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FALL 2012 NU PUTNAM SELECTION TEST

Problem A1. Prove that $(\sqrt{5}+2)^{1/3} - (\sqrt{5}-2)^{1/3} = 1$.

- *Answer:* Let $x = (\sqrt{5}+2)^{1/3} - (\sqrt{5}-2)^{1/3}$. Raising to the third power, expanding and simplifying we get that x verifies the equation

$$x^3 + 3x - 4 = 0\,.$$

On the other hand we have:

$$x^3 + 3x - 4 = (x-1)(x^2+x+4)\,.$$

The second factor has no real roots, so the only real root of $x^3 + 3x^2 - 4$ is 1, and the result follows.

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Problem A2. Let x be a real number. Prove that the sequence a_n with

$$a_n = \sum_{k=1}^{n} \cos(kx)$$

is bounded if and only if x is not a multiple of 2π.

- *Answer:* If x is a multiple of 2π then the all terms of the sum are 1, so $a_n = n$, and the sequence diverges.

On the other hand, if x is not a multiple of 2π we will show that the sequence is bounded. In fact, we have $\cos kx = \Re\{e^{ikx}\}$ = real part of e^{ikx}, hence

$$a_2 = \Re\left\{\sum_{k=1}^{n} e^{ikx}\right\} = \Re\left\{\frac{e^{i(n+1)x} - e^{ix}}{e^{ix} - 1}\right\} .$$

If x is not a multiple of 2π then the denominator is not zero, and

$$|a_n| \leq \left|\frac{e^{i(n+1)x} - e^{ix}}{e^{ix} - 1}\right| \leq \frac{|e^{i(n+1)x}| + |e^{ix}|}{|e^{ix} - 1|} = \frac{2}{|e^{ix} - 1|} ,$$

hence the sequence is bounded, Q.E.D.

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Problem A3. For certain $n \times n$-matrices A and B, it is know that $AB = A + B$. Prove that $AB = BA$.

- *Answer:* If I is the identity $n \times n$-matrix then we have:

$$(A - I)(B - I) = AB - A - B + I = AB - AB - I = I\,,$$

hence $B - I = (A - I)^{-1}$, and

$$I = (B - I)(A - I) = BA - B - A + I = BA - AB + I\,,$$

from which we get $BA - AB = 0$, and the desired result follows.

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Problem A4. Determine whether the following statement is true or false. For every finite set V of positive integers there exists a polynomial P with integer coefficients such that $P(1/n) = n$ for all n in V.

- *Answer:* It is true.

A way to find such polynomial is to notice that $xP(x) - 1$ must also be a polynomial with integer coefficients and roots at $1/n$ for $n \in V$. A polynomial with such property is $f(x) = \prod_{n \in V} (1 - nx)$, so $xP(x) - 1 = af(x)$ with a integer would solve the problem. Note that the constant term of f is $f(0) = 1$, hence we must take $a = -1$, and we get that the desired polynomial is $P(x) = \frac{1 - f(x)}{x} = \frac{1}{x}\Big(1 - \prod_{n \in V} (1 - nx)\Big)$.

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Problem A5. Suppose that $a_n > 0$, and $\sum_{n=1}^{\infty} a_n$ converges. Show that there is a sequence $\{b_n\}$ such that $0 < b_n \to \infty$, and $\sum_{n=1}^{\infty} a_n b_n$ converges.

- *Answer:* Under the given hypotheses we have that the tail of the series $d_m = \sum_{n=m}^{\infty} a_n$ is a decreasing sequence tending to zero. For each $k \geq 1$ let $m_k =$ the minimum m such that $d_m < 1/4^k$, and let $b_n = 1$ for $n < m_1$, $b_n = 2^k$ for every n such that $m_k \leq n < m_{k+1}$. Then $0 < b_n \to \infty$, and[1]

$$\begin{aligned}
\sum_{n=1}^{\infty} a_n b_n &= \sum_{n=1}^{m_1-1} a_n b_n + \sum_{k=1}^{\infty} \sum_{n=m_k}^{m_{k+1}-1} a_n b_n \\
&= \sum_{n=1}^{m_1-1} a_n + \sum_{k=1}^{\infty} \Bigg(2^k \underbrace{\sum_{n=m_k}^{m_{k+1}-1} a_n}_{<1/4^k}\Bigg) \\
&< \sum_{n=1}^{m_0-1} a_n + \sum_{k=1}^{\infty} \frac{1}{2^k} \\
&= \sum_{n=1}^{m_0-1} a_n + 1\,,
\end{aligned}$$

hence $\sum_{n=1}^{\infty} a_n b_n$ also converges, Q.E.D.

[1]Note: Empty sums (with no terms) have value zero, e.g., if $m_k = m_{k+1}$, then $\sum_{n=m_k}^{m_{k+1}-1} a_i$ is empty and its value is zero.

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Problem A6. Let a, b, c the side lengths of a triangle T. Prove that there is a triangle with side lengths a^2, b^2, and c^2 if and only if T is acute (all its angles are acute).

- *Answer:* A necessary and sufficient condition for three positive numbers x, y and z to be side lengths of some triangle is $x < y + z$, $y < z + x$, and $z < x + y$. In our case that leads to $a^2 < b^2 + c^2$ and similar inequalities obtained by rotation of a, b, c. By the law of cosines applied to triangle T we have $a^2 = b^2 + c^2 - 2ab \cos A$, where A is the angle opposite to a. Then the condition $a^2 < b^2 + c^2$ is equivalent to $\cos A > 0$, or $A < \frac{\pi}{2}$ (A acute). The same reasoning applies to the other angles.