Linear and Abstract Algebra

There are often problems in Putnam exams that can be categorized as *algebra* either linear algebra or abstract algebra (usually group theory). Here our emphasis is on linear algebra but we will mention few group theory related problems that have appeared in previous Putnam competitions.

- Linear Algebra $\star$
  - Many problems just require clever algebraic manipulations of matrix expressions.
    * Let A, B be $n \times n$ matrices with $AB = rA + sB$ where r, s are non-zero scalars. Prove that A and B commute. (Hint: Write this in the form of $XY = I_n$ and remember that $XY = I_n$ always implies $YX = I_n$.)
    * Let A, B, C be real square matrices of the same order, and suppose A is invertible. Prove that $(A - B)C = BA^{-1} \Rightarrow C(A - B) = A^{-1}B$.
    * A related Putnam problem $\star$ 1991 A2.
  - Many problems can be posed based on basic properties of vector spaces and linear transformations such as dimension and rank.
    * Let V be a 10-dimensional real vector space and U_1, U_2 two linear subspaces with $\dim U_1 = 3, \dim U_2 = 6$. Let ϵ be the set of all linear maps $T : V \to V$ which have $T(U_1) \subseteq U_1$, $T(U_2) \subseteq U_2$. Calculate the dimension of ϵ (again, all as real vector spaces). (Hint: Pick an ordered basis for U_1 and extend it first to an ordered basis of U_2 and then to an ordered basis of the whole space V. How the matrix presentation of an element of ϵ looks like in this basis?)
    * Let A, B be two $n \times n$ matrices with $\operatorname{rank}(AB - BA) = 1$. Show that $(AB - BA)^2 = 0$. (Hint: When the rank is one, there exists a vector such that each column can be described as a scalar multiple of it. Moreover, there is another constraint on matrices of the form $AB - BA$: the sum of entries along its diagonal (the so called *trace* of the matrix) vanishes. Now the direct computation of entries of $(AB - BA)^2$ yields the desired result.)
    * A related Putnam problem $\star$ 2004 B4.
  - There have been problems on 2×2 matrices in Putnam exams. An extremely useful fact about a 2×2 matrix A is that A^2 (and hence all higher powers of A) is a linear combination of I_2, A: $A^2 - \operatorname{tr}(A).A + \det(A).I_2 = 0$ where "tr" is the symbol for the trace of a square matrix, the sum of entries along the diagonal. (This is a special case of the famous *Cayley-Hamilton theorem* which asserts that a $n \times n$ matrix satisfies its characteristic equation. For a 2×2 matrix A, the characteristic polynomial is $\lambda^2 - \operatorname{tr}(A)\lambda + \det(A)$.)
    * (Putnam 2015 B3) Let S be the set of all 2×2 real matrices $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ whose entries a, b, c, d (in that order) form an arithmetic progression. Find all matrices M in S for which there is some integer $k > 1$ such that M^k is also in S. (Hint: As mentioned above, M^k is a linear combination of M and I_2. A scalar multiple of an element of S lies in S as well but the sum of an element of S with a scalar matrix is not in S unless the scalar matrix is zero. Hence M^k must be a multiple of M. Now there are two possibilities: either M is not invertible or some power of M is a scalar

matrix. In each case there are implications about eigenvalues of M. By writing down the characteristic polynomial of M in terms of the initial term and the difference of the arithmetic progression, you can deal with each case separately.)

    * Let P be a complex polynomial and let A and B be 2×2 complex matrices such that $AB \neq BA$ and $P(AB) = P(BA)$. Prove that $P(AB) = cI_2$ for some complex number c. (Hint: How characteristic polynomials of AB and BA are related? If we divide P by that polynomial, what can you say about the remainder?)
    * Few related Putnam problems $\star$ 1994 A4, 1994 B4, 1996 B4.

- Several Putnam problems on determinant are listed below. The book Problems and Theorems in Linear Algebra is a very rich source of tricky linear algebra problems. Check the first chapter for more exercises on determinants.

