M219, Final Test.

March, 20, 2003.

Books and notes should not be used during the test. You don't have to follow the order, it usually works best if you first do the problems that look easiest to you. Good luck!

(1) (20 points) Let $H \subset \mathbb{R}^3$ be the span of $\mathbf{v}_1 = (1, -1, 0)$ and $\mathbf{v}_2 = (1, 1, 2)$. Find the point in H which is closest to the point $\mathbf{x} = (1, 3, 1)$.

(2) (20 points) Find the inverse matrix to the following 3×3 matrix.

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

(3) Let A be the following 2×2 matrix.

$$\begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$$

a) (20 points) Diagonalize A, i.e. find an invertible matrix P, and a diagonal matrix D such that $P^{-1}AP = D$.

b) (20 points) Compute A^6.

(4) (25 points) $A = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]$ is a 3×3 matrix whose determinant equals -3. Find $det(B)$ where $B = [2\mathbf{a}_1, 3\mathbf{a}_1 + \mathbf{a}_2, 7\mathbf{a}_1 + 8\mathbf{a}_2 + 2\mathbf{a}_3]$.

(5) (25 points) The quadratic form Q on $\mathbb{R}^2$ is defined by $Q(x, y) = x^2 + y^2 + xy$. Find an orthogonal matrix P such that the matrix of the quadratic form $Q'(\mathbf{x}) = Q(P\mathbf{x})$ is diagonal.

(6) (20 points) Which of the following quadratic forms are positive definite:

$Q_1(x, y) = x^2 - y^2$; $Q_2(x, y) = xy$; $Q_3(x, y) = 2x^2 + y^2$; $Q_4(x, y) = y^2$?

(Explain your answer)

(7) (25 points) A is a 4×3 matrix. Its first two columns are non-zero and non-propotional; and its 3rd column is the sum of the first two. Find a basis in $Nul(A)$. Prove your answer.

(8) (25 points) Let A be the matrix

$$\begin{pmatrix} 3 & -2 \\ 4 & -3 \end{pmatrix}$$

Find the eigenvalues of the matrix $B = A^{100} - A + I_2$.

(Hint: find the eigenvalues of A first, and express the eigenvalues of B through the eigenvalues of A. You don't have to compute B.)

Bonus problem (30 points) The quadratic form Q is defined by $Q(x, y) = x^2 + y^2 + xy$. List all orthogonal matrices P such that $Q(P\mathbf{x}) = Q(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^2$. Prove your answer.