Math 214-1, Answers to Midterm Review Problems

1. Let $f(x) = \ln x, g(x) = \sqrt{x-1}$. Find the domains of definition of $f \circ g, g \circ f$ and $\frac{f}{g}$.

$(f \circ g)(x) = f(\sqrt{x-1}) = \ln \sqrt{x-1}$, so the domain is given by $x - 1 \geq 0$ and $\sqrt{x-1} > 0$, hence the domain is $(1, \infty)$.

$g \circ f(x) = g(\ln x) = \sqrt{\ln x - 1}$, so we need that $\ln x - 1 \geq 0$ and $x > 0$. Therefore $\ln x \geq 1$, that is $x \geq e$ the domain is $[e, \infty)$.

To find the domain of $\frac{f}{g}$ we need that $g \neq 0$, hence $\sqrt{x-1} \neq 0$, we get $x \neq 1$. Moreover domain $f = [0, \infty)$, and domain $g = [1, \infty)$, thus the domain of f/g is $(1, \infty)$.

2. The graph of

$$f(x) = \begin{cases} 3 - x^2 & x \leq -1 \\ 2 & -1 < x < 1 \\ 1 - x & x \geq 1 \end{cases}$$

f is not continuous at $x = 1$. Because the left and right limits of f at $x = 1$ are respectively 2 and 0, which are not equal. f is continuous at every other point on the real line.

3. (a) The graph of the function $f(x) = 2 + 3\ln(x - 1)$.

(b) Domain: $(1, +\infty)$ Range: $(-\infty, +\infty)$
(c) Find a formula for the inverse function f^{-1} of f and sketch the graph of $y = f^{-1}(x)$.
$y = 2 + 3\ln(x-1)$
$\frac{y-2}{3} = \ln(x-1)$
$e^{\frac{y-2}{3}} = x - 1$
$x = e^{\frac{y-2}{3}} + 1$
So $f^{-1}(x) = e^{\frac{x-2}{3}} + 1$

for f^{-1}, the domain is $(-\infty, +\infty)$, and the range $(1, +\infty)$

4. Solve the equations:
(a) $\log_2(\ln x) = 1$,
i.e. $2 = \ln x$,
thus $x = e^2$
(b) $2^{x-1} = 3^{x+1}$
$\ln 2^{x-1} = \ln 3^{x+1}$
$(x-1)\ln 2 = (x+1)\ln 3$
$x\ln 2 - \ln 2 = x\ln 3 + \ln 3$
$x\ln 2 - x\ln 3 = \ln 3 + \ln 2$
$x(\ln 2 - \ln 3) = \ln 3 + \ln 2$
$x\ln\frac{2}{3} = \ln 3 * 2$
$x = \frac{\ln 6}{\ln\frac{2}{3}}$

5. Radium has a half-life of 1600 years. How many years does it take for 90% of a given amount to decay?

Let $m(t)$ be the mass of radium after t years. The initial mass $m(0)$ is halved during each 1600-year period so

$$m(1600) = \frac{1}{2}m(0)$$

$$m(3200) = \frac{1}{2}\cdot\frac{1}{2}m(0)$$

From this pattern, it appears that the mass remaining after t years is

$$m(t) = (\frac{1}{2})^{\frac{t}{1600}} m(0)$$

Now find t such that:

$$m(t) = (1 - \frac{90}{100})m(0) = \frac{1}{10}m(0)$$

EITHER:

cancel $m(0)$ from both sides to get

$$\frac{1}{10} = (\frac{1}{2})^{\frac{t}{1600}} \Rightarrow 2^{\frac{t}{1600}} = 10 \Rightarrow t = 1600 \cdot \log_2(10) \sim 5315$$

OR:

Apply the natural logarithm to the above equation:

$$m(0)(\frac{1}{2})^{\frac{t}{1600}} = \frac{1}{10}m(0) \Rightarrow t = 1600(\ln 10)/\ln 2 \sim 5315$$

Therefore, it takes 5316 years for 90% of a given quantity of radium to decay.

6.

(a) $\lim_{x \to 1^-} f(x) = 4$

(b) $\lim_{x \to 1^+} f(x) = 5$

(c) $\lim_{x \to 1} f(x)$ does not exist as $\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$.

(d) $\lim_{x \to 3} f(x) = 5$

(e) At which values of x is f not continuous? f is not continuous at $x = 1$ and $x = 3$.

(f) At which values of x is f not differentiable? The only values of x where f is not differentiable are $x = 1$ and $x = 3$ where f is not continuous.

