integers N satisfy all of the following three conditions?
 - (i) N is divisible by 2020.
 - (ii) N has at most 2020 decimal digits.
 - (iii) The decimal digits of N are a string of consecutive ones followed by a string of consecutive zeros.

5. **2019-A1.** Determine all possible values of the expression

$$A^3 + B^3 + C^3 - 3ABC$$

where A, B , and C are nonnegative integers.

6. **2018-A1.** Find all ordered pairs (a, b) of positive integers for which

$$\frac{1}{a} + \frac{1}{b} = \frac{3}{2018}.$$

7. **2018-B2.** Let n be a positive integer, and let $f_n(z) = n + (n-1)z + (n-2)z^2 + \cdots + z^{n-1}$. Prove that f_n has no roots in the closed unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$.
8. **2017-A1.** Let S be the smallest set of positive integers such that
 - (a) 2 is in S ,
 - (b) n is in S whenever n^2 is in S , and

(c) $(n + 5)^2$ is in S whenever n is in S .

Which positive integers are not in S ?

(The set S is “smallest” in the sense that S is contained in any other such set.)

9. **2017-B1.** Let L_1 and L_2 be distinct lines in the plane. Prove that L_1 and L_2 intersect if and only if, for every real number $\lambda \neq 0$ and every point P not on L_1 or L_2 , there exist points A_1 on L_1 and A_2 on L_2 such that $\overrightarrow{PA_2} = \lambda \overrightarrow{PA_1}$.
10. **2016-A1.** Find the smallest positive integer j such that for every polynomial $p(x)$ with integer coefficients and for every k , the integer

$$p^{(j)}(k) = \left. \frac{d^j}{dx^j} p(x) \right|_{x=k}$$

(the j -th derivative of $p(x)$ at k) is divisible by 2016.

11. **2016-B1.** Let x_0, x_1, x_2, \dots be the sequence such that $x_0 = 1$ and for $n \geq 0$,

$$x_{n+1} = \ln(e^{x_n} - x_n)$$

(as usual, the function \ln is the natural logarithm). Show that the infinite series

$$x_0 + x_1 + x_2 + \dots$$

converges and find its sum.

12. **2016-B3.** Suppose that S is a finite set of points in the plane such that the area of triangle ΔABC is at most 1 whenever A, B , and C are in S . Show that there exists a triangle of area 4 such that (together with its interior) covers the set S .
13. **2015-A1.** Let A and B be points on the same branch of the hyperbola $xy = 1$. Suppose that P is a point lying between A and B on this hyperbola, such that the area of the triangle APB is as large as possible. Show that the region bounded by the hyperbola and the chord AP has the same area as the region bounded by the hyperbola and the chord PB .
14. **2015-B1.** Let f be a three times differentiable function (defined on \mathbb{R} and real-valued) such that f has at least five distinct real zeros. Prove that $f + 6f' + 12f'' + 8f'''$ has at least two distinct real zeros.
15. **2015-B4.**

Let T be the set of all triples (a, b, c) of positive integers for which there exist triangles with side lengths a, b, c . Express

$$\sum_{(a,b,c) \in T} \frac{2^a}{3^b 5^c}$$

as a rational number in lowest terms.

16. **2014-A1.** Prove that every nonzero coefficient of the Taylor series of

$$(1 - x + x^2)e^x$$

about $x = 0$ is a rational number whose numerator (in lowest terms) is either 1 or a prime number.

17. **2014-B1.** A *base 10 over-expansion* of a positive integer N is an expression of the form

$$N = d_k 10^k + d_{k-1} 10^{k-1} + \cdots + d_0 10^0$$

with $d_k \neq 0$ and $d_i \in \{0, 1, 2, \dots, 10\}$ for all i . For instance, the integer $N = 10$ has two base 10 over-expansions: $10 = 10 \cdot 10^0$ and the usual base 10 expansion $10 = 1 \cdot 10^1 + 0 \cdot 10^0$. Which positive integers have a unique base 10 over-expansion?

18. **2013-A1.** Recall that a regular icosahedron is a convex polyhedron having 12 vertices and 20 faces; the faces are congruent equilateral triangles. On each face of a regular icosahedron is written a nonnegative integer such that the sum of all 20 integers is 39. Show that there are two faces that share a vertex and have the same integer written on them.

19. **2013-B1.** For positive integers n , let the numbers $c(n)$ be determined by the rules $c(1) = 1$, $c(2n) = c(n)$, and $c(2n + 1) = (-1)^n c(n)$. Find the value of

$$\sum_{n=1}^{2013} c(n)c(n+2).$$

20. **2012-A1.** Let d_1, d_2, \dots, d_{12} be real numbers in the interval $(1, 12)$. Show that there exist distinct indices i, j, k such that d_i, d_j, d_k are the side lengths of an acute triangle.

21. **2012-B1.** Let S be the class of functions from $[0, \infty)$ to $[0, \infty)$ that satisfies:

- (i) The functions $f_1(x) = e^x - 1$ and $f_2(x) = \ln(x + 1)$ are in S .
- (ii) If $f(x)$ and $g(x)$ are in S , then functions $f(x) + g(x)$ and $f(g(x))$ are in S ;
- (iii) If $f(x)$ and $g(x)$ are in S and $f(x) \geq g(x)$ for all $x \geq 0$, then the function $f(x) - g(x)$ is in S .

Prove that if $f(x)$ and $g(x)$ are in S , then the function $f(x)g(x)$ is also in S .

22. **2011-B1.** Let h and k be positive integers. Prove that for every $\varepsilon > 0$, there are positive integers m and n such that

$$\varepsilon < |h\sqrt{m} - k\sqrt{n}| < 2\varepsilon.$$

23. **2010-A1.** Given a positive integer n , what is the largest k such that the numbers $1, 2, \dots, n$ can be put into k boxes so that the sum of the numbers in each box is the same? [When $n = 8$, the example $\{1, 2, 3, 6\}, \{4, 8\}, \{5, 7\}$ shows that the largest k is at least 3.]

24. 2010-B1. Is there an infinite sequence of real numbers a_1, a_2, a_3, \dots such that

$$a_1^m + a_2^m + a_3^m + \dots = m$$

for every positive integer m ?

25. 2010-B2. Given that $A, B,$ and C are noncollinear points in the plane with integer coordinates such that the distances $AB, AC,$ and BC are integers, what is the smallest possible value of AB ?

26. 2009-A1. Let f be a real-valued function on the plane such that for every square $ABCD$ in the plane, $f(A) + f(B) + f(C) + f(D) = 0$. Does it follow that $f(P) = 0$ for all points P in the plane?

27. 2009-B1. Show that every positive rational number can be written as a quotient of products of factorials of (not necessarily distinct) primes. For example,

$$\frac{10}{9} = \frac{2! \cdot 5!}{3! \cdot 3! \cdot 3!}.$$

28. 2008-A1. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a function such that $f(x, y) + f(y, z) + f(z, x) = 0$ for all real numbers $x, y,$ and z . Prove that there exists a function $g : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x, y) = g(x) - g(y)$ for all real numbers x and y .

29. 2008-A2. Alan and Barbara play a game in which they take turns filling entries of an initially empty 2008×2008 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy?

30. 2008-B1. What is the maximum number of rational points that can lie on a circle in \mathbb{R}^2 whose center is not a rational point? (A *rational point* is a point both of whose coordinates are rational numbers.)

31. 2007-A1. Find all values of α for which the curves $y = \alpha x^2 + \alpha x + \frac{1}{24}$ and $x = \alpha y^2 + \alpha y + \frac{1}{24}$ are tangent to each other.