  1992 B5, 1995 B3, 1999 B5, 2002 A4, 2009 A3, 2014 A2.

- More advanced topics in linear algebra such as *diagonalizable matrices*, *characteristics and minimal polynomials*, *eigenvalues and eigenvectors* and properties of special classes of matrices such as *nilpotent, orthogonal, unitary, symmetric* or *Hermitian* matrices may also be useful.
    * Let $A, B \in \mathrm{M_n}(\mathbb{C})$ be two $n \times n$ matrices such that $A^2B + BA^2 = 2ABA$. Prove that $AB - BA$ is nilpotent. (Hint: You can use the fact that X is nilpotent iff $\mathrm{tr}(X^k) = 0$ for all $k \in \mathbb{N}$. Now prove by induction that any $(AB - BA)^k$ is traceless. The case of $k = 1$ is a basic property of trace.)
    * Let A be a real $n \times n$ matrix satisfying $A + A^t = I_n$. Show that $\det A > 0$. (Hint: The real matrix $A - \frac{1}{2}I_n$ is anti-symmetric. What does this imply about eigenvalues of this matrix and hence those of A?)
    * (Putnam 1988, A6) If a linear transformation A on an n-dimensional vector space has $n + 1$ eigenvectors such that any n of them are linearly independent, does it follow that A is a scalar multiple of the identity? Prove your answer. (Hint: Having n linearly independent eigenvectors means that this transformation is diagonalizable. In general, what are eigenvectors of a diagonal matrix? Then, what are possible choices for the $n + 1^{\text{th}}$ eigenvector under consideration?)

- More linear algebra problems:
    * Let $A = [a_1|a_2|\cdots|a_n]$ be an $n \times n$ real matrix where $a_i \in \mathbb{R}^n$. Prove that $|\det(A)| \leq \prod_{i=1}^{n} |a_i|$ where $|.|$ denotes the Euclidean norm. (Hint: remember that $|\det(A)|$ is the volume of certain parallelepiped.)
    * Given $n + 2$ vectors $v_1, \ldots, v_{n+2}$ in $\mathbb{R}^{n+2}$, prove that there are $i \neq j$ such that $\langle v_i, v_j \rangle \geq 0$. (Hint: Induction on n.)
    * Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis for $\mathbb{R}^n$ and and $f_1, \ldots, f_n \in \mathbb{R}^n$ be vectors satisfying $\forall 1 \leq i \leq n : |f_i - e_i| \leq \epsilon$ where $\epsilon \geq 0$ and $|.|$ denotes the Euclidean norm. Show that the dimension of the subspace spanned by f_i's is at least $n(1 - \epsilon^2)$. (Hint: Denote this dimension by k and pick an orthonormal basis $\{e'_1, \ldots, e'_k\}$ for the subspace $\langle f_1, \ldots, f_n \rangle$. Describe each f_i in terms of this basis and plug it back in the inequality $|f_i - e_i| \leq \epsilon$. Raise to the power of two in order to get rid of $|.|$ and simplify inner products using orthonormality. Then add up all inequalities and try to deduce $k \geq n(1 - \epsilon^2)$.)
    * (Putnam 2015 A6) Let n be a positive integer. Suppose that A, B, and M are $n \times n$ matrices with real entries such that $AM = MB$, and such that A and B have the same characteristic polynomial. Prove that $\det(A - MX) = \det(B - XM)$ for every $n \times n$ matrix X with real entries. (Hint: It is a standard fact that there are $n \times n$ invertible matrices P, Q such that PXQ is in the block form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ where $r := \mathrm{rank}(X)$. First solve the problem for this specific X and then reduce the general case to this one by modifying A, B, X, M.)

* Among previous Putnam problems ⋆

1985 B6, 1986 B6, 1987 B5, 1990 A5, 1992 B6, 2008 A2.

- Abstract Algebra ⋆ Apparently such Putnam problems are more about group theory. Basic facts about the *order* of an element or a subgroup of a finite group and its relation to the cardinality (e.g. *Lagrange* and *Cauchy* theorems), the notions of *index of a subgroup* and *group action*, *the structure theorem of finitely generated abelian groups* and finally basic properties of *symmetric groups* may come in handy. Check the following Putnam problems:

  1990 B4, 1997 A4, 2007 A5, 2008 A6, 2009 A5, 2010 A5, 2011 A6.