NAME:

Math B90–3

Score: ____________ out of 200

June 7, 1996

Final Exam

PLEASE EXPLAIN YOUR ANSWERS.
BE CLEAR AND CONCISE.
BE NEATER THAN THE PROFESSOR IS AT THE BOARD.

Question 1. (12 points.) Find all solutions (if there are any) to the following system of linear equations by using Gauss-Jordan elimination:

$$\begin{array}{rcrcrcr} 4x_1 & + & 2x_2 & + & x_3 & = & -2 \\ 8x_1 & & & + & 6x_3 & = & 8 \\ -4x_1 & + & 2x_2 & - & 5x_3 & = & -6 \end{array}$$

Question 2. (10 points.) Let A and B be square matrices (of the same size) with $AB \neq BA$. Prove that

$$(A+B)(B+A) = (B+A)(A+B).$$

Question 3. (15 points.) Prove that for the motion of a particle described by the vector-valued function $\mathbf{r}(t)$ with velocity $\mathbf{v}(t)$, speed $v(t)$, and acceleration $\mathbf{a}(t)$, we always have

$$\mathbf{v}(t) \cdot \mathbf{a}(t) = \frac{1}{2}\frac{d}{dt}v^2(t).$$

Question 4. (15 points.) Let $C(-1,1)$ be the space of continuous (real) functions on the interval $[-1,1]$ with the usual inner product, $(f,g) = \int_{-1}^{1} f(x)g(x)dx$. Let S_1 be the set of all odd functions (f is odd if we always have $f(x) = -f(-x)$) and S_2 the set of all even functions (f is even if $f(x) = f(-x)$). We have in the past observed that both S_1 and S_2 are subspaces of $C(-1,1)$.

i) What is $S_1 \cap S_2$? (Note: we showed on a quiz once that this must be a subspace.)

ii) Prove that $S_1 \cup S_2$ is not a subspace.

iii) $L(S_1 \cup S_2)$ is certainly a subspace. What subspace is it? (Prove your answer.)

Question 5. (25 points.) Define the following terms clearly and precisely:

i) The inverse of a (square) matrix A.

ii) Zero (in a vector space).

iii) Basis (of a vector space).

iv) Line (in $\mathbf{R}^n$).

v) Perpendicular (for two planes M_1 and M_2 in $\mathbf{R}^n$).

Question 6. (24 points.) For each of the following, tell if it is a vector space under the natural actions of addition and scalar multiplication. If it is, describe the zero element. If it is not, prove that it is not.

i) The set of all differentiable real functions on the interval $[0, 1]$.

ii) The set of all vectors $x = (x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ in $\mathbf{R}^7$ which have $|x_1| \leq |x_6|$.

iii) The set of all vectors in $\mathbf{R}^7$ which are orthogonal to both $(1, 2, 3, 4, 5, 6, 7)$ and $(0, 0, 0, 0, 0, 0, 1)$.

iv) The set of all polynomials of odd degree.

v) The set of all upper–triangular $n \times n$ (real) matrices.

vi) The set of all 2×2 (real) matrices which have determinant ≥ 0.

Question 7. (18 points.) Let M be the plane $\{s(1,0,1)+t(-1,-1,-1)|\ s,t\in \mathbf{R}\}$. Recall that since M is a plane through the origin, it is a subspace of $\mathbf{R}^3$.

i) What is an orthogonal basis for the subspace M of $\mathbf{R}^3$?

ii) What vector in M is closest to the vector $(1,0,0)$?

iii) What is the distance from from the vector $(1,0,0)$ to the plane M?

iv) Write down a (nonzero) vector orthogonal to the plane M.

Question 8. (20 points.) For λ any complex number, let

$$A = \begin{bmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & 2-\lambda & 0 & 0 \\ 4 & 4 & -\lambda & 0 \\ 1 & 2 & 3 & -6-\lambda \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -7 & \lambda^2 & 0 & 0 \\ 3 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

i) What is the determinant of A?

ii) For what values (if any) of λ is A singular?

iii) What is the rank of B? (Note: this may depend on λ. If it does, then say explicitly what the rank of B is for any possible λ.)

iv) What is $B^t A$?

Question 9. (15 points.) Prove that for functions f and g in $C(0,1)$, we always have

$$\int_0^1 |f(x)+g(x)|^2 dx \leq \left(\left(\int_0^1 |f(x)|^2 dx \right)^{\frac{1}{2}} + \left(\int_0^1 |g(x)|^2 dx \right)^{\frac{1}{2}} \right)^2 .$$

(Hint: $C(0,1)$ can be thought of as a Euclidean space with a familiar inner product and norm. Interpret the quantities of interest in these terms.)

Question 10. (10 points.) What is the length of the curve given by the parametric equation $\mathbf{r}(t) = (4t^3, 3t^3, 6)$ for $0 \leq t \leq 1$? What type of curve is this? (Explain.)

Question 11. (24 points.) Let $T : P_5 \to \mathbf{R}^3$ be a linear operator on the space of polynomials of degree ≤ 5 defined as follows on the usual basis $\{1, x, x^2, x^3, x^4, x^5\}$ for P_5: $T(1) = (1,0,0)$, $T(x) = (1,2,1)$, $T(x^2) = (5,0,0)$, $T(x^3) = (-1,6,3)$, $T(x^4) = (0,0,0)$, and $T(x^5) = (0,2,1)$.

i) Write down $m(T)$, the matrix representation of T (relative to the above basis for P_5 and the usual basis of unit coordinate vectors for $\mathbf{R}^3$.

ii) What is the rank of $m(T)$?

iii) What is the dimension of $T(P_5)$?

iv) What is the dimension of $N(T)$ (the kernel of T)?

v) Give a basis for $T(P_5)$.

vi) What is $T\big((x^2+1)(x^2-1)\big)$?

Question 12. (12 points.)

i) What is the dimension of the linear span of the following set of vectors (in $\mathbf{R}^4$): $\{(1,3,4,0),(1,0,0,0),(-5,-2,-1,0),(0,4,0,0),(-1,-1,4,0),(0,0,1,0)\}$? Explain.

ii) Let S be a subset of $\mathbf{R}^4$ which contains exactly three (different) elements. What are the possible values for the dimension of $L(S)$? For each possible value of the dimension, give an example of a subset S such that $L(S)$ has this dimension.