32. 2007-B1. Let f be a polynomial with positive integer coefficients. Prove that if n is a positive integer, then $f(n)$ divides $f(f(n) + 1)$ if and only if $n = 1$. [Note: one must assume f is nonconstant.]

33. 2006-A1. Find the volume of the region of points (x, y, z) such that

$$(x^2 + y^2 + z^2 + 8)^2 \leq 36(x^2 + y^2).$$

- 34. 2006-B2.** Prove that, for every set $X = \{x_1, x_2, \dots, x_n\}$ of n real numbers, there exists a non-empty subset S of X and an integer m such that

$$\left| m + \sum_{s \in S} s \right| \leq \frac{1}{n+1}.$$

- 35. 2005-A1.** Show that every positive integer is a sum of one or more numbers of the form $2^r 3^s$, where r and s are nonnegative integers and no summand divides another. (For example, $23 = 9 + 8 + 6$.)
- 36. 2005-B1.** Find a nonzero polynomial $P(x, y)$ such that $P(\lfloor a \rfloor, \lfloor 2a \rfloor) = 0$ for all real numbers a . (Note: $\lfloor \nu \rfloor$ is the greatest integer less than or equal to ν .)
- 37. 2004-A1.** Basketball star Shanille O'Keal's team statistician keeps track of the number, $S(N)$, of successful free throws she has made in her first N attempts of the season. Early in the season, $S(N)$ was less than 80% of N , but by the end of the season, $S(N)$ was more than 80% of N . Was there necessarily a moment in between when $S(N)$ was exactly 80% of N ?
- 38. 2004-B2.** Let m and n be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m! n!}{m^m n^n}.$$

- 39. 2003-A1.** Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers, $n = a_1 + a_2 + \dots + a_k$, with k an arbitrary positive integer and $a_1 \leq a_2 \leq \dots \leq a_k \leq a_1 + 1$? For example, with $n = 4$ there are four ways: 4 , $2+2$, $1+1+2$, $1+1+1+1$.
- 40. 2003-A2.** Let a_1, a_2, \dots, a_n and b_1, b_2, \dots, b_n be nonnegative real numbers. Show that

$$(a_1 a_2 \cdots a_n)^{1/n} + (b_1 b_2 \cdots b_n)^{1/n} \leq [(a_1 + b_1)(a_2 + b_2) \cdots (a_n + b_n)]^{1/n}.$$

- 41. 2002-A1.** Let k be a fixed positive integer. The n -th derivative of $\frac{1}{x^k-1}$ has the form $\frac{P_n(x)}{(x^k-1)^{n+1}}$ where $P_n(x)$ is a polynomial. Find $P_n(1)$.
- 42. 2002-A2.** Given any five points on a sphere, show that some four of them must lie on a closed hemisphere.
- 43. 2001-A1.** Consider a set S and a binary operation $*$, i.e., for each $a, b \in S$, $a * b \in S$. Assume $(a * b) * a = b$ for all $a, b \in S$. Prove that $a * (b * a) = b$ for all $a, b \in S$.
- 44. 2000-A2.** Prove that there exist infinitely many integers n such that $n, n+1, n+2$ are each the sum of the squares of two integers. [Example: $0 = 0^2 + 0^2$, $1 = 0^2 + 1^2$, $2 = 1^2 + 1^2$.]

45. 1999-A1. Find polynomials $f(x)$, $g(x)$, and $h(x)$, if they exist, such that for all x ,

$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1 & \text{if } x < -1 \\ 3x + 2 & \text{if } -1 \leq x \leq 0 \\ -2x + 2 & \text{if } x > 0. \end{cases}$$

46. 1998-A1. A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

47. 1997-A5. Let N_n denote the number of ordered n -tuples of positive integers (a_1, a_2, \dots, a_n) such that $1/a_1 + 1/a_2 + \dots + 1/a_n = 1$. Determine whether N_{10} is even or odd.

48. 1988-B1. A *composite* (positive integer) is a product ab with a and b not necessarily distinct integers in $\{2, 3, 4, \dots\}$. Show that every composite is expressible as $xy + xz + yz + 1$, with x, y, z positive integers.

HINTS

1. The second derivative has many terms but most of them will vanish.
2. Prove that $b_k k!$ is an odd integer for all $k \geq 0$.
3. Taxicab distance.
4. —
5. Write $B = A + b$ and $C = A + c$.
6. Clear denominators and factor.
7. Look at $(z - 1)f_n(z)$.
8. Think module 5.
9. Consider a dilation (homothety) of the plane by a factor of λ with center P .
10. When is the coefficient of $\frac{d^j}{dx^j}x^n$ divisible by 2016?
11. Show that the sum telescopes and the n -th term tends to zero.
12. Consider three points A , B , and C such that triangle $\triangle ABC$ has the largest area.
13. Find the area of the triangle, maximize it, and compute the area between the cords and the hyperbola.
14. The given expression is the third derivative of a function with the same zeros as f .
15. Rewrite the sum and add the resulting geometric sums.
16. The coefficient can be computed and simplified explicitly.
17. —
18. If the the numbers of the faces having a common vertex have different numbers, what can we say about their sum?
19. Telescoping.
20. Three numbers $0 < a \leq b \leq c$ are the side lengths of an acute triangle precisely if $a^2 + b^2 > c^2$.
21. $e^{u+v} = e^u e^v$.
22. There is some rational number between $\frac{3\varepsilon}{hk}$ and $\frac{4\varepsilon}{hk}$.
23. For k to be as large as possible the “boxes” must be “small”.

24. One approach is to use the Cauchy-Schwartz inequality for some appropriately chosen values of k . Another approach is to look at how the LHS grows with m depending on the values of the a_k .
25. By looking at Pythagorean triples we get a reasonable conjecture about what the smallest possible value of AB could be. Then use $|AC - BC| \leq AB$, with equality if and only if A, B, C are collinear.
26. Find relations among the values of the function at nine points forming a 2 by 2 square grid.
27. Induction.
28. Try successively $(x, y, z) = (0, 0, 0)$, $(x, y, z) = (x, 0, 0)$, $(x, y, z) = (x, y, 0)$.
29. Try either getting two equal rows, or all rows summing zero.
30. How can we find the center of a circle if we are given some points on that circle?
31. Considered different cases depending on how each curve intersects the line $y = x$.
32. Consider $f(f(n) + 1) \bmod f(n)$.
33. Change to cylindrical coordinates.
34. Pigeonhole Principle.
35. Induction. The base case is $1 = 2^0 3^0$. The induction step depends on the parity of n . If n is even, divide by 2. If it is odd, subtract a suitable power of 3.
36. Note that $[2a] = 2[a]$ or $[2a] = 2[a] + 1$ depending on whether the fractional part of a is in $[0, 0.5)$ or $[0.5, 1)$.
37. Assume that $S(N)$ jumps abruptly from less than $4/5$ to more than $4/5$ at some point and find a contradiction.
38. Rewrite the inequality $\frac{(m+n)!}{m!n!} m^m n^n < (m+n)^{m+n}$.
39. If $0 < k \leq n$, is there any such sum with exactly k terms? How many?
40. Divide both sides by $[(a_1 + b_1) \cdots (a_n + b_n)]^{1/n}$ and use the AM-GM inequality on each of the two terms of the left hand side.
41. Differentiate $P_n(x)/(x^k - 1)^{n+1}$ and get a relation between $P_n(1)$ and $P_{n+1}(1)$.
42. Draw a great circle through two of the points.
43. Replace a by $b * a$.