7. Find the limits if they exist.

(a) $\lim\limits_{x\to 5^-} \dfrac{x^2-3x-10}{x^2-25} = \lim\limits_{x\to 5^-} \dfrac{(x-5)(x+2)}{(x-5)(x+5)} = \lim\limits_{x\to 5^-} \dfrac{x+2}{x+5} = \dfrac{7}{10}$

(b) $\lim\limits_{x\to 9} \dfrac{3-\sqrt{x}}{9-x} = \lim\limits_{x\to 9} \dfrac{3-\sqrt{x}}{(3-\sqrt{x})(3+\sqrt{x})} = \lim\limits_{x\to 9} \dfrac{1}{3+\sqrt{x}} = \dfrac{1}{3+3} = \dfrac{1}{6}$

(c) $\lim\limits_{x\to -\infty} \dfrac{3-x^3}{x^2+5x+5} = \lim\limits_{x\to -\infty} \dfrac{\frac{3}{x^2}-x}{1+\frac{5}{x}+\frac{5}{x^2}} = \dfrac{\infty}{1} = \infty$

8. Find all vertical and horizontal asymptotes of the function $f(x) = \dfrac{2x-3}{x+2}$.

$\lim\limits_{x\to -2^-} \dfrac{2x-3}{x+2} = \infty, \quad \lim\limits_{x\to -2^+} \dfrac{2x-3}{x+2} = -\infty$

Hence $x=-2$ is the only *vertical asymptote*;

$\lim\limits_{x\to +\infty} \dfrac{2x-3}{x+2} = \lim\limits_{x\to +\infty} \dfrac{2-\frac{3}{x}}{1+\frac{2}{x}} = 2, \quad \lim\limits_{x\to -\infty} \dfrac{2x-3}{x+2} = \lim\limits_{x\to -\infty} \dfrac{2-\frac{3}{x}}{1+\frac{2}{x}} = 2$

Hence $y=2$ is the only *horizontal asymptote.*

9. State the definition of the derivative in terms of limits. Use this definition to find the derivative of $f(x) = \dfrac{1}{1-x}$.

$f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \left(\dfrac{1}{1-(x+h)} - \dfrac{1}{1-x}\right)\cdot\dfrac{1}{h}$

$= \lim\limits_{h\to 0} \dfrac{1-x-(1-x-h)}{(1-x)(1-x-h)h} = \lim\limits_{h\to 0} \dfrac{1}{(1-x)(1-x-h)} = \dfrac{1}{(1-x)^2}.$

10. If $y=f(x)$ satisfies $f(2)=5$ and $f'(2)=3$, find the equation of the tangent line to $y=f(x)$ at $x=2$.

The slope $m=f'(2)=3$ and the given point is $(2,5)$. Using the Slope-point formula, we have $y-5=m(x-2)=3(x-2)$, or $y=3x-1$.

11. Evaluate the following derivatives:

(a) $\dfrac{d}{du}(u^2+3u-\dfrac{2}{u})^4 = 4(u^2+3u-\dfrac{2}{u})^3\cdot\dfrac{d}{du}(u^2+3u-\dfrac{2}{u})$

$= 4(u^2+3u-\dfrac{2}{u})^3\cdot(2u+3+\dfrac{2}{u^2})$

(b) $\dfrac{d}{dx}(\sqrt{\sin\pi x}\cdot\cos\pi x) = \dfrac{d}{dx}((\sin\pi x)^{\frac{1}{2}})\cdot\cos\pi x + (\sin\pi x)^{\frac{1}{2}}\cdot\dfrac{d}{dx}(\cos\pi x)$

$= (\dfrac{1}{2}(\sin\pi x)^{-\frac{1}{2}}\pi\cos\pi x)\cdot\cos\pi x + (\sin\pi x)^{\frac{1}{2}}\cdot(-\pi\sin\pi x)$

(c) $\dfrac{d}{dt}\dfrac{\sin t}{t^2-5} = \dfrac{(t^2-5)\frac{d}{dt}\sin t - \frac{d}{dt}(t^2-5)\sin t}{(t^2-5)^2}$

$= \dfrac{(t^2-5)\cos t - 2t\sin t}{(t^2-5)^2}$

(d) $\dfrac{d}{dt}\tan(t^2+1) = 2t\sec^2(t^2+1)$

(e) $\dfrac{d}{dx}\sqrt{e^{2x}+1} = \dfrac{1}{2}(e^{2x}+1)^{-\frac{1}{2}}(2e^{2x})$

12. Suppose that $h(x) = f(x)g(x)$ and $F(x) = f(g(x))$, where $f(2) = 3, g(2) = 5, g'(2) = 4, f'(2) = -2$, and $f'(5) = 11$. Find (a) $h'(2)$ and (b) $F'(2)$.

(a) $h'(2) = f'(2)g(2) + f(2)g'(2) = (-2)(5) + (3)(4) = 2.$

(b) $F'(2) = f'(g(2)) \cdot g'(2) = f'(5) \cdot (4) = (11)(4) = 44.$

13. At what points on the curve $y = \sin^2 x$ is the tangent horizontal?

The tangent is horizontal exactly at the points where the derivative $y'(x) = 0$. Let us calculate the derivative two ways: one can use the product rule or the chain

$$\frac{d}{dx}\sin^2 x = \frac{d}{dx}(\sin x \cdot \sin x)$$
$$= \frac{d}{dx}(\sin x) \cdot \sin x + \sin x \cdot \frac{d}{dx}(\sin x)$$
$$= \cos x \sin x + \sin x \cos x$$

rule.

$$= 2\cos x \sin x$$

$$\frac{d}{dx}\sin^2 x = \frac{d}{dx}(\sin x)^2$$
$$= 2\sin x \frac{d}{dx}(\sin x)$$
$$= 2\sin x \cos x$$

So we want x such that $2\sin x \cos x = 0$, i.e., when $\sin x \cos x = 0$. This happens when either $\sin x = 0$ or $\cos x = 0$.

