Honors Analysis - Homework 5

1. Let $l^1(\mathbf{Z})$ denote the Banach space of complex sequences $\mathbf{a} = \{a_n\}$, with $n \in \mathbf{Z}$ such that the norm

$$\|\mathbf{a}\| = \sum_{n \in \mathbf{Z}} |a_n|$$

is finite. Show that the multiplication given by convolution

$$(\mathbf{a} * \mathbf{b})_n = \sum_{k \in \mathbf{Z}} a_k b_{n-k}$$

makes $l^1(\mathbf{Z})$ into a Banach algebra. Does it have a unit element?

2. Define the operator $T : L^2[0,1] \to L^2[0,1]$ by letting

$$T(f(x)) = xf(x).$$

Find the spectrum of T.

3. Let X be a compact Hausdorff space, and $C(X)$ the algebra of continuous, complex valued functions on X.

(a) If $p \in X$, show that the set

$$\mathfrak{m}_p = \{f \in C(X) \mid f(p) = 0\}$$

is a maximal ideal in $C(X)$.

(b) Show that every maximal ideal in $C(X)$ is of the form $\mathfrak{m}_p$ for some $p \in X$. (*Hint: show that if I is a proper ideal, then $I \subset \mathfrak{m}_p$ for some p.*)

4. Define $C(S^1)$ to be the space of continuous functions $f : [-\pi, \pi] \to \mathbb{C}$ with $f(-\pi) = f(\pi)$, equipped with the sup norm. The Fourier series of f is

$$\sum_{n=-\infty}^{\infty} c_n(f) e^{int}, \text{ where } c_n(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int}\, dt.$$

Define the linear functionals $T_n : C(S^1) \to \mathbb{C}$ by

$$T_n(f) = \sum_{k=-n}^{n} c_k(f).$$

We will show that there are functions $f \in C(S^1)$ such that $T_n(f)$ does not converge, i.e. the Fourier series of f does not converge at $t = 0$.

(a) Show that the T_n are bounded linear functionals.

(b) Show that $\|T_n\| \to \infty$ as $n \to \infty$. (*Hint: the norm of T_n is the integral of a certain function. To see how to estimate this integral, graph the function.*)

(c) Conclude that there must be some $f \in C(S^1)$ such that the sequence $T_n(f)$ does not converge as $n \to \infty$. (*Hint: use the uniform boundedness principle.*)

5. On the space $L^1([0,1])$, define the product

$$(f * g)(t) = \int_0^t f(t-s)g(s)\,ds.$$

(a) Show that this product is well-defined on $L^1([0,1])$, it is commutative, and satisfies

$$\|f * g\|_1 \leqslant \|f\|_1 \, \|g\|_1.$$

Let A denote the commutative Banach algebra $L^1([0,1])$ together with this product.

(b) Define $\mathbf{1} \in A$ to be the constant 1 function. Show that polynomials in $\mathbf{1}$ (with no constant term) are dense in A. (*Hint: compute $\mathbf{1}^n$ in A.*)

(c) Show that the spectral radius of $\mathbf{1}$ is zero, and as a consequence there are no non-zero homomorphisms $A \to \mathbb{C}$.