44. Show that the equation $x^2 - y^2 = z^2 + 1$ has infinitely many integer solutions. Set $n = y^2 + z^2$.
45. Try with first degree polynomials. Some of those polynomials must change sign precisely at $x = -1$ and $x = 0$. Recall that $|u| = \pm u$ depending on whether $u \geq 0$ or $u < 0$.
46. Consider the plane containing both the axis of the cone and two opposite vertices of the cube's bottom face.
47. Discard solutions coming in pairs, such as the ones for which $a_1 \neq a_2$; so we may assume $a_1 = a_2$.
48. —

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SOLUTIONS

1. We will show that the answer is $n = 18$.

If D_x is the differentiation operator, i.e., $D_x(g) = g'$, then with the convention $D^0(g) = g$ and $D^1(g) = g'$ we have:

$$f'(x) = \sum_{i=1}^n \prod_{k=1}^n D_x^{\delta_{i,k}} \cos kx,$$

$$f''(x) = \sum_{i=1}^n \sum_{j=1}^n \prod_{k=1}^n D_x^{\delta_{i,k}} D_x^{\delta_{j,k}} \cos kx,$$

where $\delta_{i,k}$ is the Kronecker delta, equal to 1 if $i = k$ and 0 otherwise. On the other hand we have $D^1 \cos kx = (\cos kx)' = -k \sin kx$, $D^1 D^1 \cos kx = (\cos kx)'' = -k^2 \cos kx$. Since $\sin 0 = 0$ and $\cos 0 = 1$, most terms of $f''(x)$ will vanish, with the only exception of the terms for which $i = j = k$. Hence,

$$|f''(0)| = \left| - \sum_{k=1}^n k^2 \right| = \frac{n(n+1)(2n+1)}{6}.$$

So, the problem amounts to finding the smallest positive n such that $\frac{n(n+1)(2n+1)}{6} > 2023$, which is $n = 18$.

2. We prove that $b_k k!$ is an odd integer for all $k \geq 0$.

Since $e^{P(x)} = \sum_{n=0}^{\infty} \frac{(P(x))^n}{n!}$, the number $k! b_k$ is the coefficient of x^k in

$$(P(x))^k + \sum_{n=0}^{k-1} \frac{k!}{n!} (P(x))^n.$$

In particular, $b_0 = 1$ and $b_1 = a_1$ are both odd.

Now suppose $k \geq 2$; we want to show that b_k is odd. The coefficient of x^k in $(P(x))^k$ is a_1^k . It suffices to show that the coefficient of x^k in $\frac{k!}{n!} (P(x))^n$ is an even integer for any $n < k$. For k even or $n \leq k - 2$, this follows immediately from the fact that $\frac{k!}{n!}$ is an even integer. For k odd and $n = k - 1$, we have

$$\begin{aligned} \frac{k!}{(k-1)!} (P(x))^{k-1} &= k(a_1 x + a_2 x^2 + \dots)^{k-1} \\ &= k(a_1^{k-1} x^{k-1} + (k-1)a_1^{k-2} a_2 x^k + \dots) \end{aligned}$$

and the coefficient of x^k is $k(k-1)a_1^{k-2} a_2$, which is again an even integer.

3. The answer is 578.

First note that the only possible steps are given by the vectors $(\pm 3, \pm 4)$, $(\pm 5, 0)$, and $(0, \pm 5)$. Next, we need to prove that (1) 578 hops suffice, and (2) it is impossible to get to the destination with fewer hops.

(1) 578 hops suffice. In fact, $288 \cdot (3, 4) + 288 \cdot (4, 3) + (5, 0) + (0, 5) = (2021, 2021)$, hence $288 + 288 + 1 + 1 = 578$.

(2) It is impossible to get to destination with less than 578 hops. This can be proved by using the taxicab distance, defined $d((x_1, y_1), (x_2, y_2)) = |x_2 - x_1| + |y_2 - y_1|$. We note that each step of the form $(\pm 3, \pm 4)$ has a taxicab length $3 + 4 = 7$, and steps of the form $(5, 0)$ and $(0, 5)$ have taxicab length 5, so each hop takes a taxicab distance of at most 7. On the other hand, the taxicab distance between $(0, 0)$ and $(2021, 2021)$ is $2021 + 2021 = 4042$, but $577 \cdot 7 = 4039 < 4042$, hence 577 hops or less are not enough to reach the destination.

4. The values of N that satisfy (ii) and (iii) are precisely the numbers of the form $N = (10^a - 10^b)/9$ for $0 \leq b < a \leq 2020$; this expression represents the integer with a digits beginning with a string of 1's and ending with b 0's. A value N of this form is divisible by $2020 = 2^2 \cdot 5 \cdot 101$ if and only if $10^b(10^{a-b} - 1)$ is divisible by each of 3^2 , $2^2 \cdot 5$, and 101. Divisibility by 3^2 is a trivial condition since $10 \equiv 1 \pmod{9}$. Since $10^{a-b} - 1$ is odd, divisibility by $2^2 \cdot 5$ occurs if and only if $b \geq 2$. Finally, since $10^2 \equiv -1 \pmod{101}$, we see that 10^{a-b} is congruent to 10, -1 , -10 , or 1 $\pmod{101}$ depending on whether $a - b$ is congruent to 1, 2, 3, or 0 $\pmod{4}$; thus $10^{a-b} - 1$ is divisible by 101 if and only if $a - b$ is divisible by 4.

It follows that we need to count the number of (a, b) with $2 \leq b < a \leq 2020$ with $4 \mid a - b$. For given b , there are $\lfloor \frac{2020-b}{4} \rfloor$ possible values of a . Thus the answer is

$$\begin{aligned} &504 + 504 + 504 + 503 + 503 + 503 + 503 + \cdots + 1 + 1 + 1 + 1 \\ &= 4(504 + 503 + \cdots + 1) - 504 = 504 \cdot 1009 = 508536. \end{aligned}$$

5. The answer is all nonnegative integers, except multiples of 3 that are not multiple of 9. Let $X = A^3 + B^3 + C^3 - 3ABC$.

First, we show that we can make X equal to each of the claimed values.

Write $B = A + b$ and $C = A + c$, so that

$$X = (b^2 - bc + c^2)(3A + b + c).$$

Taking $(b, c) = (0, 1)$ or $(b, c) = (1, 1)$, we obtain respectively $X = 3A + 1$ and $X = 3A + 2$; consequently, as A varies, we achieve every nonnegative integer not divisible by 3. By taking $(b, c) = (1, 2)$, we obtain $X = 9A + 9$; consequently, as A varies, we achieve every positive integer divisible by 9. We may also achieve $X = 0$ by taking $(b, c) = (0, 0)$.

Next, we show that X can take *only* the claimed values.

Note that X is always nonnegative because of the arithmetic mean-geometric mean inequality:

$$\frac{A^3 + B^3 + C^3}{3} \geq \sqrt[3]{A^3 B^3 C^3} = ABC.$$

It thus only remains to show that if X is a multiple of 3, then it is a multiple of 9. We have

$$X = \underbrace{((b + c)^2 - 3bc)}_{b^2 - bc + c^2} (3A + b + c) \equiv (b + c)^3 \equiv b + c \pmod{3}.$$

Consequently, if X is divisible by 3, then $b + c$ must be divisible by 3, so each factor in $X = ((b + c)^2 - 3bc)(3A + b + c)$ is divisible by 3. This proves the claim.

6. By clearing denominators and regrouping, we see that the given equation is equivalent to

$$(3a - 2018)(3b - 2018) = 2018^2.$$

Each of the factors is congruent to 1 (mod 3). There are 6 factors of $2018^2 = 2^2 \cdot 1009^2$ that are congruent to 1 (mod 3): 1, 2^2 , 1009, $2^2 \cdot 1009$, 1009^2 , $2^2 \cdot 1009^2$. These lead to the 6 possible pairs: $(a, b) = (673, 1358114)$, $(674, 340033)$, $(1009, 2018)$, $(2018, 1009)$, $(340033, 674)$, and $(1358114, 673)$.

