Honors Analysis - Homework 4

1. Let $A \subset [0,1]$ be a measurable set, with respect to Lebesgue measure. A point $x \in A$ is a point of density of A, if

$$\lim_{\epsilon \to 0} \frac{\mu(A \cap [x-\epsilon, x+\epsilon])}{2\epsilon} = 1.$$

Show that if $D \subset A$ is the set of points of density of A, then $\mu(A \setminus D) = 0$.

2. Let $A : E \to F$ be a linear operator between Banach spaces E, F. Show that A is continuous if and only if it is bounded.

3. For Banach spaces E, F let $L(E, F)$ be the space of all bounded linear operators $A : E \to F$, equipped with the norm

$$\|A\| = \sup_{x \neq 0} \frac{\|A(x)\|}{\|x\|}.$$

Show that $L(E, F)$ is complete, i.e. it is a Banach space.

4.

(a) Let $A : E \to F$ and $B : F \to G$ be bounded linear operators between Banach spaces. Show that

$$\|BA\| \leqslant \|B\|\|A\|.$$

(b) Let $A : H \to H$ be a bounded linear operator on a Hilbert space H. Show that

$$\|A^*A\| = \|A\|^2.$$

5.

(a) Give an example of an invertible operator $A : E \to E$ between Banach spaces, so that $\|A^2\| \neq \|A\|^2$.

(b) Give an example of operators $A, B : E \to E$ for a Banach space E such that $AB = I$, but $BA \neq I$, where I is the identity.

6. Let C be a non-empty closed convex subset of a Hilbert space H, and let $x \in H$. Show that there is a unique $y \in C$ such that

$$\|x - y\| = \inf\{\|x - z\| \,:\, z \in C\}.$$

7. Let x_n be a sequence in a Banach space E. Show that the series $\sum_{n=1}^{\infty} x_n$ converges in E if the series $\sum_{n=1}^{\infty} \|x_n\|$ converges.