Math C37-2 Final Exam

Mar. 15, 1999, 9:00 AM

Each problem is worth 20 points

1. Define *group* and *Abelian group*. Prove that a group G is Abelian if and only if $(gh)^2 = g^2h^2$ for all $g, h \in G$.
2. (a) Define **normal subgroup**.
   (b) Give an example of a group G with a subgroup H which is **not** normal. Justify your answer.
   (c) Prove that the intersection of two normal subgroups is a normal subgroup.
3. Define the **center** of a group G. Prove that the center of any group is a normal subgroup.
4. State and prove the First Isomorphism Theorem for groups.
5. Let $\mathcal{R} = \{a + b\sqrt{5} \mid a, b \in \mathbb{Q}\}$, where $\mathbb{Q}$ denotes the rational numbers. Prove that $\mathcal{R}$ is a field.
   EXTRA CREDIT: There are only two ring isomorphisms of the field $\mathcal{R}$ to itself. One is the identity. What is the other? Prove that it is the only other one.
6. Define **ideal.** Prove that the kernel of a ring homomorphism is an ideal.
7. Define **prime ideal, maximal ideal** and **principal ideal.** Prove that if $\mathcal{R}$ is a commutative ring with unity then $\mathcal{I}$ is a maximal ideal if and only if $\mathcal{R}/\mathcal{I}$ is a field.
8. Consider the commutative ring $\mathcal{R} = \mathbb{Z}_4 \oplus \mathbb{Z}_8$. Find a zero divisor $(a, b) \in \mathcal{R}$. Find a zero divisor $(a, b) \in \mathcal{R}$ with $a \neq 0$ and $b \neq 0$. What is the unity of $\mathcal{R}$? List all of the units of $\mathcal{R}$, i.e., all of its elements which have multiplicative inverses.
9. Let S_n denote the symmetric group of degree n. Assume the result proven in class that if there are k 2-cycles, $\beta_1, \beta_2, \ldots, \beta_k$ in S_n whose product $\beta_1\beta_2\ldots\beta_k$ equals the identity of S_n, then k is even.
   Use this to prove that if an element x of S_n can be written in one way as a product of an even number of 2-cycles then every way of writing it as a product of 2-cycles has an even number of 2-cycles. The set of these "even 2-cycle" elements is denoted A_n. Prove it is a normal subgroup of S_n.
10. Let $\mathbb{F}$ be a field and $\mathbb{F}[x]$ be the ring of polynomials with coefficients in $\mathbb{F}$.
    (a) If $f(x) \in \mathbb{F}[x]$ and $a \in \mathbb{F}$ prove that the function $\varphi : \mathbb{F}[x] \to \mathbb{F}$ defined by $\varphi(f(x)) = f(a)$ is a ring homomorphism.
    (b) Prove that the kernel of φ is the ideal $\langle x - a \rangle$.
    (c) If $\mathbb{Q}$ denotes the field of rational numbers and $\psi : \mathbb{Q}[x] \to \mathbb{Q}$ is a non-trivial ring homomorphism prove that there is a $b \in \mathbb{Q}$ such that $\psi(f(x)) = f(b)$.