7. Note first that $f_n(1) > 0$, so 1 is not a root of f_n . Next, note that

$$(z - 1)f_n(z) = z^n + \cdots + z - n;$$

however, for $|z| \leq 1$, we have $|z^n + \cdots + z| \leq n$ by the triangle inequality; equality can only occur if z, \dots, z^n have norm 1 and the same argument, which only happens for $z = 1$. Thus there can be no root of f_n with $|z| \leq 1$.

8. The positive integers not in S are 1 and all multiples of 5.

First we prove that the set $S = \{n \in \mathbb{N} \mid n \neq 1 \text{ and } n \not\equiv 0 \pmod{5}\}$ verifies (a)–(c).

In fact, it verifies (a) because 2 is not 1 and is not a multiple of 5.

It verifies (b) because for $n \in \mathbb{Z}^+$, $n = 1 \Leftrightarrow n^2 = 1$, and if n is a multiple of 5 so is n^2 , hence if $n^2 > 1$ and *not* a multiple of 5, then $n > 1$ and *not* a multiple of 5 either.

It verifies (c) because if $n > 1$ and *not* a multiple of 5, the same holds for $(n + 5)^2$.

Next we prove that any other set T verifying conditions (a)–(c) contains S . Note that any such set verifies

(d) if $n \in T$, then $n + 5k \in T$ for all $k \geq 0$,

because if $n \in T$, by (c) we have that $(n + 5)^2 \in T$, and then by (b) $n + 5 \in T$. Hence, the following must be in T , with implications labeled by conditions (b) through (d):

$$\begin{aligned} 2 &\stackrel{c}{\Rightarrow} 49 \stackrel{c}{\Rightarrow} 54^2 \stackrel{d}{\Rightarrow} 56^2 \stackrel{b}{\Rightarrow} 56 \stackrel{d}{\Rightarrow} 121 \stackrel{b}{\Rightarrow} 11 \\ 11 &\stackrel{d}{\Rightarrow} 16 \stackrel{b}{\Rightarrow} 4 \stackrel{d}{\Rightarrow} 9 \stackrel{b}{\Rightarrow} 3 \\ 16 &\stackrel{d}{\Rightarrow} 36 \stackrel{b}{\Rightarrow} 6 \end{aligned}$$

Since $2, 3, 4, 6 \in T$, by (d) $S \subseteq T$, and so S is smallest.

9. Recall that L_1 and L_2 intersect if and only if they are not parallel.

In one direction, suppose that L_1 and L_2 intersect. Then for any P and λ , the dilation (homothety) of the plane by a factor of λ with center P carries L_1 to another line parallel to L_1 and hence not parallel to L_2 . Let A_2 be the unique intersection of L_2 with the image of L_1 , and let A_1 be the point on L_1 whose image under the dilation is A_2 ; then $\overrightarrow{PA_2} = \lambda \overrightarrow{PA_1}$.

In the other direction, suppose that L_1 and L_2 are parallel. Let P be any point in the region between L_1 and L_2 and take $\lambda = 1$. Then for any point A_1 on L_1 and any point

A_2 on L_2 , the vectors $\overrightarrow{PA_1}$ and $\overrightarrow{PA_2}$ have components perpendicular to L_1 pointing in opposite directions; in particular, the two vectors cannot be equal.

10. The answer is $j = 8$.

We have $\frac{d^j}{dx^j}x^j = j!$, so $j!$ must be a multiple of $2016 = 2^5 \cdot 3^2 \cdot 7$. Since 2^5 does not divide $7!$ we have $j \geq 8$.

Next, for any polynomial $p(x) = \sum_{i=0}^n a_i x^i$, we have

$$\left. \frac{d^8}{dx^8} p(x) \right|_{x=k} = \sum_{i=8}^n n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)(n-7) a_i k^{i-8}.$$

So, all we need to prove is that $n(n-1) \cdots (n-7)$ a multiple of 2^5 , 3^2 , and 7 .

In fact, one of any seven consecutive integers is a multiple of 7 , so $n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)$ must be a multiple of 7 .

One of every three consecutive integers is a multiple of 3 , so $n(n-1)(n-2)$ and $(n-3)(n-4)(n-5)$ are multiples of 3 , which implies $n(n-1)(n-2)(n-3)(n-4)(n-5)$ is a multiple of 3^2 .

Finally, given eight consecutive integers, four of them will be multiples of 2 , two will be multiples of 4 , and one is a multiple of 8 , hence $n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)(n-7)$ is a multiple of $2^4 \cdot 2^2 \cdot 2 = 2^7$, which is in fact more than needed.

11. First we prove that x_n has a limit as $n \rightarrow \infty$. The recurrence can be written $e^{x_{n+1}} = e^{x_n} - x_n$. For $x > 0$ we have $e^x > 1 + x$, hence by induction we get $0 < x_{n+1} < x_n \leq 1$, so the sequence is decreasing and bounded. By the Monotone Convergence Theorem that implies that x_n has in fact a limit.

Next we prove that the limit is zero. In fact, let $x = \lim_{n \rightarrow \infty} x_n$. Then $x = \ln(e^x - x)$, $e^x = e^x - x$, hence $x = 0$.

Finally we will prove that the sum is $e - 1$. From the recursive definition of x_n we have $x_n = e^{x_n} - e^{x_{n+1}}$, hence

$$\begin{aligned} x_0 + x_1 + x_2 + \cdots + x_N &= (e^{x_0} - e^{x_1}) + (e^{x_1} - e^{x_2}) + \cdots + (e^{x_N} - e^{x_{N+1}}) \\ &= e^{x_0} - e^{x_{N+1}} \xrightarrow{N \rightarrow \infty} e^1 - e^0 = e - 1. \end{aligned}$$

12. See figure 1

Pick three points A, B, C in S such that the triangle ΔABC has maximum area. Draw lines $A'B', B'C', C'A'$ parallel to AB, BC, CA respectively. For any point P above line $B'C'$, the triangle ΔPBC will have larger area than ΔABC because it has the same base but more height than ΔABC , hence all points of S must be below line $B'C'$. A similar argument leads to the conclusion that all points of S must be on the triangle $\Delta A'B'C'$. Also note that triangles ΔABC and $\Delta CB'A$ are each half of the parallelogram $ABCB'$, so they are equal. For the same reason $\Delta BAC'$ and $\Delta A'CB$ are also equal to ΔABC . Since the area of ΔABC is at most 1 , then the area of $\Delta A'B'C'$ is at most 4 , and this completes the proof.

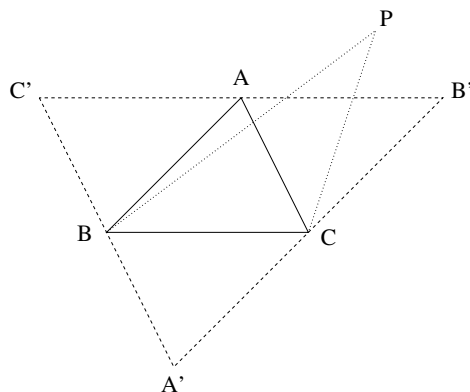


FIGURE 1

13. If $A = (x_1, \frac{1}{x_1})$, $B = (x_2, \frac{1}{x_2})$, $P = (x, \frac{1}{x})$, the area of the triangle APB is

$$T(x) = \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x \\ \frac{1}{x_1} & \frac{1}{x_2} & \frac{1}{x} \end{vmatrix}.$$

Its derivative respect to x is

$$T'(x) = \frac{x_2 - x_1}{x_1 x_2} - (x_2 - x_1) \frac{1}{x^2}.$$

Solving $T'(x) = 0$ we get a maximum at $x_0 = \sqrt{x_1 x_2}$.

