Honors Analysis - Homework 2

1. Prove that every open and every closed subset of $[0,1] \times [0,1]$ is Lebesgue measurable.

2. Let $A \subset [0,1] \times [0,1]$ be Lebesgue measurable.

(a) Prove that for every $\epsilon > 0$ there is an open set U such that $A \subset U$ and $\mu(U) < \mu(A) + \epsilon$.

(b) Prove that for every $\epsilon > 0$ there is a closed set F such that $F \subset A$ and $\mu(F) > \mu(A) - \epsilon$.

(c) Show that in part (a) we cannot replace "open" with "closed".

3. Recall that a Borel subset of $[0,1] \times [0,1]$ is any set obtained from open sets by a countable number of unions, intersections and differences. Show that every Lebesgue measurable subset of $[0,1] \times [0,1]$ is the union of a Borel set and a set of measure zero.

4. Show that if $A \subset [0,1] \times [0,1]$ is Lebesgue measurable, then for all subsets $Z \subset [0,1] \times [0,1]$ we have

$$\mu^*(Z) = \mu^*(Z \cap A) + \mu^*(Z \setminus A).$$

(Hint: first show that there is a measurable set Z' containing Z such that $\mu(Z') = \mu^*(Z)$.)

5. Suppose that $A \subset [0,1] \times [0,1]$ is Lebesgue measurable with $\mu(A) > 0$, and let $\alpha \in (0,1)$. Prove that there exists a rectangle R such that

$$\mu(A \cap R) > \alpha\mu(R).$$

(Hint: try to argue by contradiction - if the inequality fails for all rectangles, then it also fails for all elementary sets.)

6. Suppose that $A \subset [0,1]$ is Lebesgue measurable and $\mu(A) > 0$. Show that the difference set

$$A - A = \{x - y \, : \, x, y \in A\}$$

contains an open interval.
(Hint: you can use the idea from the previous problem.)

7. Suppose that $\mathcal{S}$ is an algebra of subsets of X, and m is a σ-additive measure on $\mathcal{S}$. Let μ be the Lebesgue extension of m, and let $\widetilde{\mu}$ be an arbitrary σ-additive extension of m (i.e. $\widetilde{\mu}$ is a σ-additive measure defined on an algebra containing $\mathcal{S}$, and it coincides with m on $\mathcal{S}$). Prove that if $A \subset X$ is such that both $\mu(A)$ and $\widetilde{\mu}(A)$ are defined, then $\mu(A) = \widetilde{\mu}(A)$.