Homework 3, due 2/27

1. Show that if M, N are surfaces, then the Euler characteristic of the connected sum satisfies $\chi(M\#N) = \chi(M) + \chi(N) - 2$.

2. Let K be a simplicial complex. Form the *cone* CK over K as follows: the vertices of CK are the vertices of K as well as one additional vertex o. The simplices of CK are the simplices of K as well as the cones over the simplices of K, i.e. for every simplex $[u_0, \ldots, u_p]$ of K we have an additional simplex $[o, u_0, \ldots, u_p]$ of CK. For example the cone over a p-simplex is a $(p+1)$-simplex. Show that $\tilde{H}_i(CK) = 0$ for all $i \geq 0$.

   Hint: think about decomposing each p-chain in CK into two pieces: a piece involving the vertex o (which then corresponds to a $p-1$-chain on K), and a piece not involving o (which corresponds to a p-chain on K). Work out how the boundary map behaves on these two pieces.

3. Suppose that $g : |K| \to |L|$ is a continuous map between simplicial complexes, and $f : K \to L$ is a simplicial approximation of g. Show that f and g are homotopic.

   Hint: show that there is a family of continuous maps interpolating between f and g linearly: i.e. show that $f_t(x) = tf(x) + (1-t)g(x)$ gives well defined maps for $t \in [0, 1]$.

4. Let K be the simplicial complex given by the boundary of a triangle (i.e. K has 3 vertices and 3 edges). Let us identify the geometric realization $|K|$ with angles $\theta \in [0, 2\pi)$ as in class. Let $g : |K| \to |K|$ be the map $g(\theta) = 3\theta \,(\text{mod } 2\pi)$.

   (a) Find a simplicial approximation $f : K' \to K$ of g, for some subdivision K' of K.

   (b) Compute the induced map $f_* : H_1(|K|) \to H_1(|K|)$. Note that $H_1(|K|) = \mathbb{Z}_2$.

5. Compute the Betti numbers of the projective plane $\mathbb{RP}^2$. You can find a triangulation of it (i.e. a simplicial complex with $\mathbb{RP}^2$ as its geometric realization) at e.g. `http://www.math.jhu.edu/~jmb/note/rp2tri.pdf`.