Alternatively we could also find P as the point where the tangent to the hyperbola is parallel to AB . The slope of the hyperbola at $(x, \frac{1}{x})$ is $-1/x^2$, hence

$$-\frac{1}{x_0^2} = \frac{\frac{1}{x_2} - \frac{1}{x_1}}{x_2 - x_1} \Rightarrow x_0 = \sqrt{x_1 x_2}.$$

Next, the line going through $(x_1, \frac{1}{x_1})$ and $(x_0, \frac{1}{x_0})$ is

$$\frac{x - x_1}{x_0 - x_1} = \frac{y - \frac{1}{x_1}}{\frac{1}{x_0} - \frac{1}{x_1}} \Rightarrow y = \frac{1}{x_1} + \frac{x_1 - x}{x_0 x_1}.$$

The area of the region bounded by the hyperbola and the chord AP is

$$R_A = \int_{x_1}^{x_0} \left(\frac{1}{x_1} + \frac{x_1 - x}{x_0 x_1} - \frac{1}{x} \right) dx = \frac{x_2 - x_1}{2\sqrt{x_1 x_2}} - \log \sqrt{\frac{x_2}{x_1}}.$$

If we swap x_1 and x_2 in the computations above and integrate between x_0 and x_2 we get the area of the region bounded by the hyperbola and the chord PB :

$$R_B = \int_{x_0}^{x_2} \left(\frac{1}{x_2} + \frac{x_2 - x}{x_0 x_2} - \frac{1}{x} \right) dx = \frac{x_2 - x_1}{2\sqrt{x_1 x_2}} - \log \sqrt{\frac{x_2}{x_1}}.$$

That shows that $R_A = R_B$, QED.

14. The given expression multiplied by $e^{x/2}$ is the third derivative of $g(x) = 8e^{x/2}f(x)$, which has the same zeros as $f(x)$. We know (from Rolle's theorem) that between two distinct real zeros of a function there is a zero of its derivative, hence g''' must have at least $5 - 3 = 2$ distinct real zeros, and the same is true for the given expression.
15. In order to form a triangle, a, b, c must verify $a + b > c$, $b + c > a$, and $c + a > b$. Hence the following numbers (Ravi substitution) $x = a + b - c$, $y = b + c - a$, and $z = c + a - b$ must be all positive, and all have the same parity, hence they can be written as $(x, y, z) = (2u, 2v, 2w)$, or $(x, y, z) = (2u - 1, 2v - 1, 2w - 1)$ with $u, v, w \geq 1$. Also we have $a = \frac{x+z}{2}$, $b = \frac{x+y}{2}$, $\frac{y+z}{2}$. Hence

$$\begin{aligned} \sum_{(a,b,c) \in T} \frac{2^a}{3^b 5^c} &= \sum_{u,v,w \geq 1} \frac{2^{u+w}}{3^{u+v} 5^{v+w}} + \sum_{u,v,w \geq 1} \frac{2^{u+w-1}}{3^{u+v-1} 5^{v+w-1}} \\ &= \sum_{u,v,w \geq 1} \frac{2^{u+w}}{3^{u+v} 5^{v+w}} \left(1 + \frac{2^{-1}}{3^{-1} 5^{-1}} \right) \\ &= \frac{17}{2} \sum_{u=1}^{\infty} \frac{2^u}{3^u} \sum_{v=1}^{\infty} \frac{1}{3^v 5^v} \sum_{w=1}^{\infty} \frac{2^w}{5^w} \\ &= \frac{17}{2} \frac{\frac{2}{3}}{1 - \frac{2}{3}} \frac{\frac{1}{15}}{1 - \frac{1}{15}} \frac{\frac{2}{5}}{1 - \frac{2}{5}} \\ &= \frac{17}{21}. \end{aligned}$$

16. The coefficient of x^n in the Taylor series of $(1 - x + x^2)e^x$ for $n = 0, 1, 2$ is $1, 0, \frac{1}{2}$, respectively. For $n \geq 3$, the coefficient of x^n is

$$\begin{aligned} \frac{1}{n!} - \frac{1}{(n-1)!} + \frac{1}{(n-2)!} &= \frac{1 - n + n(n-1)}{n!} \\ &= \frac{n-1}{n(n-2)!}. \end{aligned}$$

If $n-1$ is prime, then since $n-1$ is relatively prime to n and to $(n-2)!$, the lowest-terms numerator is $n-1$, which is prime. If $n-1 = ab$ is composite, then if $a \neq b$, both a and b appear separately in $(n-2)!$, and so the lowest-terms numerator is 1. If $n-1 = a^2$, then either $a = 2$, in which case the coefficient is $\frac{4}{30} = \frac{2}{15}$; or $a > 2$, in which case $n-1 = a^2 > 2a$, whence both a and $2a$ appear in $(n-2)!$, and so $n-1 = a^2$ divides $(n-2)!$ and the lowest-terms numerator is 1.

17. These are the integers with no 0's in their usual base 10 expansion. If the usual base 10 expansion of N is $d_k 10^k + \dots + d_0 10^0$ and one of the digits is 0, then there exists an $i \leq k-1$ such that $d_i = 0$ and $d_{i+1} > 0$; then we can replace $d_{i+1} 10^{i+1} + (0) 10^i$ by $(d_{i+1} - 1) 10^{i+1} + (10) 10^i$ to obtain a second base 10 over-expansion. We claim conversely that if N has no 0's in its usual base 10 expansion, then this standard form is the unique base 10 over-expansion for N . This holds by induction on

the number of digits of N : if $1 \leq N \leq 9$, then the result is clear. Otherwise, any base 10 over-expansion $N = d_k 10^k + \dots + d_1 10 + d_0 10^0$ must have $d_0 \equiv N \pmod{10}$, which uniquely determines d_0 since N is not a multiple of 10; then $(N - d_0)/10$ inherits the base 10 over-expansion $d_k 10^{k-1} + \dots + d_1 10^0$, which must be unique by the induction hypothesis.

- 18.** If the numbers on the faces having a common vertex v have different numbers $a_0 < a_1 < a_2 < a_3 < a_4$, then $a_k \geq k$ for $k = 0, 1, 2, 3, 4$, and $S_v = a_0 + a_1 + a_2 + a_3 + a_4 \geq 0 + 1 + 2 + 3 + 4 = 10$. Adding over the 12 vertices we get $\sum_v S_v \geq 12 \cdot 10 = 120$. In that sum each number occurs three times, one per each vertex of the face, so the sum of the numbers written on the faces of the icosahedron will be greater than or equal to $120/3 = 40$, contradicting the hypothesis that it is 39.

- 19.** We have:

$$\begin{aligned} c(2n)c(2n+2) + c(2n+1)c(2n+3) &= c(n)c(n+1) + (-1)^n c(n)(-1)^{n+1} c(n+1) \\ &= c(n)c(n+1) - c(n)c(n+1) = 0, \end{aligned}$$

so each term in an even position cancels with the next term, and the sum telescopes:

$$\begin{aligned} \sum_{n=1}^{2013} c(n)c(n+2) &= \underbrace{c(1)}_1 \underbrace{c(3)}_{-1} + \underbrace{\sum_{n=2}^{2013} c(n)c(n+2)}_0 \\ &= -1. \end{aligned}$$

- 20.** Assume without loss of generality that $1 < d_1 \leq d_2 \leq \dots \leq d_{12} < 12$.

