

Math 214-3 Spring 2003 Answers to Review Problems II

1. $\mathbf{a}(t) = -g\mathbf{j}$

$$\mathbf{v}(t) = -gt\mathbf{j} + \mathbf{v}_0 = \frac{128}{\sqrt{2}}\mathbf{i} + \left(\frac{128}{\sqrt{2}} - 32t\right)\mathbf{j}$$

$$\mathbf{r}(t) = \frac{128}{\sqrt{2}}t\mathbf{i} + \left(\frac{128}{\sqrt{2}}t - 16t^2\right)\mathbf{j} \quad (\text{because } \mathbf{r}(0) = \mathbf{0})$$

(a) The canonball reaches the maximum height when

$$\frac{128}{\sqrt{2}} - 32t = 0, \quad \text{so } t = 2\sqrt{2} \approx 2.83 \text{ sec.}$$

Then the maximum height is

$$\frac{128}{\sqrt{2}} \cdot 2\sqrt{2} - 16(2\sqrt{2})^2 = 128 \text{ ft.}$$

(b) Setting $\frac{128}{\sqrt{2}}t - 16t^2 = 0$, we obtain $t = 0$ or $t = 4\sqrt{2}$. That is, the canonball returns to the ground after $4\sqrt{2} \approx 5.66$ seconds.

2. $\mathbf{r}'(t) = \langle 2 \cos t, 3, -2 \sin t \rangle$, $\mathbf{r}''(t) = \langle -2 \sin t, 0, -2 \cos t \rangle$.

When $t = \frac{\pi}{2}$, $\mathbf{r}'\left(\frac{\pi}{2}\right) = \langle 0, 3, -2 \rangle$, $\mathbf{r}''\left(\frac{\pi}{2}\right) = \langle -2, 0, 0 \rangle$.

$$\mathbf{r}'\left(\frac{\pi}{2}\right) \times \mathbf{r}''\left(\frac{\pi}{2}\right) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 3 & -2 \\ -2 & 0 & 0 \end{vmatrix} = 4\mathbf{j} + 6\mathbf{k}$$

$$\kappa\left(\frac{\pi}{2}\right) = \frac{|\mathbf{r}'\left(\frac{\pi}{2}\right) \times \mathbf{r}''\left(\frac{\pi}{2}\right)|}{|\mathbf{r}'\left(\frac{\pi}{2}\right)|^3} = \frac{\sqrt{4^2 + 6^2}}{(3^2 + (-2)^2)^{3/2}} = \frac{2}{13}$$

3. (a) The surface is a plane that passes through the origin and contains the vectors $\langle 2, 1, 0 \rangle$ and $\langle 3, 0, 1 \rangle$.

(b) The surface is the unit sphere $x^2 + y^2 + z^2 = 1$.

4. The parametric equations are

$$x = x, \quad y = y, \quad z = x^2 + y^2 \quad \text{where } x^2 + y^2 \leq 4$$

or

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = r^2 \quad \text{where } r \leq 2, \quad 0 \leq \theta \leq 2\pi$$

5. (a) The domain is $\{(x, y) \mid x^2 + y^2 \geq 1\}$.

(b) Setting $f(x, y) = \sqrt{x^2 + y^2 - 1} = k$, we obtain $x^2 + y^2 = k^2 + 1$, that is, the level curves are circles centered at the origin with radius $\sqrt{k^2 + 1}$.

(c) $z^2 = x^2 + y^2 - 1 \Rightarrow x^2 + y^2 - z^2 = 1$ with $z \geq 0$. The surface is the upper half of a hyperboloid of one sheet.

$$(d) \frac{\partial f}{\partial x} = \frac{1}{2} \frac{2x}{\sqrt{x^2 + y^2 - 1}} = \frac{x}{\sqrt{x^2 + y^2 - 1}}, \quad \frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2 + y^2 - 1}}$$

$$\text{When } x = 1, y = 2, \quad \frac{\partial f}{\partial x}(1, 2) = \frac{1}{2}, \quad \frac{\partial f}{\partial y}(1, 2) = 1.$$

The equation of the tangent plane is

$$z = 2 + \frac{1}{2}(x - 1) + 1 \cdot (y - 2) \quad \text{or} \quad \frac{1}{2}x + y - z - \frac{1}{2} = 0.$$

6. As $(x, y) \rightarrow (0, 0)$ along x -axis, $f(x, y) = \frac{5x^2}{x^2} = 5$.

$$\text{As } (x, y) \rightarrow (0, 0) \text{ along } y\text{-axis, } f(x, y) = \frac{-y^2}{y^2} = -1.$$

Since $f(x, y)$ has two different limits along two different paths, the limit does not exist.