$\sin x = 0 \Rightarrow x = 2\pi n; \quad n = 0, \pm 1, \pm 2, ...$

$\cos x = 0 \Rightarrow x = 2\pi n + \dfrac{\pi}{2}; \quad n = 0, \pm 1, \pm 2, ...$

So the x where we have a horizontal tangent is the union of the two sets above, that is $x = \cdots - 5\pi/2, -2\pi, -3\pi/2, -\pi, -\pi/2, 0, \pi/2, \pi, 3\pi/2, 2\pi, 5\pi/2 \ldots$

14. Make a careful sketch of $f(x) = \sin x$ and below it make a rough sketch of the graph of an antiderivative F, given that $F(0) = 0$.

$y = f(x)$

$y = F(x)$

15. Let $f(x) = x^3 - 6x^2 + 9x - 5$.

(a) & (b) At what values of x does f have a local maximum or minimum? On what intervals is f increasing or decreasing?

A local maximum or minimum happens at a point when the graph of f changes from increasing to decreasing or from decreasing to increasing respectively. So we have:

$f'(x) = 3x^2 - 12x + 9$

The graph is decreasing when $f'(x) < 0$ and increasing when $f'(x) > 0$. Note that $f'(x) = 3(x - 3)(x - 1)$ is a parabola with roots at $x = 1, 3$. Check that $f'(x) < 0$ on $(1, 3)$ and $f'(x) > 0$ on $(-\infty, 1) \cup (3, \infty)$.

At $x = 1$, the graph of f changes from increasing to decreasing, thus $f(1) = -1$ is a local maximum. Similarly, for $x = 3$, the graph of f changes from decreasing to increasing, thus $f(3) = -5$ is a local minimum.

(c) On what intervals is the graph of f concave up or down?

$f"(x) = 6x - 12$

$f"(x) > 0$ for $x > 2$ and $f"(x) < 0$ for $x < 2$, thus the graph is concave down for $x < 2$ and concave up for $x > 2$.

An inflection point occurs at $x = 2$.

(d) Make a rough sketch of the graph of f.

Now we can use the solutions (a) - (c) to give a rough sketch of the graph of f.

16. The Census figures for the US population (in millions) are listed in the table below. Let f be the function such that $P = f(t)$ is the population (in millions) in year t.

t	1950	1960	1970	1980	1990
$f(t)$	150.7	179.0	205.0	226.5	248.7

(a) Estimate the rate of change of the population for the year 1960.

Average Change for $1950 \leq t \leq 1960 = \dfrac{179 - 150.7}{1960 - 1950} = 2.83$ million people per year.

Average Change for $1960 \leq t \leq 1970 = \dfrac{205 - 179}{1970 - 1960} = 2.6$ million people per year.

Thus $f'(1960)$ is approximately the average of 2.83 and 2.6, that is $f'(1960) \approx 2.715$ million people per year.

(b) What does the derivative of $f(t)$ at $t = 1960$ represent?

The population in 1960 is increasing at a rate of 2.6 million people per year.

17. The following graphs are the graph of a function $y = f(x)$ and its first derivative $y = f'(x)$.

The derivative f':

The graph of the function f:

Which graph is which? Give reasons for your answer.

18. Suppose that the cost, in dollars, for a company to produce x pairs of a new line of jeans is

$C(x) = 1000 + 2x + 0.1x^2 + 0.01x^3$

(a) $C'(x) = 2 + 0.2x + 0.03x^2$

(b) $C'(10) = 2+2+3 = 7$ dollars per pair of jeans, i.e. the cost to produce the 11th pair of jeans is approximately 7 dollars or "the marginal cost at the production level of 10 pairs is \$7 per pair.

19. In December 2001, a Swedish pharmaceutical company released one of its best-selling anti-allergic drugs, X, in U.S. market. The company's monthly revenue $R(t)$ (in millions of dollars) from selling X in U.S. is approximately given by the formula: $R(t) = 25(1 - e^{-0.2t})$ where t represents the number of months since December 2001.

(a) $R'(t) = 5e^{-0.2t}$ and $R"(t) = -e^{-0.2t} < 0$. Hence $R'(t)$ is a decreasing function and the graph of R is concave down.

(b) In the future, we expect to see the increase in revenue to slow down. To be more precise, the revenue will level off and approach a limit of 25 (million dollars) since $\lim_{t\to\infty} R(t) = 25$.