Note that three numbers $0 < a \leq b \leq c$ are the side lengths of an acute triangle precisely if $a^2 + b^2 > c^2$, so if not such three indices exist we would have $d_i^2 + d_{i+1}^2 \leq d_{i+2}^2$ for $i = 1, \dots, 10$. Consequently $1 < d_1, d_2, d_3^2 \geq d_1^2 + d_2^2 > 2, d_4^2 \geq d_2^2 + d_3^2 > 3$, and analogously $d_5^2 > 5, d_6^2 > 8, d_7^2 > 13, d_8^2 > 21, d_9^2 > 34, d_{10}^2 > 55, d_{11}^2 > 89, d_{12}^2 > 144$, but this last inequality implies $d_{12} > 12$, which is a contradiction.

- 21.** Assume $f(x)$ and $g(x)$ are in S . Then by (ii) the following function is in S :

$$\begin{aligned} f_2(f(x)) + f_2(g(x)) &= \ln(f(x) + 1) + \ln(g(x) + 1) \\ &= \ln((f(x) + 1)(g(x) + 1)) \\ &= \ln(f(x)g(x) + f(x) + g(x) + 1). \end{aligned}$$

Next, by (i) the following function is in S :

$$\begin{aligned} f_1(\ln(f(x)g(x) + f(x) + g(x) + 1)) &= e^{\ln(f(x)g(x) + f(x) + g(x) + 1)} - 1 \\ &= f(x)g(x) + f(x) + g(x). \end{aligned}$$

Finally we can apply (iii), subtract $f(x)$, then $g(x)$, and we get that the following function is in S :

$$(f(x)g(x) + f(x) + g(x)) - f(x) - g(x) = f(x)g(x),$$

where the use of (iii) is justified because the final difference $f(x)g(x)$ is nonnegative.

22. Since the rational numbers are dense in the reals, we can find positive integers a, b such that

$$\frac{3\varepsilon}{hk} < \frac{b}{a} < \frac{4\varepsilon}{hk}.$$

By choosing a and b large enough we can also ensure that $3a^2 > b$. We then have

$$\frac{\varepsilon}{hk} < \frac{b}{3a} < \frac{b}{\sqrt{a^2 + b} + a} = \sqrt{a^2 + b} - a$$

and

$$\sqrt{a^2 + b} - a = \frac{b}{\sqrt{a^2 + b} + a} \leq \frac{b}{2a} < 2\frac{\varepsilon}{hk}.$$

We may then take $m = k^2(a^2 + b), n = h^2a^2$.

23. The largest such k is $\lfloor \frac{n+1}{2} \rfloor = \lceil \frac{n}{2} \rceil$. For n even, this value is achieved by the partition

$$\{1, n\}, \{2, n-1\}, \dots;$$

for n odd, it is achieved by the partition

$$\{n\}, \{1, n-1\}, \{2, n-2\}, \dots$$

One way to see that this is optimal is to note that the common sum can never be less than n , since n itself belongs to one of the boxes. This implies that $k \leq (1 + \dots + n)/n = (n+1)/2$. Another argument is that if $k > (n+1)/2$, then there would have to be two boxes with one number each (by the pigeonhole principle), but such boxes could not have the same sum.

24. - *First solution:* No such sequence exists. If it did, then the Cauchy-Schwartz inequality would imply

$$\begin{aligned} 8 &= (a_1^2 + a_2^2 + \dots)(a_1^4 + a_2^4 + \dots) \\ &\geq (a_1^3 + a_2^3 + \dots)^2 = 9, \end{aligned}$$

contradiction.

- *Second solution:* Suppose that such a sequence exists. If $a_k^2 \in [0, 1]$ for all k , then $a_k^4 \leq a_k^2$ for all k , and so

$$4 = a_1^4 + a_2^4 + \dots \leq a_1^2 + a_2^2 + \dots = 2,$$

contradiction. There thus exists a positive integer k for which $a_k^2 > 1$. However, in this case, for m large, $a_k^{2m} > 2m$ and so $a_1^{2m} + a_2^{2m} + \dots \neq 2m$.

25. The smallest distance is 3, achieved by $A = (0, 0), B = (3, 0), C = (0, 4)$. To check this, it suffices to check that AB cannot equal 1 or 2. (It cannot equal 0 because if two of the points were to coincide, the three points would be collinear.)

The triangle inequality implies that $|AC - BC| \leq AB$, with equality if and only if A, B, C are collinear. If $AB = 1$, we may assume without loss of generality that $A = (0, 0), B = (1, 0)$. To avoid collinearity, we must have $AC = BC$, but this forces $C = (1/2, y)$ for some $y \in \mathbb{R}$, a contradiction.

If $AB = 2$, then we may assume without loss of generality that $A = (0, 0)$, $B = (2, 0)$. The triangle inequality implies $|AC - BC| \in \{0, 1\}$. Also, for $C = (x, y)$, $AC^2 = x^2 + y^2$ and $BC^2 = (2 - x)^2 + y^2$ have the same parity; it follows that $AC = BC$. Hence $c = (1, y)$ for some $y \in \mathbb{R}$, so y^2 and $y^2 + 1 = BC^2$ are consecutive perfect squares. This can only happen for $y = 0$, but then A, B, C are collinear, a contradiction again.

26. Yes, it does follow. Let P be any point in the plane. Let $ABCD$ be any square with center P . Let E, F, G, H be the midpoints of the segments AB, BC, CD, DA , respectively. The function f must satisfy the equations

$$\begin{aligned} 0 &= f(A) + f(B) + f(C) + f(D) \\ 0 &= f(E) + f(F) + f(G) + f(H) \\ 0 &= f(A) + f(E) + f(P) + f(H) \\ 0 &= f(B) + f(F) + f(P) + f(E) \\ 0 &= f(C) + f(G) + f(P) + f(F) \\ 0 &= f(D) + f(H) + f(P) + f(G). \end{aligned}$$

If we add the last four equations, then subtract the first equation and twice the second equation, we obtain $0 = 4f(P)$, whence $f(P) = 0$.

27. Every positive rational number can be uniquely written in lowest terms as a/b for a, b positive integers. We prove the statement in the problem by induction on the largest prime dividing either a or b (where this is considered to be 1 if $a = b = 1$). For the base case, we can write $1/1 = 2!/2!$. For a general a/b , let p be the largest prime dividing either a or b . Assume p divides a (the other case, with p dividing b , is analogous). Then $a/b = p^k a'/b$ for some $k > 0$ and positive integers a', b whose largest prime factors are strictly less than p . Writing $p = p!/(p-1)!$ we have $a/b = (p!)^k \frac{a'}{(p-1)!^k b}$, and all prime factors of a' and $(p-1)!^k b$ are strictly less than p . By the induction hypothesis, $\frac{a'}{(p-1)!^k b}$ can be written as a quotient of products of prime factorials, and so $a/b = (p!)^k \frac{a'}{(p-1)!^k b}$ can as well. This completes the induction.
28. The function $g(x) = f(x, 0)$ works. Substituting $(x, y, z) = (0, 0, 0)$ into the given functional equation yields $f(0, 0) = 0$, whence substituting $(x, y, z) = (x, 0, 0)$ yields $f(x, 0) + f(0, x) = 0$. Finally, substituting $(x, y, z) = (x, y, 0)$ yields $f(x, y) = -f(y, 0) - f(0, x) = g(x) - g(y)$.
29. **First solution:** Pair each entry of the first row with the entry directly below it in the second row. If Alan ever writes a number in one of the first two rows, Barbara writes the same number in the other entry in the pair. If Alan writes a number anywhere other than the first two rows, Barbara does likewise. At the end, the resulting matrix will have two identical rows, so its determinant will be zero.
- Second solution:** Whenever Alan writes a number x in an entry in some row, Barbara writes $-x$ in some other entry in the same row. At the end, the resulting matrix will have all rows summing to zero, so it cannot have full rank.

