Homework 1, due 1/30

1. Define a topology $\mathcal{T}$ on $X = \mathbf{R}$ as follows: a subset U is in $\mathcal{T}$ if either $U = \emptyset$, or $X \setminus U$ is finite.
   (a) Show that $\mathcal{T}$ defines a topology.
   (b) Which of the following functions $f : X \to X$ are continuous (using the topology $\mathcal{T}$ on both copies of X):
      (i) $f(x) = \sin x$,
      (ii) $f(x) = x^2$.

2. Consider the set $Y = [-1, 1]$ as a subspace of $\mathbf{R}$, with the subspace topology induced by the standard topology on $\mathbf{R}$ Which of the following sets are open in Y? Which are open in $\mathbf{R}$?

$$\begin{aligned} A &= \{x : 1/2 < |x| < 1\}, \\ B &= \{x : 1/2 < |x| \leq 1\}, \\ C &= \{x : 1/2 \leq |x| < 1\}, \\ D &= \{x : 1/2 \leq |x| \leq 1\}, \\ E &= \{x : 0 < |x| < 1 \text{ and } 1/x \notin \mathbf{Z}_+\}. \end{aligned}$$

3. Let $A \subset X$ and let A' be the limit points of A. Prove that $\overline{A} = A \cup A'$. Note: this is Theorem 17.6 in Munkres. You can read the proof there if you are stuck, but write it up in your own words.

4. Let A, B, A_α be subsets of X. Prove that
   (a) $\overline{A} \cup \overline{B} = \overline{A \cup B}$
   (b) $\bigcup_\alpha \overline{A_\alpha} \subset \overline{\bigcup_\alpha A_\alpha}$. Give an example when equality fails.

5. What is wrong with the following "proof" that $\overline{\bigcup A_\alpha} \subset \bigcup \overline{A_\alpha}$: if $\{A_\alpha\}$ is a collection of sets in X and if $x \in \overline{\bigcup A_\alpha}$, then every neighborhood U of x intersects $\bigcup A_\alpha$. Thus U must intersect some A_α, so that x must belong to the closure of some A_α. Therefore, $x \in \bigcup \overline{A_\alpha}$.