7. Let $F(x, y, z) = xy + z^3x - 2yz$. Then from Equation (7) in p. 795, we have

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{y + z^3}{3xz^2 - 2y}$$

$$\text{At } (x, y, z) = (1, 1, 1), \quad \partial z / \partial x = -2.$$

Alternatively, we can differentiate implicitly with respect to x ,

$$y + 3z^2 \frac{\partial z}{\partial x} x + z^3 - 2y \frac{\partial z}{\partial x} = 0.$$

Solving this equation for $\partial z / \partial x$, we obtain

$$\frac{\partial z}{\partial x} = \frac{y + z^3}{2y - 3xz^2}.$$

8. The Chain Rule gives

$$w_x = \frac{\partial w}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial x} = x \frac{\partial w}{\partial u} + y \frac{\partial w}{\partial v}$$

$$\begin{aligned} w_{xx} &= \frac{\partial w}{\partial u} + x \left[\frac{\partial^2 w}{\partial u^2} \frac{\partial u}{\partial x} + \frac{\partial^2 w}{\partial u \partial v} \frac{\partial v}{\partial x} \right] + y \left[\frac{\partial^2 w}{\partial u \partial v} \frac{\partial u}{\partial x} + \frac{\partial^2 w}{\partial v^2} \frac{\partial v}{\partial x} \right] \\ &= \frac{\partial w}{\partial u} + x^2 \frac{\partial^2 w}{\partial u^2} + 2xy \frac{\partial^2 w}{\partial u \partial v} + y^2 \frac{\partial^2 w}{\partial v^2} \end{aligned}$$

Similarly

$$w_{yy} = -\frac{\partial w}{\partial u} + y^2 \frac{\partial^2 w}{\partial u^2} - 2xy \frac{\partial^2 w}{\partial u \partial v} + x^2 \frac{\partial^2 w}{\partial v^2}$$

Hence

$$w_{xx} + w_{yy} = (x^2 + y^2) \left(\frac{\partial^2 w}{\partial u^2} + \frac{\partial^2 w}{\partial v^2} \right) = 0.$$

9. Let $z = f(x, y)$. The differential dz is

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = -\frac{dx}{(1+x+y)^2} - \frac{dy}{(1+x+y)^2}.$$

Putting $x = 3$, $y = 6$, $dx = 0.02$ and $dy = 0.05$, we get

$$dz = -0.01(0.02) - 0.01(0.05) = -0.0007.$$

Hence $f(3.02, 6.05) \approx f(3, 6) + dz = 0.1 - 0.0007 = 0.0993$.

10. Let V be the volume, h the height and r the radius. Since $V = \frac{1}{3}\pi r^2 h$, the Chain Rule gives

$$\frac{dV}{dt} = \frac{\partial V}{\partial r} \frac{dr}{dt} + \frac{\partial V}{\partial h} \frac{dh}{dt} = \frac{2}{3}\pi r h \frac{dr}{dt} + \frac{1}{3}\pi r^2 \frac{dh}{dt}.$$

Putting $h = 100$, $\frac{dh}{dt} = 3$, $r = 40$ and $\frac{dr}{dt} = 2$, we get

$$\frac{dV}{dt} = \frac{20800\pi}{3} \text{ in}^3/\text{min}$$

that is, the volume is increasing at $20800\pi/3$ cubic inches per minute at that instant.

11. Set $f(x, y) = \sqrt{1 - x^2 - 2y^2} = k$. We have

$$1 - x^2 - 2y^2 = k^2 \quad \Rightarrow \quad x^2 + 2y^2 = 1 - k^2 \quad \Rightarrow \quad \frac{x^2}{1 - k^2} + \frac{y^2}{\left(\frac{1 - k^2}{2}\right)} = 1$$

that is, the level curves are a family of ellipses.

12. (A)–(II), (B)–(IV), (C)–(I), (D)–(III)

13. $f_x(1, 1) = 1$, $f_y(1, 1) = -2$. $f_{xx}(1, 1) = 0$, $f_{xy}(1, 1) = 0$, $f_{yy}(1, 1) < 0$.

14. The parametric equations for the line segment C are

$$x = 3 - t, \quad y = 1 - t, \quad z = t, \quad 0 \leq t \leq 1$$

Therefore, the work done is

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C y dx - \int_C z dy + \int_C x dz \\ &= \int_0^1 (1 - t)(-1) dt - \int_0^1 t(-1) dt + \int_0^1 (3 - t) dt = \frac{5}{2}. \end{aligned}$$

15. Positive.