- 30.** There are at most two such points. For example, the points $(0, 0)$ and $(1, 0)$ lie on a circle with center $(1/2, x)$ for any real number x , not necessarily rational.

On the other hand, with three point A, B, C , we could find the center of the circle as the intersection of the perpendicular bisectors of the segments AB and BC . If A, B , and C are rational, the middle points of AB and BC will be rational, the bisectors will be rational lines (representable by equations with rational coefficients), and their intersection will be rational.

- 31.** The only such α are $2/3, 3/2, (13 \pm \sqrt{601})/12$.

In fact, let C_1 and C_2 be the curves $y = \alpha x^2 + \alpha x + \frac{1}{24}$ and $x = \alpha y^2 + \alpha y + \frac{1}{24}$, respectively, and let L be the line $y = x$. We consider three cases.

If C_1 is tangent to L , then the point of tangency (x, x) satisfies

$$2\alpha x + \alpha = 1, \quad x = \alpha x^2 + \alpha x + \frac{1}{24};$$

by symmetry, C_2 is tangent to L there, so C_1 and C_2 are tangent. Writing $\alpha = 1/(2x + 1)$ in the first equation and substituting into the second, we must have

$$x = \frac{x^2 + x}{2x + 1} + \frac{1}{24},$$

which simplifies to $0 = 24x^2 - 2x - 1 = (6x + 1)(4x - 1)$, or $x \in \{1/4, -1/6\}$. This yields $\alpha = 1/(2x + 1) \in \{2/3, 3/2\}$. If C_1 does not intersect L , then C_1 and C_2 are separated by L and so cannot be tangent.

If C_1 intersects L in two distinct points P_1, P_2 , then it is not tangent to L at either point. Suppose at one of these points, say P_1 , the tangent to C_1 is perpendicular to L ; then by symmetry, the same will be true of C_2 , so C_1 and C_2 will be tangent at P_1 . In this case, the point $P_1 = (x, x)$ satisfies

$$2\alpha x + \alpha = -1, \quad x = \alpha x^2 + \alpha x + \frac{1}{24};$$

writing $\alpha = -1/(2x + 1)$ in the first equation and substituting into the second, we have

$$x = -\frac{x^2 + x}{2x + 1} + \frac{1}{24},$$

or $x = (-23 \pm \sqrt{601})/72$. This yields $\alpha = -1/(2x + 1) = (13 \pm \sqrt{601})/12$.

If instead the tangents to C_1 at P_1, P_2 are not perpendicular to L , then we claim there cannot be any point where C_1 and C_2 are tangent. Indeed, if we count intersections of C_1 and C_2 (by using C_1 to substitute for y in C_2 , then solving for y), we get at most four solutions counting multiplicity. Two of these are P_1 and P_2 , and any point of tangency counts for two more. However, off of L , any point of tangency would have a mirror image which is also a point of tangency, and there cannot be six solutions. Hence we have now found all possible α .

- 32.** The problem fails if f is allowed to be constant, e.g., take $f(n) = 1$. We thus assume that f is nonconstant. Write $f(n) = \sum_{i=0}^d a_i n^i$ with $a_i > 0$. Then

$$\begin{aligned} f(f(n) + 1) &= \sum_{i=0}^d a_i (f(n) + 1)^i \\ &\equiv f(1) \pmod{f(n)}. \end{aligned}$$

If $n = 1$, then this implies that $f(f(n) + 1)$ is divisible by $f(n)$. Otherwise, $0 < f(1) < f(n)$ since f is nonconstant and has positive coefficients, so $f(f(n) + 1)$ cannot be divisible by $f(n)$.

- 33.** We change to cylindrical coordinates, i.e., we put $r = \sqrt{x^2 + y^2}$. Then the given inequality is equivalent to

$$r^2 + z^2 + 8 \leq 6r,$$

or

$$(r - 3)^2 + z^2 \leq 1.$$

This defines a solid of revolution (a solid torus); the area being rotated is the disc $(x - 3)^2 + z^2 \leq 1$ in the xz -plane. By Pappus's theorem, the volume of this equals the area of this disc, which is π , times the distance through which the center of mass is being rotated, which is $(2\pi)3$. That is, the total volume is $6\pi^2$.

- 34.** Let $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of x . For $i = 0, \dots, n$, put $s_i = x_1 + \dots + x_i$ (so that $s_0 = 0$). Sort the numbers $\{s_0\}, \dots, \{s_n\}$ into ascending order, and call the result t_0, \dots, t_n . Since $0 = t_0 \leq \dots \leq t_n < 1$, the differences

$$t_1 - t_0, \dots, t_n - t_{n-1}, 1 - t_n$$

are nonnegative and add up to 1. Hence (as in the pigeonhole principle) one of these differences is no more than $1/(n+1)$; if it is anything other than $1 - t_n$, it equals $\pm(\{s_i\} - \{s_j\})$ for some $0 \leq i < j \leq n$. Put $S = \{x_{i+1}, \dots, x_j\}$ and $m = \lfloor s_i \rfloor - \lfloor s_j \rfloor$; then

$$\begin{aligned} \left| m + \sum_{s \in S} s \right| &= |m + s_j - s_i| \\ &= |\{s_j\} - \{s_i\}| \\ &\leq \frac{1}{n+1}, \end{aligned}$$

as desired. In case $1 - t_n \leq 1/(n+1)$, we take $S = \{x_1, \dots, x_n\}$ and $m = -\lfloor s_n \rfloor$, and again obtain the desired conclusion.

- 35.** We proceed by induction, with base case $1 = 2^0 3^0$. Suppose all integers less than $n - 1$ can be represented. If n is even, then we can take a representation of $n/2$ and multiply each term by 2 to obtain a representation of n . If n is odd, put $m = \lfloor \log_3 n \rfloor$,

so that $3^m \leq n < 3^{m+1}$. If $3^m = n$, we are done. Otherwise, choose a representation $(n - 3^m)/2 = s_1 + \cdots + s_k$ in the desired form. Then

$$n = 3^m + 2s_1 + \cdots + 2s_k,$$

and clearly none of the $2s_i$ divide each other or 3^m . Moreover, since $2s_i \leq n - 3^m < 3^{m+1} - 3^m$, we have $s_i < 3^m$, so 3^m cannot divide $2s_i$ either. Thus n has a representation of the desired form in all cases, completing the induction.

- 36.** Take $P(x, y) = (y - 2x)(y - 2x - 1)$. To see that this works, first note that if $m = \lfloor a \rfloor$, then $2m$ is an integer less than or equal to $2a$, so $2m \leq \lfloor 2a \rfloor$. On the other hand, $m + 1$ is an integer strictly greater than a , so $2m + 2$ is an integer strictly greater than $2a$, so $\lfloor 2a \rfloor \leq 2m + 1$.
- 37.** Yes. Suppose otherwise. Then there would be an N such that $S(N) < 80\%$ and $S(N + 1) > 80\%$; that is, O'Keal's free throw percentage is under 80% at some point, and after one subsequent free throw (necessarily made), her percentage is over 80%. If she makes m of her first N free throws, then $m/N < 4/5$ and $(m + 1)/(N + 1) > 4/5$. This means that $5m < 4N < 5m + 1$, which is impossible since then $4N$ is an integer between the consecutive integers $5m$ and $5m + 1$.
- 38.** We have

$$(m + n)^{m+n} > \binom{m+n}{m} m^m n^n$$

because the binomial expansion of $(m + n)^{m+n}$ includes the term on the right as well as some others. Rearranging this inequality yields the claim.

- 39.** There are n such sums. More precisely, there is exactly one such sum with k terms for each of $k = 1, \dots, n$ (and clearly no others). To see this, note that if $n = a_1 + a_2 + \cdots + a_k$ with $a_1 \leq a_2 \leq \cdots \leq a_k \leq a_1 + 1$, then

$$\begin{aligned} ka_1 &= a_1 + a_1 + \cdots + a_1 \\ &\leq n \leq a_1 + (a_1 + 1) + \cdots + (a_1 + 1) \\ &= ka_1 + k - 1. \end{aligned}$$

However, there is a unique integer a_1 satisfying these inequalities, namely $a_1 = \lfloor n/k \rfloor$. Moreover, once a_1 is fixed, there are k different possibilities for the sum $a_1 + a_2 + \cdots + a_k$: if i is the last integer such that $a_i = a_1$, then the sum equals $ka_1 + (i - 1)$. The possible values of i are $1, \dots, k$, and exactly one of these sums comes out equal to n , proving our claim.

40. Assume without loss of generality that $a_i + b_i > 0$ for each i (otherwise both sides of the desired inequality are zero). Then the AM-GM inequality gives

$$\begin{aligned} \left(\frac{a_1 \cdots a_n}{(a_1 + b_1) \cdots (a_n + b_n)} \right)^{1/n} &\leq \frac{1}{n} \left(\frac{a_1}{a_1 + b_1} + \cdots + \frac{a_n}{a_n + b_n} \right), \\ \left(\frac{b_1 \cdots b_n}{(a_1 + b_1) \cdots (a_n + b_n)} \right)^{1/n} &\leq \frac{1}{n} \left(\frac{b_1}{a_1 + b_1} + \cdots + \frac{b_n}{a_n + b_n} \right). \end{aligned}$$

Adding these two inequalities we get

$$\left(\frac{a_1 \cdots a_n}{(a_1 + b_1) \cdots (a_n + b_n)} \right)^{1/n} + \left(\frac{b_1 \cdots b_n}{(a_1 + b_1) \cdots (a_n + b_n)} \right)^{1/n} = 1.$$

Clearing denominators yields the desired result.

41. By differentiating $P_n(x)/(x^k - 1)^{n+1}$, we find that

$$P_{n+1}(x) = (x^k - 1)P_n'(x) - (n+1)kx^{k-1}P_n(x).$$

Substituting $x = 1$ yields $P_{n+1}(1) = -(n+1)kP_n(1)$. Since $P_0(1) = 1$, an easy induction gives $P_n(1) = (-k)^n n!$ for all $n \geq 0$.

Note: one can also argue by expanding in Taylor series around 1. Namely, we have

$$\frac{1}{x^k - 1} = \frac{1}{k(x-1) + \cdots} = \frac{1}{k}(x-1)^{-1} + \cdots,$$

so

$$\frac{d^n}{dx^n} \frac{1}{x^k - 1} = \frac{(-1)^n n!}{k(x-1)^{-n-1}}$$

and

$$\begin{aligned} P_n(x) &= (x^k - 1)^{n+1} \frac{d^n}{dx^n} \frac{1}{x^k - 1} \\ &= (k(x-1) + \cdots)^{n+1} \\ &\quad \left(\frac{(-1)^n n!}{k} (x-1)^{-n-1} + \cdots \right) \\ &= (-k)^n n! + \cdots. \end{aligned}$$

42. Draw a great circle through two of the points. There are two closed hemispheres with this great circle as boundary, and each of the other three points lies in one of them. By the pigeonhole principle, two of those three points lie in the same hemisphere, and that hemisphere thus contains four of the five given points.
43. The hypothesis implies $((b * a) * b) * (b * a) = b$ for all $a, b \in S$ (by replacing a by $b * a$), and hence $a * (b * a) = b$ for all $a, b \in S$ (using $(b * a) * b = a$).
44. - First solution: Let a be an even integer such that $a^2 + 1$ is not prime. (For example, choose $a \equiv 2 \pmod{5}$, so that $a^2 + 1$ is divisible by 5.) Then we can write $a^2 + 1$ as a difference of squares $x^2 - b^2$, by factoring $a^2 + 1$ as rs with $r \geq s > 1$, and setting

$x = (r+s)/2$, $b = (r-s)/2$. Finally, put $n = x^2 - 1$, so that $n = a^2 + b^2$, $n+1 = x^2 + 0$, $n+2 = x^2 + 1$.

- Second solution: It is well-known that the equation $x^2 - 2y^2 = 1$ has infinitely many solutions (the so-called ‘‘Pell’’ equation). Thus setting $n = 2y^2$ (so that $n = y^2 + y^2$, $n+1 = x^2 + 0^2$, $n+2 = x^2 + 1^2$) yields infinitely many n with the desired property.

- Third solution: As in the first solution, it suffices to exhibit x such that $x^2 - 1$ is the sum of two squares. We will take $x = 3^{2^n}$, and show that $x^2 - 1$ is the sum of two squares by induction on n : if $3^{2^n} - 1 = a^2 + b^2$, then

$$\begin{aligned} (3^{2^{n+1}} - 1) &= (3^{2^n} - 1)(3^{2^n} + 1) \\ &= (3^{2^{n-1}}a + b)^2 + (a - 3^{2^{n-1}}b)^2. \end{aligned}$$

45. Note that if $r(x)$ and $s(x)$ are any two functions, then

$$\max(r, s) = (r + s + |r - s|)/2.$$

Therefore, if $F(x)$ is the given function, we have

$$\begin{aligned} F(x) &= \max\{-3x - 3, 0\} - \max\{5x, 0\} + 3x + 2 \\ &= (-3x - 3 + |3x - 3|)/2 \\ &\quad - (5x + |5x|)/2 + 3x + 2 \\ &= |(3x - 3)/2| - |5x/2| - x + \frac{1}{2}, \end{aligned}$$

so we may set $f(x) = (3x - 3)/2$, $g(x) = 5x/2$, and $h(x) = -x + \frac{1}{2}$.

46. Consider the plane containing both the axis of the cone and two opposite vertices of the cube’s bottom face. The cross section of the cone and the cube in this plane consists of a rectangle of sides s and $s\sqrt{2}$ inscribed in an isosceles triangle of base 2 and height 3, where s is the side-length of the cube. (The $s\sqrt{2}$ side of the rectangle lies on the base of the triangle.) Similar triangles yield $s/3 = (1 - s\sqrt{2}/2)/1$, or $s = (9\sqrt{2} - 6)/7$.

47. We may discard any solutions for which $a_1 \neq a_2$, since those come in pairs; so assume $a_1 = a_2$. Similarly, we may assume that $a_3 = a_4$, $a_5 = a_6$, $a_7 = a_8$, $a_9 = a_{10}$. Thus we get the equation

$$2/a_1 + 2/a_3 + 2/a_5 + 2/a_7 + 2/a_9 = 1.$$

Again, we may assume $a_1 = a_3$ and $a_5 = a_7$, so we get $4/a_1 + 4/a_5 + 2/a_9 = 1$; and $a_1 = a_5$, so $8/a_1 + 2/a_9 = 1$. This implies that $(a_1 - 8)(a_9 - 2) = 16$, which by counting has 5 solutions. Thus N_{10} is odd.

48. Let $x = a - 1$, $y = b - 1$, $z = 1$. Then:

$$\begin{aligned}xy + xz + yz + 1 &= (a - 1)(b - 1) + (a - 1) \cdot 1 + (b - 1) \cdot 1 + 1 \\ &= ab - a - b + 1 + a - 1 + b - 1 + 1 \\ &= ab.\end{aligned}$$